Ex:

\[ i_g(t) = 5 - 5e^{-2t}u(t) \text{ A} \]

Note: The first 5A portion of \( i_g(t) \) is always on.

a) Write the Laplace transform \( I_g(s) \) of \( i_g(t) \).

b) Draw the \( s \)-domain equivalent circuit, including source \( I_g(s) \), components, initial conditions for \( L \) and \( C \), and terminals for \( V_o(s) \).

c) Write the Laplace transform \( V_o(s) \) of \( v_o(t) \). Write your answer as a ratio of polynomials in \( s \) with numerical coefficients.

d) Write a numerical time-domain expression for \( v_o(t) \) where \( t \geq 0 \).

Sol’N: a) The 5 A portion of the current source acts like \( 5u(t) \) for \( t = 0^- \) to \( \infty \) where the Laplace transform is computed.

\[ I_g(s) = \frac{5}{s} - \frac{5}{s+2} \text{ A} \]

b) To find initial conditions, we use the time-domain circuit at \( t = 0^- \) with the \( C \) treated as an open circuit and the \( L \) treated as a wire. The 5 A portion of the current source will be on, and the second term of the current source will be off.

\( t = 0^- \):

\[ i_g(t) = 5 \text{ A} \]

The current will all flow in the wire that models the \( L \).

\[ i_L(0^-) = 5 \text{ A} \]
Given the parallel form of the circuit, using a parallel current source for the initial conditions of the \( L \) is prudent. The current source corresponds to a step function that turns on a current in parallel with the \( L \) at time zero.

\[
\frac{i_L(0^-)}{s}
\]

The capacitor will have zero initial conditions since it is shorted out by the wire modeling the \( L \).

\[
V_C(0^-) = 0 \text{ V}
\]

The source for initial conditions on the \( C \) may be omitted.

c) We sum the current sources, which results in some convenient cancellation. The output voltage will be the total current times the total parallel impedances.

\[
V_o(s) = -\frac{5}{s+2} A \left( sL \parallel \frac{1}{sC} \parallel R \right) = -\frac{5}{s+2} \left( \frac{1}{sL + sC + \frac{1}{R}} \right)
\]

We clear the denominator of the denominator and get a coefficient of unity for the \( s^2 \) coefficient.

\[
V_o(s) = -\frac{5}{s+2} \left( \frac{s/C}{\frac{1}{sL} + sC + \frac{1}{R}} \right) = -\frac{5}{s+2} \left( \frac{s/C}{s^2 + \frac{1}{RC} + \frac{1}{LC}} \right)
\]

or

\[
V_o(s) = -\frac{5}{s+2} \left( \frac{s/C}{\frac{1}{LC} + \frac{1}{RC} + \frac{1}{s^2}} \right) = -\left( \frac{5}{s+2} \right) \left( \frac{s/C}{s^2 + \frac{1}{RC} + \frac{1}{sLC}} \right)
\]
or
\[ V_o(s) = -\frac{5s}{(s+2)\left(\frac{s^2}{2} + \frac{1}{0.5}s + \frac{1}{0.2}\right)} = -\frac{5s}{(s+2)\left(s^2 + 2s + 5\right)} \]

d) We factor the quadratic polynomial in the denominator so it matches the form found in the Laplace transforms of decaying cosine and sine.
\[ s^2 + 2s + 5 = (s+1)^2 + 2^2 = (s+a)^2 + \omega^2 \]
\[
\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2 + \omega^2}
\]
\[
\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}
\]
We expand the output voltage in a modified partial fraction form.
\[ V_o(s) = -\frac{5s}{(s+2)\left(\frac{s^2}{2} + \frac{1}{0.5}s + \frac{1}{0.2}\right)} = \frac{A}{s+2} + \frac{B(s+1)+C(2)}{(s+1)^2 + 2^2} \]

We use the method of multiplying by the pole and evaluating at the pole to find the value of \( A \).

\[
A = V_o(s)(s+2)\bigg|_{s=-2} = -\frac{5s}{(s+1)^2 + 2^2}\bigg|_{s=-2} = -\frac{5(-2)}{(-2+1)^2 + 2^2}
\]
or
\[
A = V_o(s)(s+2)\bigg|_{s=-2} = -\frac{10}{5} = 2
\]
Having found \( A \), we put terms over a common denominator.
\[ V_o(s) = \frac{2(s^2 + 2s + 5) + [B(s+1)+C(2)](s+2)}{(s+2)\left(\frac{s^2}{2} + \frac{1}{0.5}s + \frac{1}{0.2}\right)} \]
\[ = -\frac{5s}{(s+2)\left(s^2 + 2s + 5\right)} \]
For the polynomials in the numerator to be equal, the coefficients of each power of \( s \) must be equal.

\[
2(s^2 + 2s + 5) + [B(s + 1) + C(2)](s + 2) = -5s
\]

We group the coefficients of powers of \( s \).

\[
2(s^2 + 2s + 5) + B(s^2 + 3s + 2) + C(2s + 4) = -5s
\]

or

\[
(2 + B)s^2 + (4 + 3B + 2C)s + (10 + 2B + 4C) = -5s
\]

From the \( s^2 \) coefficient, which equals zero (on right side of equation), we find the value of \( B \).

\[
B = -2
\]

From the constant coefficient, which equals zero (on right side of equation), we find the value of \( C \).

\[
10 + 2(-2) + 4C = 0
\]

or

\[
C = -\frac{3}{2}
\]

We check our answer by trying some values of \( s \) in our original expression for \( V_o(s) \) and in our modified partial fraction expression for \( V_o(s) \).

\[
V_o(s)\bigg|_{s=0} = \left[ -\frac{5s}{(s + 2)(s + 1)^2 + 2^2} \right]_{s=0} = 0
\]

\[
\left[ \frac{2}{s + 2} + \frac{-2(s + 1) - \frac{3}{2}(2)}{s^2 + 2s + 5} \right]_{s=0} = \frac{2}{2} + \frac{-2 - 3}{5} = 0 \text{ works } \checkmark
\]

and
Having verified our expansion, we take the inverse Laplace transform.

\[ v_o(t) = L^{-1}\left\{ \frac{2}{s+2} + \frac{-2(s+1) - \frac{3}{2}(2)}{s^2 + 2s + 5} \right\} \]

or

\[ v_o(t) = [2e^{-2t} - 2e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t]u(t) \text{ V} \]