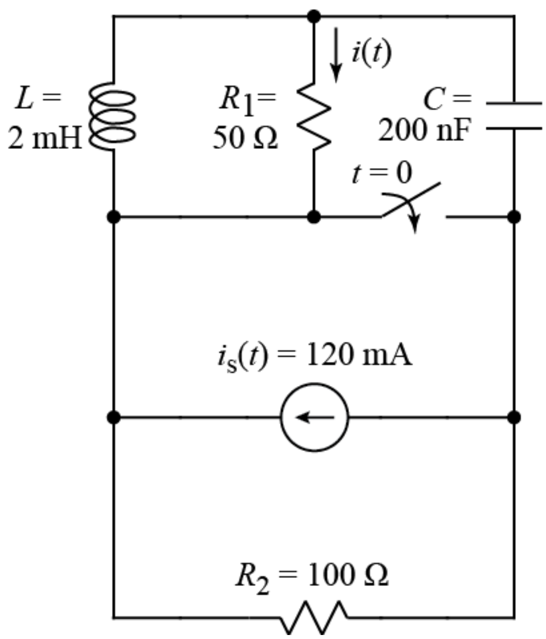
Ex:



After being open for a long time, the switch closes at $t = 0$.

a) Find the characteristic roots of the circuit, and state whether $i(t)$ is under-damped, over-damped, or critically-damped.

b) Write a numerical time-domain expression for $i(t)$, $t > 0$, the current through R_1. This expression must not contain any complex numbers.

SOL'N: a) We use the circuit for $t > 0$ when finding the characteristic roots. Thus, the switch is closed, and the bottom half of the circuit is isolated from the top half. This leaves a parallel RLC circuit with resistance R_1. We use the equations for roots of a parallel RLC.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

where

$$\alpha = \frac{1}{2R_1C} = \frac{1}{2(50)200\,\text{ns}} = \frac{1\text{G}}{20\,\text{ks}} = 50\,\text{k/s}$$

and

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{2\text{m}200\text{n}} = \frac{1}{(20\mu)^2} = (50\,\text{kr/s})^2$$

Using component values, we compute the numerical values of the roots.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -50\,\text{kr/s} \pm \sqrt{(50\,\text{k})^2 - (50\,\text{k})^2} = -50\,\text{kr/s}$$

The roots are both the same, so we have critical damping.

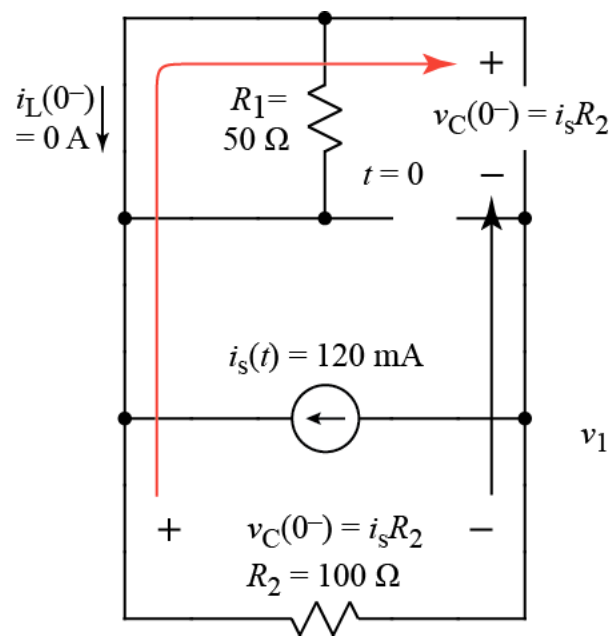
b) We use the critical damping solution.

$$i(t) = A_1 e^{st} + A_2 t e^{st} + A_3$$

The value of A_3 is the final value of $i(t)$, which is found by assuming the circuit has stabilized and C = open, L = wire. The switch is closed as t approaches infinity, so the bottom half of the circuit is irrelevant and we have no power source in the top half. Thus, the cirucit will run down to zero. This causes the final value to be zero.

$$A_3 = 0\,\text{A}$$

Now we find initial conditions so we can solve for A_1 and A_2. We consider $t = 0^{-}$. The switch is open, and the C acts like an open and the L acts like a wire. We find v_C and i_L.

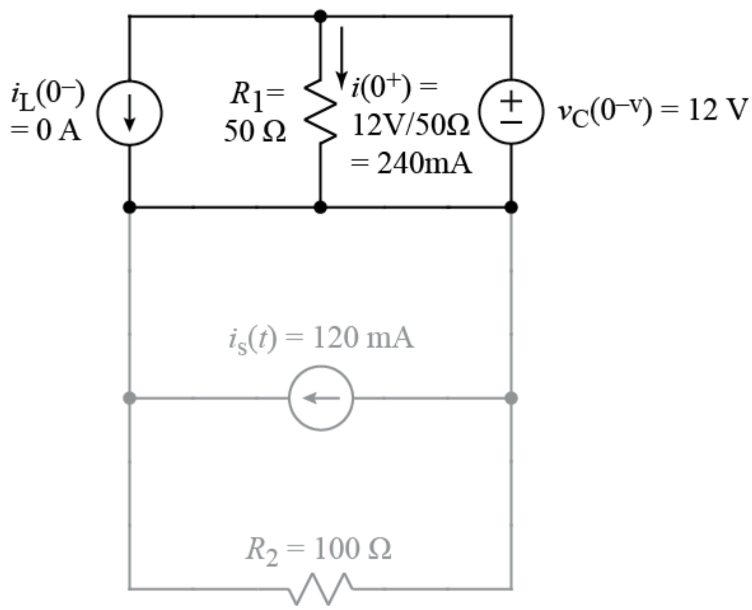


Because of the open circuits, no current can flow in the L, so $i_L = 0$ A. As indicated by the arrows in the above diagram, the voltage drop across C is the same as the voltage drop across R_2, which is $i_s R_2$.

$$v_C(0^-) = i_s R_2 = 120\,\text{mA} \cdot 100\,\Omega = 12\,\text{V}$$

We use the initial values for the L and C as the values of sources in the model at $t = 0^+$.

$t = 0^+$:



Since the voltage of the capacitor is directly across R_1, we use Ohm's law to find the initial value of $i(0^+)$.

$$i(0^+) = \frac{12\,\text{V}}{50\,\Omega} = 240\,\text{mA}$$

We match this value to the value our symbolic solution gives at time 0^+.

$$i(0^+) = A_1 = 240\,\text{mA}$$

To find A_2, we use the derivative of $i(t)$ at $t = 0^+$. To find the value of the derivative of $i(t)$ at $t = 0^+$, we first write $i(t)$ in terms of energy variables i_L and v_C. Since R_1 is across v_C, we just use Ohm's law again.

$$i(t) = \frac{v_C(t)}{R_1}$$

Now we differentiate the equation.

$$\frac{di(t)}{dt} = \frac{1}{R_1}\frac{dv_C}{dt}$$

We can now substitute for the derivative of the capacitor voltage by using the component equation for the capacitor. (This is why we wrote $i(t)$ in terms of energy variables.)

$$\frac{dv_C}{dt} = \frac{i_C}{\mathrm{C}}$$

so

$$\frac{di(t)}{dt} = \frac{1}{R_1}\frac{i_C}{C}$$

Now we evaluate the derivative at time 0^+.

$$\left.\frac{di(t)}{dt}\right|_{t=0^+} = \frac{1}{R_1}\frac{i_C(0^+)}{C}$$

Since the L carries no current at $t = 0^+$, the current in R_1 must be supplied by the C. The current flows up from the C and down through R_1. The direction of the current is opposite the passive sign convention, so it is the negative of the current in R_1.

$$i_C(0^+) = -240\,\text{mA}$$

Now we can compute the value of the derivative of $i(t)$ at time 0^+.

$$\left.\frac{di(t)}{dt}\right|_{t=0^+} = \frac{1}{50\,\Omega}\frac{-240\,\text{mA}}{200\,\text{nF}} = -\frac{240\ \text{m}}{10\,\text{kn}} = -24\,\text{kA/s}$$

We match the derivative of our symbolic solution to this value.

$$A_1 s + A_2 = -24\,\text{kA/s}$$

or

$$240\,\text{mA}\cdot(-50\,\text{k/s}) + A_2 = -24\,\text{kA/s}$$

or

$$A_2 = -24\,\text{kA/s} - (240\,\text{mA})(-50\,\text{k/s}) = -12\,\text{kA/s}$$

We have our answer.

$$i(t) = 240\,\mathrm{mA}\, e^{-50\mathrm{k}\,t} - 12\,\mathrm{kA/s}\, t\, e^{-50\mathrm{k}\,t}$$