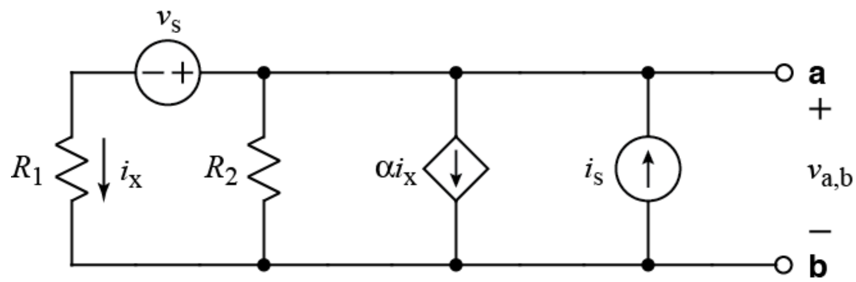
HOMEWORK #10 prob 1 solution

Ex:



a) Using superposition, derive an expression for $v_{a,b}$ that contains no circuit quantities other than i_s, v_s, R_1, R_2, and α. Current i_x must not appear in your solution. **Note:** $\alpha \geq 0$.

b) Make a consistency check on your expression for $v_{a,b}$ by setting resistors and sources to numerical values for which the value of $v_{a,b}$ is obvious. State the values of resistors and sources for your consistency check, and show that your expression for $v_{a,b}$ is satisfied for these values. (In other words, plug the values into your expression from part (a) and show that it agrees with the value from your consistency check.)

c) Find the Thevenin equivalent circuit at terminals **a** and **b**. Express the Thevenin voltage, v_{Th}, and Thevenin resistance, R_{Th} in terms of no circuit quantities other than i_s, v_s, R_1, R_2, and α. i_x must not appear in your solution. **Note:** $\alpha \geq 0$.

d) Find an expression for the value of R_L connected from **a** to **b** that would absorb maximum power. Your answer must be written in terms of no circuit quantities other than i_s, v_s, R_1, R_2, and α. **Note:** $\alpha \geq 0$.

SOL'N: a) In superposition, we turn on one independent source at a time, and we keep dependent sources on all the time.

Case I: Turn on v_s and turn off i_s (which becomes an open circuit).

Using the node-voltage method, we label the entire top wire to the right of v_s as $v_{a,b}$ and we label the bottom wire as reference. We then write an expression for i_x in terms of node-voltage $v_{a,b}$ and write an equation for the sum of currents out of the $v_{a,b}$ node.

$$i_x = \frac{v_{a,b1} - v_s}{R_1}$$

Now the sum-of-currents equation:

$$\frac{v_{a,b1} - v_s}{R_1} + \frac{v_{a,b1}}{R_2} + \alpha \frac{v_{a,b1} - v_s}{R_1} = 0\,\text{A}$$

We factor out $v_{a,b}$ and move constants to the other side of the equation:

$$v_{a,b1}\left(\frac{1}{R_1} + \frac{1}{R_2} + \alpha\frac{1}{R_1}\right) = v_s\left(\frac{1}{R_1} + \alpha\frac{1}{R_1}\right)$$

or

$$v_{a,b1} = v_s \frac{\frac{1+\alpha}{R_1}}{\frac{1+\alpha}{R_1} + \frac{1}{R_2}} = v_s \frac{(1+\alpha)R_2}{(1+\alpha)R_2 + R_1}$$

An alternative approach is to model the dependent source as a resistor, R_{eq}. We observe that $v_{a,b}$ may then be computed by using a voltage divider:

$$v_{a,b1} = v_s \frac{R_2 \parallel R_{Eq}}{R_1 + R_2 \parallel R_{Eq}}$$

To find the equivalent resistance of the dependent source, we observe that current i_x flows through R_2 in parallel with R_{eq}, so we have a current divider with currents $(1 + \alpha)i_x$ in R_2 and $-\alpha i_x$ in R_{eq}. The ratio of the currents is the inverse of the ratio of the resistances:

$$\frac{(1+\alpha)i_x}{-\alpha i_x} = \frac{R_{eq}}{R_2}$$

or

$$R_{eq} = -R_2 \frac{1+\alpha}{\alpha}$$

Now we can compute the parallel resistance of R_2 and R_{eq}.

$$R_2 \parallel R_{eq} = R_2 \parallel -\frac{R_2(1+\alpha)}{\alpha} = R_2 \cdot 1 \parallel -\frac{1+\alpha}{\alpha} = R_2 \frac{-\frac{1+\alpha}{\alpha}}{1 - \frac{1+\alpha}{\alpha}}$$

or

$$R_2 \parallel R_{eq} = R_2 \frac{-(1+\alpha)}{\alpha-(1+\alpha)} = R_2(1+\alpha)$$

We use this result in the voltage divider, obtaining the same result as before:

$$v_{a,b1} = v_s \frac{R_2 \parallel R_{Eq}}{R_1 + R_2 \parallel R_{Eq}} = v_s \frac{R_2(1+\alpha)}{R_1 + R_2(1+\alpha)}$$

Case II: Turn off v_s (which becomes a wire) and turn on i_s.

Using node voltage, we proceed as in Case I but have a simpler equations because v_s is off.

$$i_{x2} = \frac{v_{a,b2}}{R_1}$$

Now the sum-of-currents equation:

$$\frac{v_{a,b2}}{R_1} + \frac{v_{a,b2}}{R_2} + \alpha\frac{v_{a,b2}}{R_1} - i_s = 0\,\text{A}$$

We factor out $v_{a,b}$ and move constants to the other side of the equation:

$$v_{a,b2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \alpha\frac{1}{R_1}\right) = i_s$$

or

$$v_{a,b2}\left[R_2(1+\alpha) + R_1\right] = i_s R_1 R_2$$

or

$$v_{a,b2} = i_s \frac{R_1 R_2}{(1+\alpha)R_2 + R_1}$$

An alternative approach is to model the dependent source as a resistor. We have the dependent source in parallel with R_1, and we may calculate the voltage across R_1 as $i_x R_1$. This voltage is also across the dependent source, allowing us to define an equivalent resistance for the dependent source using Ohm's law:

$$R_{Eq2} = \frac{i_x R_1}{\alpha i_x} = \frac{R_1}{\alpha}$$

It is interesting to note that this equivalent resistance is different than the equivalent resistance from Case I.

Now we have current source i_s in parallel with three resistors that we can combine into one resistance, and Ohm's law gives $v_{a,b2}$ in terms of current times resistance:

$$v_{a,b2} = i_s \cdot R_1 \parallel R_2 \parallel R_{Eq2} = i_s \cdot R_1 \parallel R_2 \parallel \frac{R_1}{\alpha}$$

or

$$v_{a,b2} = i_s \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{\alpha}{R_1}} = i_s \frac{R_1 R_2}{(1+\alpha)R_2 + R_1}$$

Now we sum the two $v_{a,b}$'s to get the total $v_{a,b}$:

$$v_{a,b} = v_{a,b1} + v_{a,b2} = v_s \frac{R_2(1+\alpha)}{R_1 + R_2(1+\alpha)} + i_s \frac{R_1 R_2}{(1+\alpha)R_2 + R_1}$$

b) One consistency check is to set $v_s = 0$ V and set $\alpha = 1$, causing R_1 and the dependent source to be in parallel and have the same current, implying that they are the same resistance. In parallel, they have resistance $R_1/2$.

In this case, our circuit will have the following output voltage:

$$v_{a,b} = i_s \cdot \frac{R_1}{2} \parallel R_2 = i_s \cdot \frac{\dfrac{R_1}{2} R_2}{\dfrac{R_1}{2} + R_2} = i_s \cdot \frac{R_1 R_2}{R_1 + 2R_2}$$

Now we check what value our formula from part (a) gives:

$$v_{a,b} = 0 \cdot \frac{R_2(1+1)}{R_1 + R_2(1+1)} + i_s \frac{R_1 R_2}{(1+1)R_2 + R_1} = i_s \frac{R_1 R_2}{2R_2 + R_1}$$

This agrees with what we expect, so the consistency check is satisfied. Many other checks are possible.

c) The voltage $v_{a,b}$ found in part (a) is the Thevenin equivalent voltage, so all we need now is R_{Th}. Perhaps the simplest way to find R_{Th} is to turn off the independent sources and connect a current source to the output. We then determine $v_{a,b}$ across the current source and use Ohm's law to find R_{Th}. For the source, we could use a value of i_s, in which case we have exactly Case II of the superposition from part (a). Our voltage will then be $v_{a,b2}$. Thus, we have the following value for R_{Th}:

$$R_{Th} = \frac{v_{a,b2}}{i_s} = \frac{i_s \dfrac{R_1 R_2}{(1+\alpha)R_2 + R_1}}{i_s} = \frac{R_1 R_2}{(1+\alpha)R_2 + R_1}$$

An alternative approach to finding R_{Th} is to use the short-circuit current, i_{sc}, that flows from **a** to **b** when a wire is connected across those terminals. In that case, the voltage on the top and bottom rails is zero. This means there is no voltage drop across R_2, and we may ignore R_2. Also, we have voltage $-v_s$ on the top end of R_1, giving the current for i_x directly:

$$i_x = -\frac{v_s}{R_1}$$

Now we can write a current summation for the top rail:

$$-\frac{v_s}{R_1} + \alpha\left(-\frac{v_s}{R_1}\right) - i_s + i_{sc} = 0\,\text{A}$$

or

$$i_{sc} = \frac{v_s}{R_1} - \alpha\left(-\frac{v_s}{R_1}\right) + i_s = v_s \frac{1+\alpha}{R_1} + i_s$$

Using this current, we find R_{Th}:

$$R_{Th} = \frac{v_{a,b}}{i_{sc}} = \frac{v_s \dfrac{R_2(1+\alpha)}{R_1 + R_2(1+\alpha)} + i_s \dfrac{R_1 R_2}{(1+\alpha)R_2 + R_1}}{v_s \dfrac{1+\alpha}{R_1} + i_s}$$

or

$$R_{\mathrm{Th}} = \frac{v_{\mathrm{a,b}}}{i_{\mathrm{sc}}} = \frac{R_1 R_2}{R_1 + R_2(1+\alpha)}$$

d) Maximum power is obtained by setting $R_{\mathrm{L}} = R_{\mathrm{Th}}$:

$$R_{\mathrm{L}} = R_{\mathrm{Th}} = \frac{R_1 R_2}{(1+\alpha)R_2 + R_1}$$