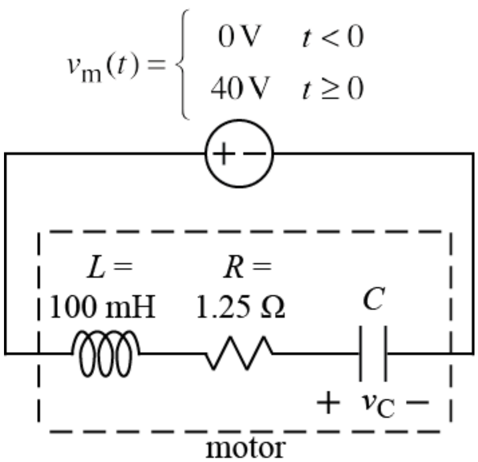
FINAL EXAM prob 2 solution

Ex:



The above circuit is a model of a motor driven by a voltage source. The inductor and resistor represent the inductance and resistance of the windings of the motor, and the capacitor models the back emf (voltage) generated by the windings when the motor is rotating. (When the motor rotates, the windings experience a changing magnetic field that creates a voltage drop in the windings. Thus, the capacitor in this model is considered to be part of the windings, along with the inductor and resistor.) The capacitor value is related to motor parameters:

$$C = \frac{J}{K^2}$$

where $J \equiv$ moment of inertia of motor
$K \equiv$ emf or torque constant of motor

The motor is at rest initially. After being off (0 V) for a long time, the voltage source steps to 40 V at $t = 0$.

a) Find the value of C given the following characteristic roots for the RLC model of the motor:

$$s_1 = -1.25\text{ r/s},\ \ s_2 = -11.25\text{ r/s}$$

b) Using your C value from (a), find a numerical expression for the back emf, $v_C(t)$, for $t > 0$.

bonus) The angular velocity of the motor rotation is related to the voltage across C by the following equation:

$$\omega(t) = \frac{v_C(t)}{K}$$

If J = 10/9 ≈ 1.111 in SI units, find the velocity of the motor in rotations per second at time $t = 1$ s.

SOL'N: a) We have a series *RLC*.

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Taking the difference of the roots yields an equation we can solve for the value of *C*.

$$s_1 - s_2 = -1.25 - (-11.25) = 10 = 2\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

We divide both sides by 2.

$$\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = 5$$

Squaring both sides eliminates the square root.

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 25$$

Substituting the known values for *L* and *C* moves us closer to a solution.

$$\left(\frac{R}{2L}\right)^2 - 25 = \left(\frac{1.25}{2(100\text{m})}\right)^2 - 25 = 6.25^2 - 5^2 = 3.75^2 = \frac{1}{LC}$$

Rearranging to solve for *C*, yields our answer.

$$C = \frac{1}{3.75^2 L} = \frac{1}{3.75^2(100\text{m})} = \frac{32}{45} \approx 0.711\text{F}$$

b) Since the roots are real and distinct, the solution is over-damped.

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3$$

A_3 is the final value (*t* approaching infinity) of v_C, which will be 40 V since the *L* will look like a wire and the *R* will have no voltage drop as a result of zero current flow.

$$A_3 = 40\text{ V}$$

Now we match our solution to initial conditions.

At $t = 0^-$, we no voltage source, the L looks like a wire, the C looks like an open circuit, and there is no current flow. Consequently, all voltages and currents are zero, including the voltage on the C.

$$v_C(0^+) = v_C(0^-) = 0\,\text{V} = \text{A}_1 + A_2 + A_3 = \text{A}_1 + A_2 + 40\,\text{V}$$

The derivative of the capacitor voltage may be expressed directly in terms of the capacitor current, and the capacitor has the same current as the inductor because they are in series with each other. The inductor current at $t = 0^+$ is the same as at $t = 0^-$, which is zero. We match this value from the circuit to the symbolic expression for the derivative.

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = 0\,\text{A} = A_1 s_1 + A_2 s_2$$

We now have two equations to solve:

$$-40\,\text{V} = \text{A}_1 + A_2$$

$$0\,\text{A} = A_1(-1.25) + A_2(-11.25) = A_1(-1.25) + A_2(9)(-1.25)$$

Dividing the second equation by 1.25, we have an equation that we may use to express A_1 in terms of A_2.

$$0\,\text{A} = A_1 + 9A_2$$

or

$$A_1 = -9A_2$$

Substituting into the first of the two equations, we can solve for A_2.

$$-40\,\text{V} = -9\text{A}_2 + A_2 = -8A_2$$

or

$$A_2 = 5\,\text{V}$$

From an equation above, A_1 is –9 times A_2.

$$A_1 = -45\,\text{V}$$

Now we have values for our expression for v_C.

$$v_C(t) = -45e^{-1.25t/\text{s}} + 5e^{-11.25t/\text{s}} + 40\,\text{V}$$

bonus) We use one of the equations given to find the value of K.

$$C = \frac{J}{K^2}$$

or

$$K^2 = \frac{J}{C} = \frac{\frac{10}{9}}{\frac{32}{45}} = \frac{10}{9}\frac{45}{32} = \frac{25}{16}$$

or

$$K = \frac{5}{4} = 1.25$$

Using our solution to (b), we use t = 1s in our expression for v_C.

$$\omega(1\text{s}) = \frac{v_C(1\text{s})}{K} = \frac{-45e^{-1.25} + 5e^{-11.25} + 40}{1.25} \approx \frac{-45e^{-1.25} + 40}{1.25}$$

or

$$\omega(1\text{s}) \approx 21.7 \text{ r/s}$$

Finally, we divide by 2π to get revolutions per second.

$$\frac{\omega}{2\pi} \approx \frac{21.7}{6.28} \approx 3.46 \,\text{rps}$$