The voltage source in the above circuit is off (0 V) for \( t < 0 \).

An engineer wishes to use the above circuit to create two decaying sinusoidal signals 120° out-of-phase to drive a three-phase motor for a short time. (A third signal that is 120° out-of-phase with the first two may be created by an additional op-amp circuit, not shown, that computes \(-v_0 - v_1\).) The signal at \( v_0(t) \) will necessarily be a decaying sinusoid of the following form:

\[
v_0(t) = -v_m e^{-\alpha t} \sin(\beta t)
\]

where \( v_m, \alpha, \) and \( \beta \) are positive real-valued constants.

The design problem now is to create a \( v_1(t) \) signal that is 120° out-of-phase with \( v_0(t) \).

a) Find a symbolic expression for the Laplace-transformed output, \( V_1(s) \), in terms of not more than \( R_1, R_2, R_3, L, C, \) and values of sources or constants.

b) Choose a numerical value for \( C \) to make

\[
v_1(t) = v_m e^{-\alpha t} \cos(\beta t - 30°)
\]

Hint: \( \cos(A - B) = \cos A \cos B + \sin A \sin B \)

bonus) Why could the desired \( v_1(t) \) not be obtained if the positions of the \( L \) and \( C \) were reversed?

**SOL’N:** a) The input voltage source is a step function that Laplace transforms to \( 1/s \).

\[
V_i(s) = \frac{1}{s}
\]

Before time zero, the input voltage is zero and it follows that initial conditions for both the \( L \) and \( C \) are zero.
At the – input of the op-amp, we have the same voltage (because of the negative feedback) as at the + input, namely zero volts.

We can express the current flowing toward the – input as the input voltage divided by the sum of impedances up to the – input. This is true in the Laplace domain and is just an example of Ohm's law.

\[
I(s) = \frac{1}{s} \frac{1}{sL + R + \frac{1}{sC}} = \frac{1}{s} \frac{1}{sL + \frac{R}{L} + \frac{1}{LC}}
\]

where \( R \) represents \( R_1 + R_2 \).

To find \( V_1(s) \), we observe that we may use the voltage drop across \( L \) and \( R_2 \). Again, we use Ohm's law, multiplying the impedances of \( L \) and \( R_2 \) by \( I(s) \).

\[
V_1(s) = I(s)(sL + R_2) = \frac{s + \frac{R_2}{L}}{s^2 + \frac{R}{L} + \frac{1}{LC}} = \frac{s + \frac{R_2}{L}}{\left(s + \frac{R}{2L}\right)^2 + \frac{1}{LC} - \left(\frac{R}{2L}\right)^2}
\]

The second form for the answer will figure into our solution to (b).

b) Using the hint, we rewrite the expression for \( v_1 \) in terms of sine and cosine.

\[
v_1(t) = v_m e^{-\alpha t} [\cos(\beta t)\cos(30^\circ) + \sin(\beta t)\sin(30^\circ)]
\]

or

\[
v_1(t) = v_m e^{-\alpha t} \left[\cos(\beta t)\frac{\sqrt{3}}{2} + \sin(\beta t)\frac{1}{2}\right]
\]

We Laplace transform the expression for \( v_1(t) \).

\[
V_1(s) = v_m \left[\frac{\sqrt{3}}{2} \frac{s + \alpha}{(s + \alpha)^2 + \beta^2} + \frac{1}{2} \frac{\beta}{(s + \alpha)^2 + \beta^2}\right]
\]

Matching the denominator to our answer from (a), we identify the values of \( \alpha \) and \( \beta \).
\[ \alpha = \frac{R}{2L} \]
\[ \beta^2 = \frac{1}{LC} - \left( \frac{R}{2L} \right)^2 \]

We can calculate the numerical value of \( \alpha \).
\[ \frac{R}{2L} = \frac{4 + 8}{2(2m)\frac{r}{s}} = 3k \]

Now we turn our attention to the numerator of \( V_1(s) \).
\[ V_1(s) = v_m \frac{1}{2} \left[ \frac{\sqrt{3}(s + \alpha) + \beta}{(s + \alpha)^2 + \beta^2} \right] = \left[ \frac{v_m \frac{\sqrt{3}}{2} s + v_m \frac{\sqrt{3}}{2} \alpha + v_m \frac{1}{2} \beta}{(s + \alpha)^2 + \beta^2} \right] \]

From the solution to (a), the coefficient of \( s \) is unity, which dictates the necessary value of \( v_m \).
\[ v_m = \frac{2}{\sqrt{3}} \]

Now we consider the constant term of the numerator, which must map the solution from (a). Using our value of \( v_m \) and the solution to (a) gives the following equation.
\[ \alpha + \frac{1}{\sqrt{3}} \beta = \frac{R_2}{L} \]

or
\[ \frac{R_1 + R_2}{2L} + \frac{1}{\sqrt{3}} \beta = \frac{R_2}{L} \]

or, if we subtract \( R_2/2L \) from both sides, we have the following equation:
\[ \frac{R_1}{2L} + \frac{1}{\sqrt{3}} \beta = \frac{R_2}{2L} \]

A few calculations:
\[ \frac{R_1}{2L} = \frac{4}{2(2m)}\frac{r}{s} = 1k \quad \text{and} \quad \frac{R_2}{2L} = \frac{8}{2(2m)}\frac{r}{s} = 2k \]
Using these values, we have an equation for $\beta$.

$$1k + \frac{1}{\sqrt{3}} \beta = 2k$$

or

$$\frac{1}{\sqrt{3}} \beta = 1k$$

or

$$\frac{1}{3} \beta^2 = 1M$$

or, using the expression for $\beta$ from earlier, we have the following:

$$\beta^2 = \frac{1}{LC} \left( \frac{R}{2L} \right)^2 = 3M$$

or

$$\frac{1}{LC} - (3k)^2 = 3M$$

or

$$\frac{1}{2mC} = 12M$$

Finally, we can solve for $C$.

$$C = \frac{1}{2m12M} = \frac{1}{24k} \approx 41.7 \mu F$$

**bonus)** With the $C$ on the right, $v_1(t)$ would end up at 1 V as the $C$ would charge. Thus, the signal could not be a decaying sinusoid. It would have a DC offset.