Ex:

$R_1 = 18 \, \Omega \quad R_2 = 48 \, \Omega \quad R_3 = 144 \, \Omega \quad C = 31.25 \, \mu F \quad L = 2 \, mH$

a) What type of filter is the above circuit: a band-pass or a band-reject?

**Hint:** Use a Thevenin equivalent to combine all the R’s into one.

For the filter shown above, calculate the following quantities:

b) $\omega_o$  
c) $\omega_{C1}$ and $\omega_{C2}$  
d) $\beta$ and $Q$

**sol’n:**

a) The order of the branches between the rails may be changed without affecting the filter’s transfer function. Thus, we may use the following schematic:

We treat $V_i$ as a $V$-source and take the Thevenin equivalent to the left of terminals $a$ and $b$. 

$$V_{th} = \text{output } V_i \text{ no load}$$

$$V_{th} = V_i \frac{R_2 \parallel R_3}{R_1 + R_2 \parallel R_3}$$
\[ R_{Th} = R_1 \parallel R_2 \parallel R_3 \]

So we have the following picture:

Note: \( \frac{R_2 \parallel R_3}{R_1 + R_2 \parallel R_3} = \frac{R_1 \parallel R_2 \parallel R_3}{R_1} = \frac{R_{Th}}{R_1} \)

So \( V_{Th} = \frac{V_i \cdot R_2 \parallel R_3}{R_1 + R_2 \parallel R_3} \cdot \frac{R_{Th}}{R_1} \)

Calculated values:

\[ R_1 \parallel R_2 \parallel R_3 = \frac{1}{\frac{1}{18\Omega} + \frac{1}{48\Omega} + \frac{1}{144\Omega}} = 12\Omega \]

\[ \frac{R_1 \parallel R_2 \parallel R_3}{R_1} = \frac{12\Omega}{18\Omega} = \frac{2}{3} \]
We effectively have a 2-stage system. The first stage multiplies $V_i$ by $\frac{2}{3}$, and the second stage is an RLC filter.

$$H(j\omega) \equiv \frac{V_o}{V_i} = \frac{\frac{2}{3}}{\frac{j\omega L + \frac{1}{j\omega C}}{\frac{R_m + j\omega L + \frac{1}{j\omega C}}{V_i}}}$$

We put this in standard form:

$$H(j\omega) = k \frac{1}{1 \pm j\chi \frac{R_m}{R_{Th}}}$$

where $k = $ real constant
$\chi = \omega L - \frac{1}{\omega C}$

Here, $k = \frac{2}{3}$ and $\chi = \omega L - \frac{1}{\omega C}$.

$$H(j\omega) = \frac{\frac{2}{3}}{\frac{1}{1 - j\omega L \frac{R_{Th}}{\omega L - \frac{1}{\omega C}}}}$$

We will use $H(j\omega)$ below. Returning to the question at hand, however, we are asked to determine the type of filter we have. We do this by considering $\omega = 0$, $\omega \to \infty$, and $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$.

$\omega = 0$: $\frac{j\omega L + j\omega C}{j\omega C} = \frac{1}{j\omega C} = \frac{1}{j\omega L} = -j\omega = \text{open}$

$$V_v^2 = \frac{R_{Th}}{V_0} = \frac{V_i}{\frac{2}{3}} = \frac{V_i}{\frac{2}{3}} = \frac{2}{3}$$

$$H(j\omega) = \frac{V_o}{V_i} = \frac{V_i \left( \frac{2}{3} \right)}{V_i} = \frac{2}{3}$$

No current, so no V-drop across $R_{Th}$. 

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$\omega = 0$: $\frac{j\omega L + j\omega C}{j\omega C} = \frac{1}{j\omega C} = \frac{1}{j\omega L} = -j\omega = \text{open}$
\( \omega \to \infty : \quad j \omega L = j \infty = \text{open} \)
\[
\frac{1}{j \omega C} = \frac{1}{j \infty} = -j0 = \text{wire}
\]

![Circuit Diagram]

\[ H(j \omega) = \frac{V_o}{V_i} = V_i \left( \frac{\frac{z}{3}}{\frac{2}{3}} \right) = \frac{2}{3} \]

\[ \omega = \frac{1}{V_L C} = \omega_0 : \quad j \omega L = -\frac{1}{j \omega C} \]

so \( j \omega L + \frac{1}{j \omega C} = 0 \text{\ ohm} = \text{wire} \)

![Circuit Diagram]

\[ H(j \omega_0) = \frac{0}{V_i} = 0 \]

We can now sketch \(|H| vs \omega\):

![Graph of |H| vs \omega]

We have a band-reject filter.
b) \( w_0 = \frac{1}{LC} = \frac{1}{\sqrt{2m(31.25\mu)}} \frac{r/s}{r/s} \)

\[ = \frac{1}{\sqrt{62.5n}} \frac{r/s}{r/s} = \frac{1}{(250\mu)^2} \]

\[ = \frac{1}{250\mu} = 4k \frac{r/s}{r/s} \]

c) \( \omega_{1,2} \) are where \( |H(j\omega)| = \frac{1}{\sqrt{2}} \max \omega |H(j\omega)| \)

Here, \( H(j\omega) = \frac{2}{3} \cdot \frac{1}{1 - j \frac{R_{th}}{\omega L - \frac{1}{\omega C}}} \)

\[ |H(j\omega)| = \frac{2}{3} \cdot \frac{1}{|1 - j \frac{R_{th}}{\omega L - \frac{1}{\omega C}}|} \]

\[ = \frac{2}{3} \cdot \frac{1}{\sqrt{1^2 + \left( \frac{R_{th}}{\omega L - \frac{1}{\omega C}} \right)^2}} \]

The smallest value the \( \sqrt{\cdot} \) can possibly be is \( \sqrt{1^2} = 1 \). Is it actually this small for any \( \omega \)?

Yes, for \( \omega = 0 \) or \( \omega \to \infty \) we get \( \frac{R_{th}}{\omega L - \frac{1}{\omega C}} = 0 \), which we saw earlier when plotting \( |H|/\omega \) vs \( \omega \).

So \( \max_\omega |H(j\omega)| = \frac{2}{3} \).
So we want to solve \( |H(j\omega)| = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} \):

\[
\frac{2}{3} \cdot \frac{1}{|1 + j \frac{R_{th}}{\omega L - \frac{1}{\omega C}}|} = \frac{1}{\sqrt{2}} \cdot \frac{2}{3}
\]

or

\[
|1 + j \frac{R_{th}}{\omega L - \frac{1}{\omega C}}| = \sqrt{2}
\]

We observe that \( |1 + j| = \sqrt{2} \). So we want

\[
\frac{R_{th}}{\omega L - \frac{1}{\omega C}} = \pm 1
\]

Flip both sides upside-down:

\[
\omega L - \frac{1}{\omega C} = \pm 1
\]

or

\[
\omega \frac{L - 1}{\omega C} = \pm R_{th}
\]

or

\[
\omega L \pm R_{th} - \frac{1}{\omega C} = 0
\]

or

\[
\omega^2 L \pm \omega R_{th} - \frac{1}{C} = 0
\]

or

\[
\omega^2 \pm \omega \frac{R_{th}}{L} - \frac{1}{LC} = 0
\]

or

\[
\omega_{cl,1,2} = \pm \frac{1}{2} \frac{R_{th}}{L} \pm \frac{1}{\sqrt{\left(\frac{1}{2} \frac{R_{th}}{L}\right)^2 + \frac{1}{LC}}}
\]

Use roots where \( \omega_{cl,1,2} > 0 \).
The $\sqrt{i}$ term must be larger than $\frac{R}{2L}$ since the term inside the $\sqrt{i}$ is at least as big as $(\frac{R}{2L})^2$.

So we use $+\sqrt{i}$ terms:

$$\omega_{c1,2} = \pm \frac{R_{th}}{2L} + \sqrt{\left(\frac{R_{th}}{2L}\right)^2 + \frac{1}{LC}}$$

**Values:**

$$\frac{R_{th}}{2L} = \frac{12 \pi}{2 \cdot 2 m H} = 3 k \text{ r/s}$$

$$\frac{1}{LC} = (4k \text{ r/s})^2$$

$$\omega_{c1,2} = \mp 3k + \sqrt{(3k)^2 + (4k)^2} \text{ r/s}$$

$$\omega_{c1} = 2k \text{ r/s}$$

$$\omega_{c2} = 8k \text{ r/s}$$

\(d) \quad \beta \equiv \omega_{c2} - \omega_{c1} = \frac{R_{th}}{2L} + \sqrt{i} - (-\frac{R_{th}}{2L} + \sqrt{i})$$

or

$$\beta = \frac{R_{th}}{L}$$

$$\beta = \frac{12 \pi}{2 m H} = 6 k \text{ r/s}$$

$$Q \equiv \frac{\omega_0}{\beta} = \frac{4k \text{ r/s}}{6k \text{ r/s}} = \frac{2}{3} \text{ (unitless)}$$

**Note:** $\omega_0 = \sqrt{\omega_{c1} \omega_{c2}} = \sqrt{2k \cdot 8k} \text{ r/s}$

$$\omega_0 = 4k \text{ r/s} \text{ (sometimes useful)}$$