EX: Find the Laplace transform of the following waveform:

$$(t-a)\cos(t-b)u(t-a) \quad \text{where } \ a>0$$

SOL'N: Use the delay identity:

$$\mathcal{L}\{v(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{v(t)\} = e^{-as}V(s)$$

To recognize $(t-a)\cos(t-b)$ as a function of (t – a), we substitute t = (t - a) + a where necessary.

$$(t-a)\cos(t-b)u(t-a) = (t-a)\cos((t-a)+a-b)u(t-a)$$

From the right side, we identify $v(t$ - $a)$:

$$v(t-a) = (t-a)\cos((t-a)+a-b)$$

We obtain $v(t)$ by substituting t for $t-a$:

$$v(t) = t\cos(t+a-b)$$

Now we want to find $\mathcal{L} = \{v(t)\} \equiv V(s)$. We apply a trigonometric identity to the cosine to obtain a form we can transform:

$$\cos(t+a-b) = \cos t\cos(a-b) - \sin t\sin(a-b)$$

We transform this expression and then apply an identity to handle the multiplication by t.

$$\mathcal{L}\{\cos t\cos(a-b) - \sin t\sin(a-b)\} = \cos(a-b)\frac{s}{s^2+1} - \sin(a-b)\frac{1}{s^2+1}$$

Now apply the identity for multiplication by t:

$$\mathcal{L}\{tv(t)\} = -\frac{dV(s)}{ds}$$

The product rule, the rule for polynomial differentiation, and the chain rule yield the derivative.

$$-\frac{d}{ds}\left[\cos(a-b)\frac{s}{s^2+1} - \sin(a-b)\frac{1}{s^2+1}\right]$$

$$= -\frac{d}{ds}\left[\cos(a-b)s(s^2+1)^{-1} - \sin(a-b)(s^2+1)^{-1}\right]$$

or

$$-\frac{d}{ds}\left[\cos(a-b)\frac{s}{s^2+1}-\sin(a-b)\frac{1}{s^2+1}\right]$$

$$=-\cos(a-b)[(s^2+1)^{-1}+s(-1)(s^2+1)^{-2}2s]$$

$$+\sin(a-b)[(-1)(s^2+1)^{-2}2s]$$

$$=-\cos(a-b)\left[\frac{1}{s^2+1}-\frac{2s^2}{(s^2+1)^2}\right]$$

$$-\sin(a-b)\left[\frac{2s}{(s^2+1)^2}\right]$$

$$=\cos(a-b)\left[\frac{2s^2-(s^2+1)}{(s^2+1)^2}\right]$$

$$-\sin(a-b)\left[\frac{2s}{(s^2+1)^2}\right]$$

$$=\cos(a-b)\left[\frac{s^2-1}{(s^2+1)^2}\right]$$

$$-\sin(a-b)\left[\frac{2s}{(s^2+1)^2}\right]$$

Finally, we add the e^{-as} from the delay identity:

$$\mathcal{L}\{(t-a)\cos(t-b)u(t-a)\}$$

$$=e^{-as}\left\{\cos(a-b)\left[\frac{s^2-1}{(s^2+1)^2}\right]-\sin(a-b)\left[\frac{2s}{(s^2+1)^2}\right]\right\}$$