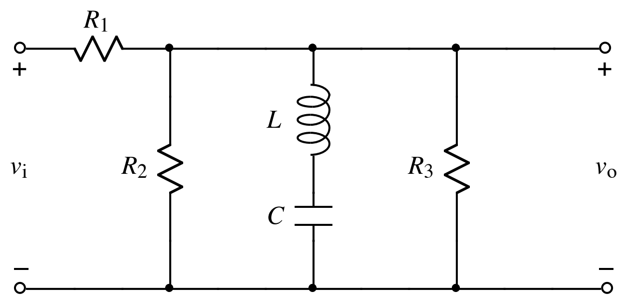
Ex:



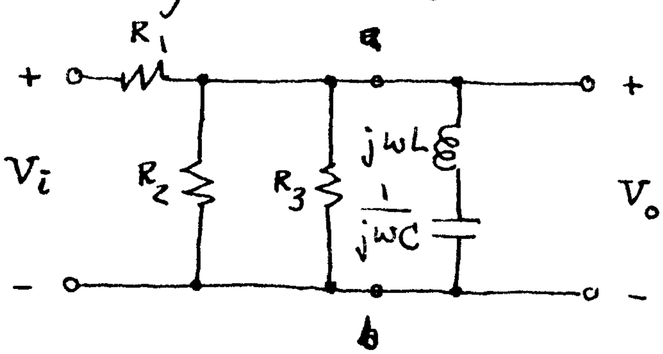
$R_1 = 18\ \Omega \quad R_2 = 48\ \Omega \quad R_3 = 144\ \Omega \quad C = 31.25\ \mu\text{F} \quad L = 2\text{ mH}$

a) What type of filter is the above circuit: a band-pass or a band-reject?
Hint: Use a Thevenin equivalent to combine all the R's into one.

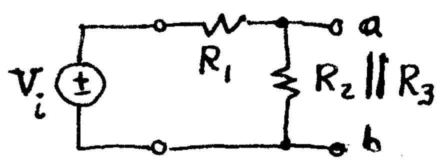
For the filter shown above, calculate the following quantities:

b) ω_o c) ω_{C1} and ω_{C2} d) β and Q

sol'n: a) The order of the branches between the rails may be changed without affecting the filter's transfer function. Thus, we may use the following schematic:

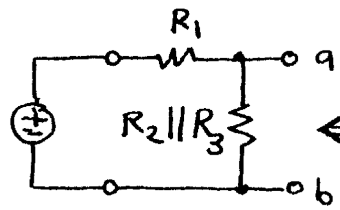


We treat V_i as a V-source and take the Thevenin equivalent to the left of terminals a and b.



V_{Th} = output V, no load

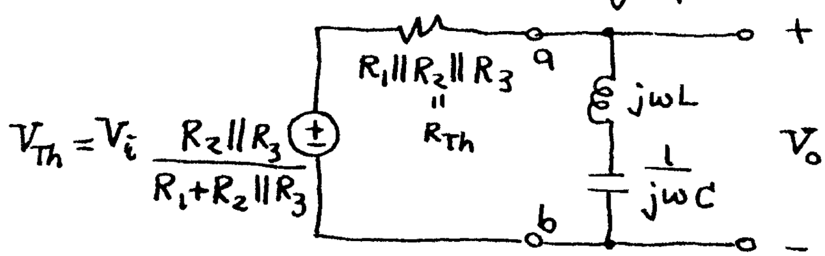
$= V_i \dfrac{R_2 \| R_3}{R_1 + R_2 \| R_3}$



$\Leftarrow R_{Th}$ = R seen looking into a, b with V-src off (V-src off = wire)

$$R_{Th} = R_1 \| R_2 \| R_3$$

So we have the following picture:



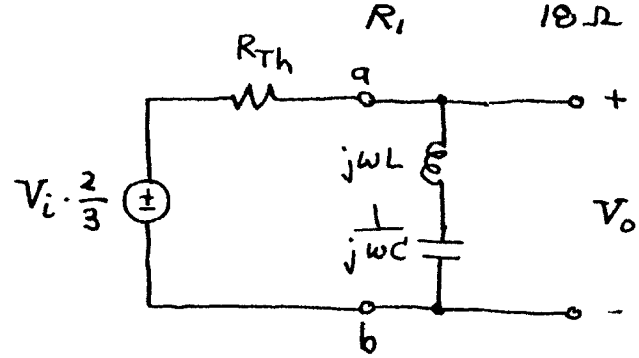
Note: $$\frac{R_2 \| R_3}{R_1 + R_2 \| R_3} = \frac{R_1 \| R_2 \| R_3}{R_1} = \frac{R_{Th}}{R_1}$$

So $$V_{Th} = V_i \frac{R_1 \| R_2 \| R_3}{R_1}.$$

Calculated values: $$R_1 \| R_2 \| R_3 = \frac{1}{\frac{1}{18\Omega} + \frac{1}{48\Omega} + \frac{1}{144\Omega}}$$

$$R_1 \| R_2 \| R_3 = 12\,\Omega$$

$$\frac{R_1 \| R_2 \| R_3}{R_1} = \frac{12\,\Omega}{18\,\Omega} = \frac{2}{3}$$



We effectively have a 2-stage system. The first stage multiplies V_i by $\frac{2}{3}$, and the second stage is an RLC filter.

$$H(j\omega) \equiv \frac{V_o}{V_i} = \frac{\cancel{V_i}\left(\frac{2}{3}\right)}{\cancel{V_i}} \cdot \frac{j\omega L + \frac{1}{j\omega C}}{R_{Th} + j\omega L + \frac{1}{j\omega C}}$$

We put this in standard form:

$$H(j\omega) = K \frac{1}{1 \pm jX}$$ where K = real const, X = real term

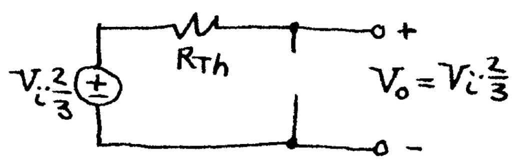
Here, $K = \frac{2}{3}$ and $X = \frac{R_{Th}}{\omega L - \frac{1}{\omega C}}$.

$$H(j\omega) = \frac{2}{3} \cdot \frac{1}{1 - j\dfrac{R_{Th}}{\omega L - \frac{1}{\omega C}}}$$

We will use $H(j\omega)$ below. Returning to the question at hand, however, we are asked to determine the type of filter we have. We do this by considering $\omega = 0$, $\omega \to \infty$, and $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$.

$\omega = 0$: $j\omega L = j0\,\Omega$ = wire

$\frac{1}{j\omega C} = \frac{1}{j0\,\Omega} = -j\infty$ = open

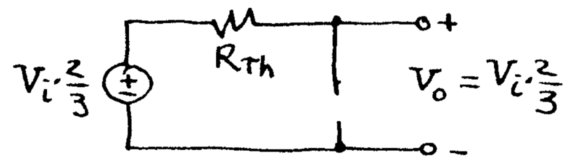


No current, so no V-drop across R_{Th}.

$$H(j0) = \frac{V_o}{V_i} = \frac{V_i\left(\frac{2}{3}\right)}{V_i} = \frac{2}{3}$$

$\omega \to \infty$: $\quad j\omega L = j\infty = \text{open}$

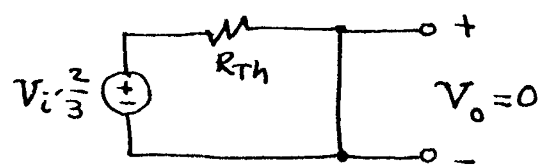
$$\frac{1}{j\omega C} = \frac{1}{j\infty} = -j0 = \text{wire}$$



$$H(j\infty) = \frac{V_o}{V_i} = \frac{V_i\left(\frac{2}{3}\right)}{V_i} = \frac{2}{3}$$

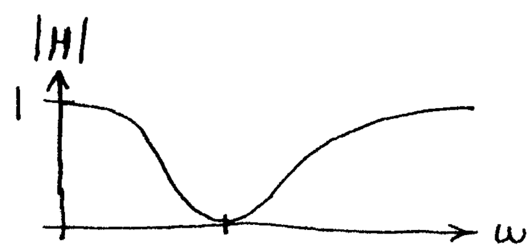
$\omega = \frac{1}{\sqrt{LC}} = \omega_o$: $\quad j\omega L = -\frac{1}{j\omega C}$

$$\text{so } j\omega L + \frac{1}{j\omega C} = 0\,\Omega = \text{wire}$$



$$H(j\omega_o) = \frac{0}{V_i} = 0$$

We can now sketch $|H|$ vs ω:



We have a band-reject filter.

b) $\omega_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{2m(31.25\mu)}}$ r/s

$= \dfrac{1}{\sqrt{62.5n}}$ r/s $= \dfrac{1}{\sqrt{(250\mu)^2}}$ r/s

$= \dfrac{1}{250\mu} = 4k$ r/s

c) $\omega_{c1,2}$ are where $|H(j\omega)| = \dfrac{1}{\sqrt{2}} \max_{\omega} |H(j\omega)|$

Here, $H(j\omega) = \dfrac{2}{3} \cdot \dfrac{1}{1 - j\dfrac{R_{Th}}{\omega L - \frac{1}{\omega C}}}$

$$|H(j\omega)| = \frac{2}{3} \cdot \frac{1}{\left|1 + j\dfrac{R_{Th}}{\omega L - \frac{1}{\omega C}}\right|}$$

$$= \frac{2}{3} \cdot \frac{1}{\sqrt{1^2 + \left(\dfrac{R_{Th}}{\omega L - \frac{1}{\omega C}}\right)^2}}$$

The smallest value ~~the~~ $\sqrt{\ }$ can possibly be is $\sqrt{1^2} = 1$. Is it actually this small for any ω? Yes, for $\omega = 0$ or $\omega \to \infty$ we get

$\dfrac{R_{Th}}{\omega L - \frac{1}{\omega C}} = 0$, which we saw

earlier when plotting $|H|$ vs ω.

So $\max_{\omega} |H(j\omega)| = \dfrac{2}{3}$.

So we want to solve $|H(j\omega)| = \frac{1}{\sqrt{2}} \cdot \frac{2}{3}$:

$$\cancel{\frac{2}{3}} \cdot \frac{1}{\left|1 + j\frac{R_{Th}}{\omega L - \frac{1}{\omega C}}\right|} = \frac{1}{\sqrt{2}} \cdot \cancel{\frac{2}{3}}$$

or $$\left|1 + j\frac{R_{Th}}{\omega L - \frac{1}{\omega C}}\right| = \sqrt{2}$$

We observe that $|1 \pm j| = \sqrt{2}$. So we want

$$\frac{R_{Th}}{\omega L - \frac{1}{\omega C}} = \pm 1$$

Flip both sides upside-down:

$$\frac{\omega L - \frac{1}{\omega C}}{R_{Th}} = \pm 1$$

or

$$\omega L - \frac{1}{\omega C} = \pm R_{Th}$$

or

$$\omega L \pm R_{Th} - \frac{1}{\omega C} = 0$$

or

$$\omega^2 L \pm \omega R_{Th} - \frac{1}{C} = 0$$

or

$$\omega^2 \pm \omega \frac{R_{Th}}{L} - \frac{1}{LC} = 0$$

or

$$\omega_{c1,2} = \pm\frac{1}{2}\frac{R_{Th}}{L} \pm \sqrt{\left(\frac{1}{2}\frac{R_{Th}}{L}\right)^2 + \frac{1}{LC}}$$

Use roots where $\omega_{c1,2} > 0$.

The $\sqrt{\;}$ term must be larger than $\frac{R}{2L}$ since the term inside the $\sqrt{\;}$ is at least as big as $\left(\frac{R}{2L}\right)^2$.

So we use $+\sqrt{\;}$ terms:

$$\omega_{c1,2} = \pm \frac{R_{Th}}{2L} + \sqrt{\left(\frac{R_{Th}}{2L}\right)^2 + \frac{1}{LC}}$$

Values: $\frac{R_{Th}}{2L} = \frac{12\Omega}{2 \cdot 2mH} = 3k\ r/s$

$$\frac{1}{LC} = (4k\ r/s)^2$$

$$\omega_{c1,2} = \pm 3k + \underbrace{\sqrt{(3k)^2 + (4k)^2}}_{5k}\ r/s$$

$\omega_{c1} = 2k\ r/s$
$\omega_{c2} = 8k\ r/s$

d) $\beta \equiv \omega_{c2} - \omega_{c1} = \frac{R_{Th}}{2L} + \sqrt{\;} - \left(-\frac{R_{Th}}{2L} + \sqrt{\;}\right)$

or $\beta = \frac{R_{Th}}{L}$

$$\beta = \frac{12\Omega}{2mH} = 6k\ r/s$$

$$Q \equiv \frac{\omega_0}{\beta} = \frac{4k\,r/s}{6k\,r/s} = \frac{2}{3} \quad \text{(unitless)}$$

Note: $\omega_0 = \sqrt{\omega_{c1}\,\omega_{c2}} = \sqrt{2k \cdot 8k}\ r/s$

$\omega_0 = 4k\,r/s$ (sometimes useful)