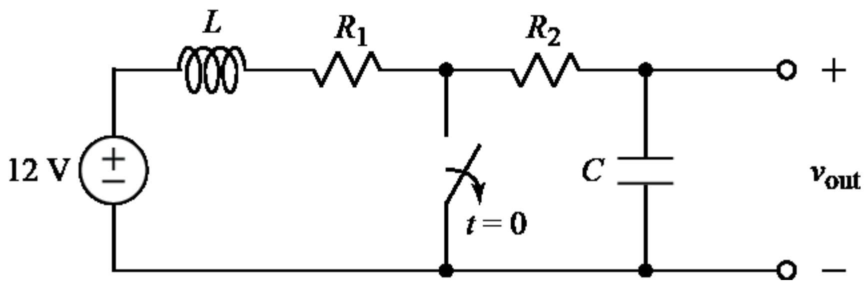
Ex:



A 12 V power supply drives a long wire, (modeled as L and R_1), followed by a short wire, R_2, and a smoothing capacitor, C. There is a safety switch, located before the smoothing capacitor, to turn off the output at the remote end. The switch is closed for a long time before opening at $t = 0$.

$L = 2\ \mu H \qquad R_1 = 2.0\ \Omega \qquad R_2 = 0.1\ \Omega \qquad C = 200\ \mu F$

a) Find the characteristic roots, s_1 and s_2, for the above circuit.

b) Find v_{out} for $t > 0$.

sol'n: For $t > 0$, the switch is open and the two R's may be combined as $R = R_1 + R_2$.

To determine whether we have a series or parallel RLC, we turn off any sources, (a V-src becomes a wire, and an i-src becomes an open). Here, we turn off the 12 V source and find that we have a series RLC.

Thus, $\alpha = \dfrac{R}{2L} = \dfrac{R_1 + R_2}{2L} = \dfrac{2\Omega + 0.1\Omega}{2(2\mu H)}$

$\alpha = \dfrac{2.1}{4\mu} /s = 525k/s$

As always, $\omega_0^2 = \dfrac{1}{LC} = \dfrac{1}{2\mu \cdot 200\mu} /s^2 = \left(\dfrac{1}{20\mu}\right)^2$

$\omega_0^2 = (50k)^2 /s^2$

Our characteristic roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_{1,2} = -525k \pm \sqrt{(525k)^2 - (50k)^2} \ /s$$

The circuit is overdamped and has one root close to zero (because $50k << 525k$).

While we could just use a calculator, it is instructive to use an approximation method that will work even when $\alpha^2 >> \omega_0^2$.

We first write the $\sqrt{\ }$ in the form $\sqrt{1-x}$ where $0 < x << 1$.

$$s_{1,2} = -525k \pm 525k \sqrt{1 - \left(\frac{50}{525}\right)^2}$$

The next step is to use a truncated Taylor series to write an approximation.

$$f(x) = f(0) + \left.\frac{df(x)}{dx}\right|_{x=0} \cdot x + \ldots$$

for $f(x) = \sqrt{1-x}$

We have $\left.\frac{df(x)}{dx}\right|_{x=0} = \left.\frac{1}{2}(1-x)^{-1/2}(-1)\right|_{x=0}$

$$= -\frac{1}{2}$$

Since $f(0) = \sqrt{1-0} = 1$ we have

$$\sqrt{1-x} = 1 - \frac{1}{2}x + \ldots$$

We might be tempted to use $\sqrt{1-x} \approx 1$, but this approximation would be too coarse. It would lead us to conclude that one characteristic root is zero, which would imply a sol'n that never decays. Furthermore, we would be unable to match initial conditions for both L and C, owing to having too few terms in our sol'n. ($A_2 e^{0t} = A_2$ could be absorbed by the A_3 term, see below)

So we use $\sqrt{1-x} \approx 1 - \frac{1}{2}x$, dropping the smaller, higher order terms in x^2, x^3, etc. These terms will be very small since x is small.

$$s_{1,2} \approx -525k \pm 525k\left(1 - \frac{1}{2}\left(\frac{50}{525}\right)^2\right)$$

$$s_1 \approx -525k + 525k(0.995) \quad 0.004535$$

or $s_1 \approx -525k(1 - 0.995) = -525k(0.0045)$

or $s_1 \approx -2.4k$ r/s

and $s_2 \approx -525k - 525k(0.995)$

$$s_2 \approx -525k(2) = -1.05M \text{ r/s}$$

Note: We may use the approximation $\sqrt{1-x} \approx 1$ for s_2 because our error will be small percentage-wise relative to the size of s_2.

Summary: $s_1 \approx -2.4k$ r/s $\quad s_2 \approx -1.05M$ r/s

We have two real and distinct roots: under-damped.

b) For the underdamped case our sol'n is

$$v_{out}(t>0) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3$$

Because $e^{s_1 t}$ and $e^{s_2 t}$ are decaying exponentials, as $t \to \infty$ we have $v_{out} \to A_3$.

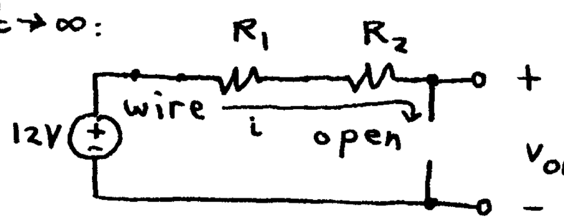
So $A_3 = v_{out}(t \to \infty)$

In our circuit model for $t \to \infty$, we assume i's and v's stabilize. Since i's and v's are not changing, we have

$$v_L = L\frac{di}{dt} = L \cdot 0 = 0 \Rightarrow$$ L acts like a wire (carries i with no v-drop)

$$i_C = C\frac{dv}{dt} = C \cdot 0 = 0 \Rightarrow$$ C acts like an open (v-drop but no i)

$t \to \infty$:



$v_{out}(t \to \infty) = 12V$

Because of the open C, the current flowing around the circuit is zero. This means the voltage drops across R_1 and R_2 are zero by Ohm's law. Consequently, we have a 12V drop across C, meaning

$$A_3 = v_{out}(t \to \infty) = 12\,V$$

To find A_1 and A_2, we use initial conditions.

$$v_{out}(t=0^+) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 \Big|_{t=0^+}$$

$$= A_1 e^{0^+} + A_2 e^{0^+} + 12V$$

$$= A_1 + A_2 + 12V, \text{ since } e^{0^+} = 1$$

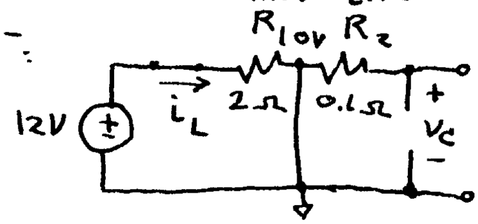
From our circuit, we determine the value of $v_{out}(0^+)$. To do so, we need to know what the L and C are doing. Fortunately, i_L and v_C are energy variables: $w_L = \frac{1}{2} L i_L^2$ and $w_C = \frac{1}{2} C v_C^2$.

Energy cannot change instantly, so

$$i_L(0^+) = i_L(0^-) \text{ and } v_C(0^+) = v_C(0^-).$$

At $t=0^-$, we may assume the circuit has stabilized, and we may treat the L as a wire and the C as an open.

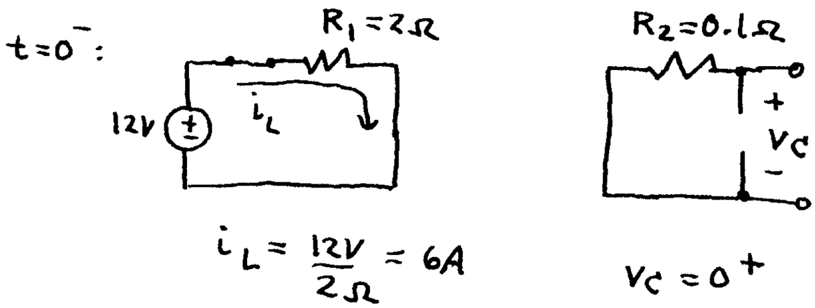
$t=0^-$:



Note: We only find i_L and v_C at $t=0^-$ since any other i or v may change when the switch moves.

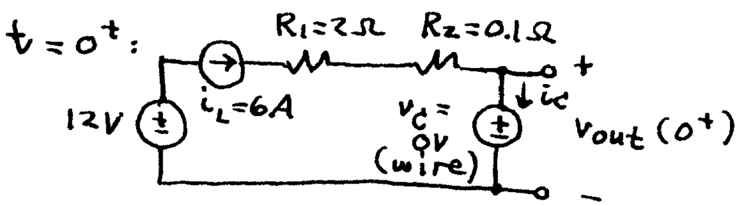
The switch in the middle allows us to separate the circuit into two sides, as it creates two circuits in parallel across a v-src. In this case, the v-src is 0V for the short circuit formed by the switch.

We can also say we have 0V between R_1 & R_2 as our node voltage.



$i_L = \frac{12V}{2\Omega} = 6A$ $\qquad$ $v_C = 0^+$

At $t=0^+$, we can model the L and C as sources, since $i_L(0^+) = i_L(0^-)$, $v_C(0^+) = v_C(0^-)$, and any component whose i or v is known may be replaced by a source with that i or v value.



Since $v_{out} = v_C$, we have $v_{out}(0^+) = 0V$.

Thus, $A_1 + A_2 + 12V = 0V$

We need one more eq'n in order to have two eq'ns in two unknowns that we can solve for A_1 and A_2.

What we need is a sol'n for $v_{out}(t)$ that matches the initial conditions for L and C. Our $v_{out}(0^+)$ matches the initial conditions for C, but we need it to also give the correct i_L.

We observe that $i_L = i_C = C\frac{dv_C}{dt} = C\frac{d\,v_{out}}{dt}$

So we have $\left.\frac{dv_{out}}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{6A}{200\mu F} = 30kV/s$

Our sol'n form for $v_{out}(t)$ gives

$$\frac{d}{dt} v_{out}(t)\bigg|_{t=0^+} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}\bigg|_{t=0^+}$$

$$= A_1 s_1 + A_2 s_2$$

So we have $A_1 s_1 + A_2 s_2 = 30\text{k V/s}$.
and, from before, $A_1 + A_2 + A_3 = 0V$, $A_3 = 12V$.

Solving the 2nd eq'n gives

$$A_2 = -12V - A_1$$

substituting this into the first eq'n gives

$$A_1 s_1 + (-12V - A_1) s_2 = 30\text{k V/s}$$

or

$$A_1 (s_1 - s_2) = 12V\, s_2 + 30\text{kV/s}$$

We can get a more accurate answer by analyzing $s_1 - s_2$:

$$s_1 - s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} - \left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)$$

$$= 2\sqrt{\alpha^2 - \omega_0^2} \approx 2(525k)(0.995) \text{ r/s}$$

$$s_1 - s_2 \approx 1.05\text{ M}\,(0.995) \text{ r/s}$$

So $A_1 \approx \dfrac{12V(-\cancel{1.05M}\text{ r/s}) + 30\text{k r/s}}{(\cancel{1.05M})(0.995)\text{ r/s}}$ small (<1% err if ignored)

$$A_1 \approx -12.06V$$

From earlier, $A_1 + A_2 + 12V = 0V$ so $A_2 = 0.06V$.

Thus, $v_{out}(t>0) = -12.06V e^{-2.4k/s\, t} + 0.06 e^{-1.05M/s\, t} + 12V$

Alternative view:

In the above discussion, the sol'n was matched to initial conditions for the L and C explicitly. It turns out the key to finding coefficients A_1 and A_2 is always to determine the value of dv/dt (or di/dt) from the circuit and match it to the symbolic sol'n.

The problem is that one must find the derivative of the sol'n before knowing the sol'n! This seeming contradiction is resolved by observing that the component eq'ns for L and C relate derivatives to non-derivatives.

$$v_L = L\frac{di_L}{dt} \qquad i_C = C\frac{dv_C}{dt}$$

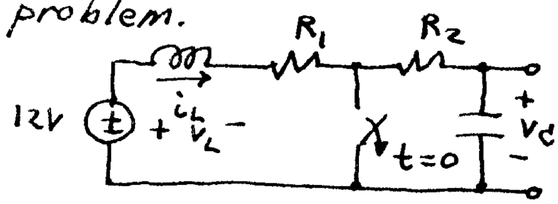
Using these eq'ns is the way we find the derivative of our sol'n. If we are solving for i_L or v_C, our task is simple: we need only find v_L or i_C:

$$\left.\frac{di_L}{dt}\right|_{0^+} = \frac{v_L(0^+)}{L}, \quad \left.\frac{dv_C}{dt}\right|_{0^+} = \frac{i_C}{C}$$

If, however, we have an arbitrary i or v, the idea is to write that i or v in terms of i_L and v_C, (plus component or source values). Then, we can differentiate to get di/dt or dv/dt in terms of di_L/dt and or dv_C/dt.

There are some subtleties to be observed here: 1) the expression for i or v in terms of i_L and v_C must be valid over $t>0$, not just at $t=0^+$; and 2) the expression for i or v must be derivative-free, otherwise it will contain 2nd derivatives when differentiated. Condition (1) is necessary in order for the derivative to be valid, and condition (2) is necessary to avoid creating terms which we are unable evaluate.

ex: Suppose we were asked to find $v_L(t)$ for $t>0$ in this problem.



We want to write v_L in terms of i_L and/or v_C. We turn to Kirchhoff's and Ohm's Laws.

Using a v-loop and the fact that i_L flows thru R_1 and R_2, we have

$$v_L = 12V - i_L(R_1+R_2) - v_C$$

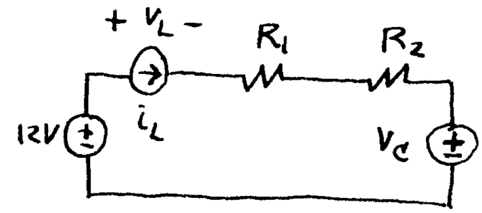
so

$$\left.\frac{dv_L}{dt}\right|_{0^+} = \left.\frac{di_L}{dt}\right|_{0^+}\cdot(R_1+R_2) - \left.\frac{dv_C}{dt}\right|_{0^+}$$

or

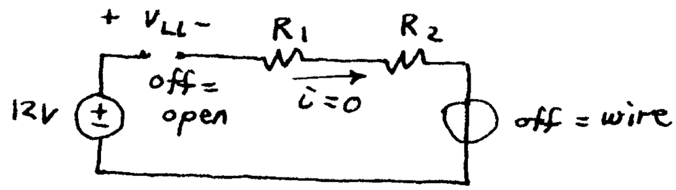
$$\left.\frac{dv_L}{dt}\right|_{0^+} = \frac{v_L(0^+)}{L}(R_1+R_2) - \frac{i_C(0^+)}{C}$$

A visual aid to finding the eq'n is to replace the L and C with sources:



Using superposition, we find v_L:

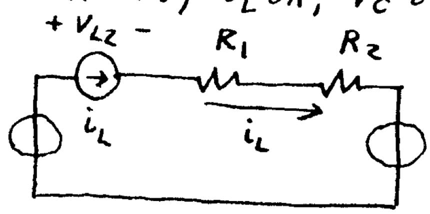
case I: 12V, i_L off, v_c off



Since no current flows, there is no drop across the R's, and all the voltage is dropped across the gap at v_{L1}.

$$v_{L1} = 12V$$

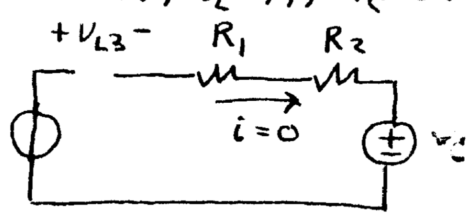
case II: 12V off, i_L on, v_c off



$$-v_{L2} - i_L R_1 - i_L R_2 = 0V$$

$$v_{L2} = -i_L (R_1 + R_2)$$

case III: 12V off, i_L off, v_c on



Since no current flows, there is no v-drop across the R's and v_L must be $-v_c$ so that the v-drops around the loop sum to zero.

$$v_{L3} = -v_c$$

Combining results, we have

$$v_L = v_{L1} + v_{L2} + v_{L3} = 12 - i_L(R_1 + R_2) - v_c$$

This result is the eq'n stated earlier, as promised. We then take d/dt of this entire eq'n, as explained earlier.