Ex: After being open for a long time, the switch closes at $t = 0$. Find $i(t)$ for $t > 0$.

**SOL’N:** We calculate characteristic roots using the circuit for $t > 0$. We set the source to zero to find $R_{\text{Thev}}$ for the roots, which will be the parallel value of the two resistors:

$$480 \text{ m}\Omega \parallel 960 \text{ m}\Omega = 480 \text{ m}\Omega \cdot 1 \parallel 2 = 480 \text{ m}\Omega \cdot \frac{2}{3} = 320 \text{ m}\Omega$$

As for all RLC circuits, we have the following formula for the characteristic roots:

$$s_{1,2} = -\alpha \pm \alpha^2 - \omega_0^2$$

For a series RLC circuit, the value of $\alpha$ is one-half the inverse $L/R$ time constant:

$$\alpha = \frac{R}{2L} = \frac{320 \text{ m}\Omega}{2 \cdot 1 \mu \text{H}} = 160 \text{ k/s}$$

The resonant frequency, $\omega_0$, is the inverse of the square root of the product of $L$ and $C$:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{1 \mu \text{H} \cdot 25 \mu \text{F}} = \left(\frac{1}{5 \mu} \text{ r/s}\right)^2 = (200 \text{ kr/s})^2$$

We find that, since $\alpha < \omega_0$, the roots are complex:

$$s_{1,2} = -160 \text{ kr/s} \pm \sqrt{(160 \text{ kr/s})^2 - (200 \text{ kr/s})^2} = -160 \text{ kr/s} \pm j120 \text{ kr/s}$$

Because the roots are complex, the circuit is under-damped:
\[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(200k)^2 - (160k)^2} \text{ rad/s} = 120 \text{ kr/s} \]

We use the general form of solution for an under-damped circuit:

\[ i(t) = A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t) + A_3 \]

\[ A_3 = \text{final value} \]

For \( t \to \infty \), \( L = \text{wire} \), \( C = \text{open} \), switch closed.

We have a current divider:

\[ i(t \to \infty) = 1A \cdot \frac{480 \Omega}{480 \Omega + 960 \Omega} = \frac{1}{3} A \]

Now find \( i(0^+) \) and \( \frac{di(t)}{dt} \bigg|_{t=0^+} \)

Start at \( t=0^- \) and find \( i_L(0^-), v_c(0^-) \).
(Then we’ll use \( i_L(0^+) = i_L(0^-), v_c(0^+) = v_c(0^-) \).)

At \( t=0^- \), \( L = \text{wire} \), \( C = \text{open} \), switch open.

\[ i_L(0^-) = 0A \]

For \( t=0^+ \), one approach is to take a

Therein equivalent of the current source and \( R \)'s.
\[ V_{th} = V_{a',b'} \text{ with nothing attached to a',b'.} \]

\[ V_{th} = 1A \cdot 480 \text{m}\Omega \parallel 960 \text{m}\Omega = 1A \cdot 320 \text{m}\Omega = 320 \text{mV} \]

\[ R_{th} = \text{resistance seen looking into a',b' with 1A source turned off} \]

\[ R_{th} = 480 \text{m}\Omega \parallel 960 \text{m}\Omega = 320 \text{m}\Omega \text{ (as noted above)} \]

We now find \( v(t) \) in our new circuit and use \( i(t) = v(t)/960 \text{m}\Omega \text{ from Ohm's law} \).

At \( t=0^+ \) we have \( i_L(0^+) = i_L(0^-) = 0A \)

\[ V_c(0^+) = V_c(0^-) = 0V. \]

\( V_{th} = 320 \text{mV} \)

\( V(0^+) = 320 \text{mV} \) from above circuit

We match this to symbolic \( v(0^+) \):
\[ v(t) = A_1 e^{-\alpha t} \cos(\omega_1 t) + A_2 e^{-\alpha t} \sin(\omega_1 t) + A_3 \]

\[ v(0^+) = A_1 + A_3 \]

What is \( A_3 \) for \( v(t) \)? It will be the \( A_3 \) we found for \( i(t) \) multiplied by \( 960 \text{ m}\Omega \) (by Ohm's Law).

\[ A_3 = \frac{1}{3} A \cdot 960 \text{ m}\Omega = 320 \text{ mV} \quad (= \frac{1A \cdot R_{th}}{v}) \]

Back to \( v(0^+) \), we have

\[ v(0^+) = A_1 + A_3 = 320 \text{ mV} \text{ from circuit} \quad \therefore 320 \text{ mV} \]

\[ \therefore A_1 = 0 \]

Now we find \( \frac{dv(t)}{dt} \) by writing

\[ v(t) \text{ in terms of state vars } i_L \text{ and } v_C. \]

We must not plug in values until after we take \( d/dt \).

\[ v(t) = v_{Th} - R_{th} i_L \text{ works since } i_L \text{ is state var} \]

\[ \frac{dv(t)}{dt} = \frac{dv_{Th}}{dt} - R_{th} \frac{di_L}{dt} \]

\[ 0 \text{ since } v_{Th} = \text{const} \]

Now use \( \frac{di_L}{dt} = \frac{v_L}{L} \), (and \( \frac{dv_C}{dt} = \frac{i_C}{C} \) usually).
\[
\frac{dv(t)}{dt} = -R_{Th} \frac{V_L}{L}
\]

\[
\left. \frac{dv(t)}{dt} \right|_{t=0^+} = -R_{Th} \frac{V_L(0^+)}{L} = -\frac{320 \text{ mV}}{1 \mu \text{H}} \cdot 320 \text{ mV}
\]

From symbolic \( v(t) \) we have

\[
\left. \frac{dv(t)}{dt} \right|_{t=0^+} = A_1(-\infty) + A_2 \omega_d
\]

Thus,

\[
A_1(-\infty) + A_2 \omega_d = -\frac{320 \text{ mV}}{1 \mu \text{H}} \cdot 320 \text{ mV}
\]

But \( A_1 = 0 \). \therefore \( A_2 = -\frac{320 \text{ mV}}{1 \mu \text{H} \cdot (\omega_d = 120 \text{kr/s})} \)

\[
v(t) = -\frac{320 \text{ mV} \cdot 320 \text{ mV}}{1 \mu \text{H} \cdot 120 \text{ kr/s}} e^{-160kt} \sin(120kt) + 320 \text{ mV}
\]

\[
i(t) = \frac{v(t)}{960 \text{ mV} \cdot 12}
\]

Since \( v \) is across 960 mΩ

\[
i(t) = -\frac{320 \text{ mV} \cdot 1/3}{1 \mu \text{H} \cdot 120 \text{ kr/s}} e^{-160kt} \sin(120kt) + \frac{1}{3} A
\]

\[
i(t) = -\frac{8 A e^{-160kt}}{9} \sin(120kt) + \frac{1}{3} A
\]