EX: Find the inverse Laplace transform of $\dfrac{24s}{(s+5)^4}$. Note: $\mathcal{L}\{t^n e^{-at}\} = \dfrac{n!}{(s+a)^{n+1}}$

SOL'N: This problem may be solved in a variety of ways. First, we solve it by writing the numerator in terms of (s + 5):

$$\mathcal{L}^{-1}\left\{\frac{24s}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{24[(s+5)-5]}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{24(s+5)}{(s+5)^4} - \frac{24(5)}{(s+5)^4}\right\}$$

or

$$\mathcal{L}^{-1}\left\{\frac{24s}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{24}{(s+5)^3} - \frac{120}{(s+5)^4}\right\}$$

$$= \left(\frac{24}{2!}t^2e^{-5t} - \frac{120}{3!}t^3e^{-5t}\right)u(t)$$

or

$$\mathcal{L}^{-1}\left\{\frac{24s}{(s+5)^4}\right\} = \left(12t^2e^{-5t} - 20t^3e^{-5t}\right)u(t)$$

Second, we solve the problem by using partial fractions.

$$\mathcal{L}^{-1}\left\{\frac{24s}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{(s+5)^4} + \frac{B}{(s+5)^3} + \frac{C}{(s+5)^2} + \frac{D}{s+5}\right\}$$

We use the derivative method to find the coefficients.

$$A = \left.\frac{24s}{(s+5)^4}(s+5)^4\right|_{s=-5} = 24s|_{s=-5} = -120$$

and

$$B = \left.\left[\frac{d}{ds}\frac{24s}{(s+5)^4}(s+5)^4\right]\right|_{s=-5} = \left.\left[\frac{d}{ds}24s\right]\right|_{s=-5} = 24|_{s=-5} = 24$$

Further derivatives give zero, but in general the next coefficient would be found by the following calculation:

$$C = \frac{1}{2}\left.\left[\frac{d^2}{ds^2}\frac{24s}{(s+5)^4}(s+5)^4\right]\right|_{s=-5} = \frac{1}{2}\left.\left[\frac{d^2}{ds^2}24s\right]\right|_{s=-5} = 0$$

and

$$D = \frac{1}{3!}\left[\frac{d^3}{ds^3}\frac{24s}{(s+5)^4}(s+5)^4\right]\Bigg|_{s=-5} = \frac{1}{6}\left[\frac{d^3}{ds^3}24s\right]\Bigg|_{s=-5} = 0$$

The values of A and B are the same as the coefficients found in the first method, so our final answer will be the same.

Third, we observe that we can use the identity for the derivative.

$$\mathcal{L}\left\{\frac{dv(t)}{dt}\right\} = sV(s) - v(t = 0^-)$$

Thus, we have the following, assuming the inner inverse transform yields a function with a value of zero at $t = 0^-$:

$$\frac{24s}{(s+5)^4} = \mathcal{L}\left\{\frac{d}{dt}\mathcal{L}^{-1}\left\{\frac{24}{(s+5)^4}\right\}\right\}$$

Let's see what happens when we inverse transform the inner function.

$$\mathcal{L}^{-1}\left\{\frac{24}{(s+5)^4}\right\} = \frac{24}{3!}t^3 e^{-5t} = 4t^3 e^{-5t}$$

Fortunately, the result is a function whose value is indeed zero at $t = 0^-$.

$$\mathcal{L}^{-1}\left\{\frac{24}{(s+5)^4}\right\} = \frac{24}{3!}t^3 e^{-5t} = 4t^3 e^{-5t}$$

Now we take the derivative in the time domain, (and multiply by $u(t)$ to remind ourselves that the values in the time domain are unknown for $t < 0$.

$$\mathcal{L}^{-1}\left\{\frac{24s}{(s+5)^4}\right\} = \left[\frac{d}{dt}4t^3 e^{-5t} = 4(3)t^2 e^{-5t} + 4t^3(-5)e^{-5t}\right]u(t)$$
$$= \left[12t^2 e^{-5t} - 20t^3 e^{-5t}\right]u(t)$$

We get the same result as before.