# 1.6 Second-Order Transients

ECE 2210   A.Stolp 4/1/01, 10/21/02, 10/22/03, 3/3/05, 10/24/05

A circuit with both a capacitor and an inductor is like a mechanical system with both a mass and a spring. When there are two different types of energy-storage elements, the transient responses can be much more interesting than the simple exponential curves that we've seen so far. Many of these systems can oscillate or "ring" when a transient is applied. When you analyze a circuit with a capacitor and an inductor you get a second-order differential equation, so the transient voltages and currents are called second-order transients.

Series RLC circuit, traditional way: Look at the circuit at right. The same current flows through all three elements. That current will begin to flow after time t=0, when the switch is closed. Using basic circuit laws:

$$V_{in} = v_R + v_L + v_C$$

$$= i \cdot R + L \cdot \frac{d}{dt} i + \frac{1}{C} \cdot \int_{-\infty}^{t} i_C \, dt \quad \text{Making the obvious substitutions.}$$
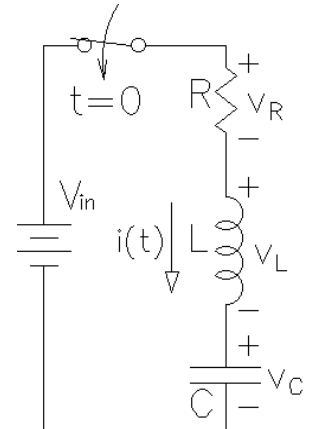
The next step here would be to differentiate both sides of the equation, but we've been through this before with the RC circuit. If you're a little more clever, there's an easier way.

Make this substitution instead $\quad i = i_C = C \cdot \frac{d}{dt} v_C, \quad$ to get $\quad V_{in} = R \cdot C \cdot \frac{d}{dt} v_C + L \cdot C \cdot \frac{d^2}{dt^2} v_C + v_C$

Rearrange this equation to get $\quad V_{in} = L \cdot C \cdot \frac{d^2}{dt^2} v_C + R \cdot C \cdot \frac{d}{dt} v_C + v_C \quad$ and $\quad \frac{V_{in}}{L \cdot C} = \frac{d^2}{dt^2} v_C + \frac{R \cdot C}{L \cdot C} \frac{d}{dt} v_C + \frac{1}{L \cdot C} \cdot v_C$

This is the classical second-order differential equation and it is solved just like the first-order differential equation, by guessing a solution of the right form and then finding the particulars of that solution.

Standard differential equation answer: $\quad v_C(t) = A + B \cdot e^{s \cdot t} \quad$ Although now it will turn out that there will be two s's (s_1 and s_2), and two B's. For

Differentiate: $\quad \frac{d}{dt} v_C = B \cdot s \cdot e^{s \cdot t} \quad$ now I'll leave out that added complexity.

And again: $\quad \frac{d^2}{dt^2} v_C = B \cdot s^2 \cdot e^{s \cdot t}$

Substitute these back into the original equation: $\quad \frac{V_{in}}{L \cdot C} = \frac{d^2}{dt^2} v_C + \frac{R}{L} \frac{d}{dt} v_C + \frac{1}{L \cdot C} \cdot v_C$

$$= B \cdot s^2 \cdot e^{s \cdot t} + \frac{R}{L} \cdot B \cdot s \cdot e^{s \cdot t} + \frac{1}{L \cdot C} \cdot \left(A + B \cdot e^{s \cdot t} \right)$$

$$= B \cdot s^2 \cdot e^{s \cdot t} + \frac{R}{L} \cdot B \cdot s \cdot e^{s \cdot t} + \frac{1}{L \cdot C} \cdot B \cdot e^{s \cdot t} + \frac{1}{L \cdot C} \cdot A$$

We can separate this equation into two parts, one which is time dependent and one which is not. Each part must still be an equation.

Time independent (forced) part: $\quad V_{in} = A, \quad A = V_{in} = $ final condition $= v_C(\infty) \quad$ just like before

Time dependent (transient) part: $\quad 0 = B \cdot s^2 \cdot e^{s \cdot t} + \frac{R}{L} \cdot B \cdot s \cdot e^{s \cdot t} + \frac{1}{L \cdot C} \cdot B \cdot e^{s \cdot t}$

Divide both sides by $\quad B \cdot e^{s \cdot t} \quad$ to get: $\quad 0 = s^2 + \frac{R}{L} \cdot s + \frac{1}{L \cdot C} \quad$ **= characteristic equation**

This equation is important. It is called the characteristic equation and we'll need to find one like it for every second-order circuit that we analyze. Luckily, there's a much easier way to get it, using impedances similar to those we used in phasor analysis. I'll talk about that in the next section, in the meantime, let's continue with this problem.

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Once you have the characteristic equation

characteristic equation: $\quad s^2 + \dfrac{R}{L} \cdot s + \dfrac{1}{L \cdot C} = 0$

Solutions to the characteristic equation:

$$s_1 = -\dfrac{R}{2 \cdot L} + \dfrac{1}{2} \cdot \sqrt{\left(\dfrac{R}{L}\right)^2 - \dfrac{4}{L \cdot C}} \qquad\qquad s_2 = -\dfrac{R}{2 \cdot L} - \dfrac{1}{2} \cdot \sqrt{\left(\dfrac{R}{L}\right)^2 - \dfrac{4}{L \cdot C}}$$

This results in three possible types of solutions, depending on what's under the radical.

| The characteristic equation is solved using the quadratic equation, recall:

$\quad$ if $\quad a \cdot x^2 + b \cdot x + c = 0$

$\quad$ there are two solutions

$$x_1 = \dfrac{-b + \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

and

$$x_2 = \dfrac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

Overdamped

if $\quad \left(\dfrac{R}{L}\right)^2 - \dfrac{4}{L \cdot C} > 0 \quad$ then s_1 and s_2 are both real and $\quad s_1 \neq s_2 \quad$ and our guessed solution $v_C(t) = A + B \cdot e^{s \cdot t}$

will become $\quad v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t} \quad$ and is simply the combination of two exponentials.

Also, unless we find a negative resistor, both s_1 and s_2 will always be negative. This is the overrdamped case, like a class of students on a Monday morning. Pretty dull and soon to be asleep.

Underdamped

if $\quad \left(\dfrac{R}{L}\right)^2 - \dfrac{4}{L \cdot C} < 0 \quad$ then s_1 and s_2 are both complex and and can be expressed as

$$s_1 = \alpha + j \cdot \omega \qquad \text{and} \qquad s_2 = \alpha - j \cdot \omega$$

Well, if you start putting complex numbers in exponentials, what do you get? Euler's equation says you'll get sines and cosines. In this case its much easier to rephrase the guessed solution like this.

$$v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot \left(B_2 \cdot \cos(\omega \cdot t) + D_2 \cdot \sin(\omega \cdot t)\right)$$

This form can be derived directly from $v_C(t) = A + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$

using Euler's equation, $e^{j \cdot \theta} = \cos(\theta) + j \cdot \sin(\theta)$, but we won't bother to here.

In fact, although B_2 and D_2 are not the same as B and D, I'll drop the 2 subscripts because we'll never actually need to convert between these two forms and the extra subscripts just become annoying.

$$v_C(t) = v_C(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

α and ω come from the s_1 and s_2 solutions to the characteristic equation. ω is known as the natural frequency of the circuit. The underdamped circuit will "ring", that is, it will oscillate at this natural frequency in response to a transient. Because α will always be negative the $e^{\alpha t}$ term insures that the transient ringing dies out in time. This is the underdamped case, like students on spring break in Fort Lauderdale.

Critically damped

if $\quad \left(\dfrac{R}{L}\right)^2 - \dfrac{4}{L \cdot C} = 0 \quad$ then s_1 and s_2 are both real and exactly the same. Now our guessed solution must be

modified to $\quad v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t} \quad$ and can result in a single overshoot.

This is actually a trivial case since it relies on an exact equality which will never happen in reality. The best use of the critically damped case is as a conceptual border between the over- and under-damped cases.

Damping Ratio

If you write s_1 and s_2 like this: $\quad s_1 = \alpha + \sqrt{\alpha^2 - \omega_o^2} \quad$ and $\quad s_2 = \alpha - \sqrt{\alpha^2 - \omega_o^2} \quad$ then $\quad \zeta = \dfrac{\alpha}{\omega}$

ζ is known as the damping ratio.

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RLC examples

Let's use some component values in the RLC circuit and see what happens.

Overdamped Example $V_{in} := 12 \cdot V$ $R := 90 \cdot \Omega$ $L := 20 \cdot mH$ $C := 10 \cdot \mu F$
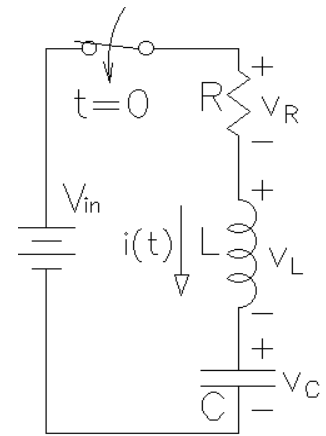
$$\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C} > 0 \qquad s_1 \text{ and } s_2 \text{ are real and negative, overdamped.}$$

$$s_1 := -\frac{R}{2 \cdot L} + \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_1 = -2000 \cdot sec^{-1}$$

$$s_2 := -\frac{R}{2 \cdot L} - \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_2 = -2500 \cdot sec^{-1}$$

$$v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$$

$v_C(\infty)$ = final condition = $12 \cdot V$, The capacitor will eventually charge up to V_{in}.

Initial conditions

In second-order circuits you will need to find two initial conditions, $i_L(0)$ and $v_C(0)$. You'll then use these to find the initial condition (and later the initial slope) of your variable of interest. In this case $v_C(0) = v_C(0)$ -- not too tough here. Finding your initial condition can be a little harder if you're interested in a variable other than $i_L(t)$ or $v_C(t)$.

At time t = 0 $v_C(0) = v_C(\infty) + B + D = 0$ (assuming no initial charge)

Rearranging: $D = -12 \cdot V - B$ But this equation has two unknowns. The initial slope will give us the needed second

equation. Start by finding the initial slope of your variable of interest, $\frac{d}{dt} v_C(0)$ in this example.

Usually that means that you rearrange a basic relation like $i_C = C \cdot \frac{d}{dt} v_C(t)$ to $\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C}$

or, $v_L = L \cdot \frac{d}{dt} i_L(t)$ to $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L}$

Both $i_C(0)$ or $v_L(0)$ **can change instantly**, so you **must** find them from $i_L(0)$ and $v_C(0)$.

From initial conditions: $\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C} = \frac{i_L(0)}{C} = \frac{0 \cdot A}{C} = 0 \cdot \frac{V}{sec}$ since there's no initial current.

Differentiate the solution: $\frac{d}{dt} v_C(t) = B \cdot s_1 \cdot e^{s_1 \cdot t} + D \cdot s_2 \cdot e^{s_2 \cdot t}$ At time t = 0: $\frac{d}{dt} v_C(0) = B \cdot s_1 + D \cdot s_2$

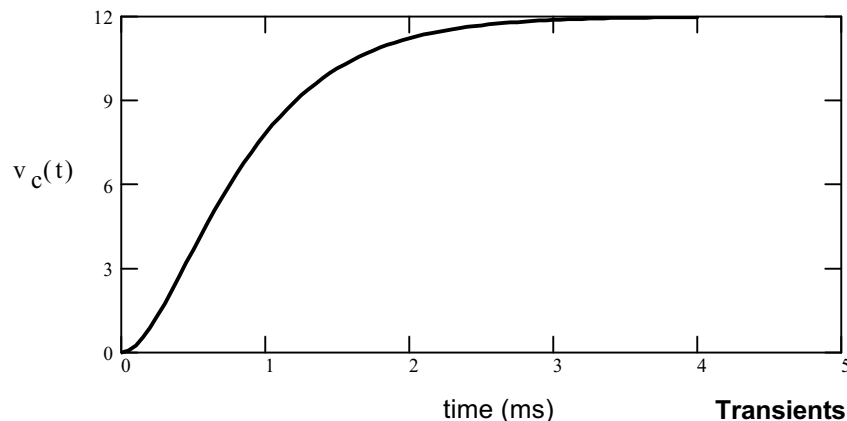Combining: $0 \cdot \frac{V}{sec} = B \cdot s_1 + D \cdot s_2$

Solve simultaneously for B and D: $0 \cdot \frac{V}{sec} = B \cdot s_1 + (-12 \cdot V - B) \cdot s_2$ $B = s_2 \cdot \frac{12 \cdot V}{(s_1 - s_2)} = -60 \cdot V$

$$D = -12 \cdot V - B = -12 \cdot V - -60 \cdot V = 48 \cdot V$$

Substitute back in: $v_C(t) := -60 \cdot V \cdot e^{-\frac{2000}{sec} \cdot t} + 48 \cdot V \cdot e^{-\frac{2500}{sec} \cdot t} + 12 \cdot V$

Notice that this is not a simple exponential curve, although admittedly it's not much more interesting.

Overdamped Example, find i(t) instead of $v_C(t)$

But what if you didn't want $v_C(t)$? What if you were more interested in the current? There are two ways to find i(t).

1) Find $v_C(t)$ and use it to find i(t):

$$i_C(t) = C \cdot \frac{d}{dt} v_C(t) = C \cdot \frac{d}{dt}\left(-60 \cdot V \cdot e^{-\frac{2000}{sec} \cdot t} + 48 \cdot V \cdot e^{-\frac{2500}{sec} \cdot t} + 12 \cdot V\right) = C \cdot -60 \cdot V \cdot \left(-\frac{2000}{sec}\right) \cdot e^{-\frac{2000}{sec} \cdot t} + C \cdot 48 \cdot V \cdot \left(-\frac{2500}{sec}\right) \cdot e^{-\frac{2500}{sec} \cdot t}$$

$$C \cdot -60 \cdot V \cdot -\frac{2000}{sec} = 1.2 \cdot A \qquad C \cdot 48 \cdot V \cdot -\frac{2500}{sec} = -1.2 \cdot A \quad \text{and} \quad i(t) := 1.2 \cdot e^{-\frac{2000}{sec} \cdot t} - 1.2 \cdot e^{-\frac{2500}{sec} \cdot t}$$

2) find $i_L(t)$ directly:

$$i(t) = i(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t} \qquad i(\infty) = \text{final condition (steady-state or forced solution)} = 0 \cdot A$$
$$\text{Capacitor will charge up and current will stop.}$$

$$i(0) = i(\infty) + B + D = 0, \quad \text{Initial current though inductor is 0} \qquad B = -D$$

$$\frac{d}{dt} i(0) = \frac{12 \cdot V}{L} \quad \text{Since all initial voltage will all be across inductor}$$

$$\frac{d}{dt} i(0) = B \cdot s_1 + D \cdot s_2 = \frac{12 \cdot V}{L}$$

Solve simultaneously for B and D $\qquad \frac{12 \cdot V}{L} = B \cdot s_1 - B \cdot s_2 \qquad B = \frac{12 \cdot V}{L \cdot (s_1 - s_2)} = 1.2 \cdot A \qquad D = -B$

Substitute back in: $\quad i(t) := 1.2 \cdot e^{-\frac{2000}{sec} \cdot t} - 1.2 \cdot e^{-\frac{2500}{sec} \cdot t}$, same as answer as the first method.

However you get to it, at least this curve is slightly more interesting than the $v_C(t)$.

time (ms)
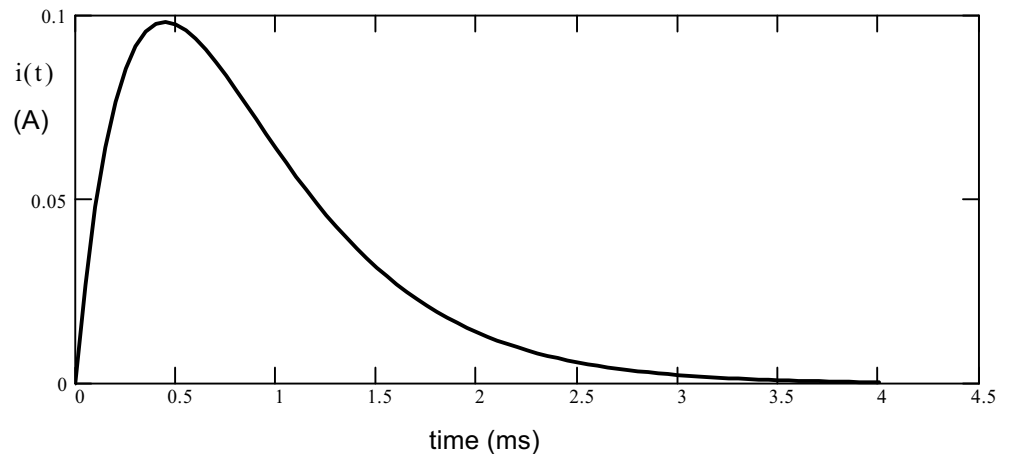
Underdamped Example $\quad R := 10 \cdot \Omega \qquad L := 20 \cdot mH \qquad C := 10 \cdot \mu F$

$$s_1 := -\frac{R}{2 \cdot L} + \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_1 = -250 + 2222i \cdot sec^{-1} \qquad s_2 := -\frac{R}{2 \cdot L} - \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \qquad s_2 = -250 - 2222i \cdot sec^{-1}$$

$$\alpha := -250 \cdot \frac{1}{sec} \qquad \omega := \left|\frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}}\right| \qquad \omega = 2.222 \cdot 10^3 \cdot \frac{rad}{sec} \qquad \text{OR:} \qquad f = \frac{1}{2 \cdot \pi} \cdot \left|\frac{1}{2} \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}}\right| = 353.7 \cdot Hz$$

Let's find the current again this time.

$$i(t) = i(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

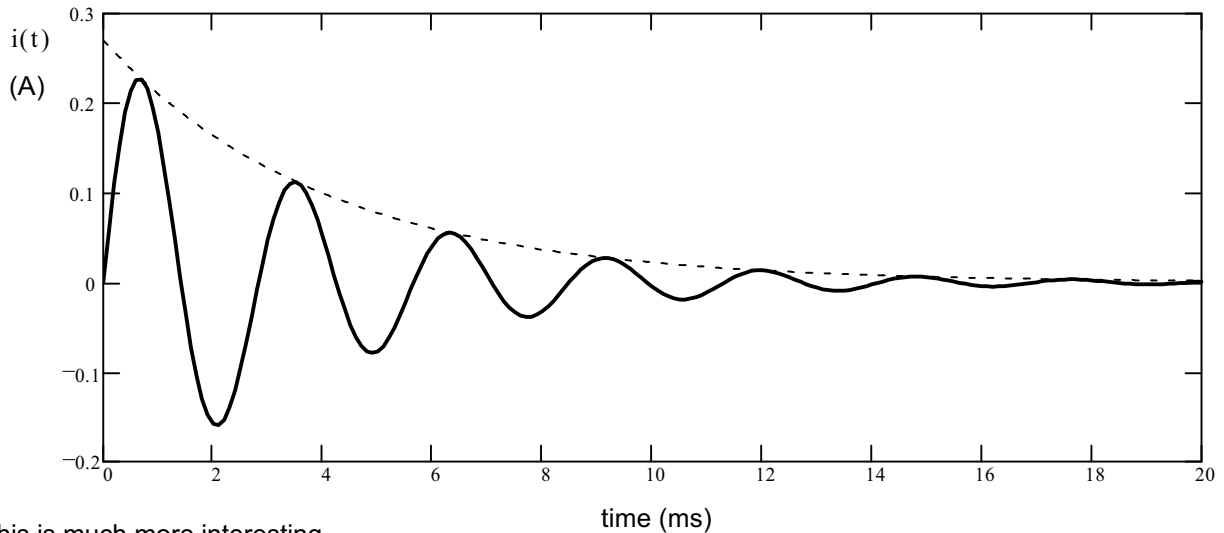$$i(0) = i(\infty) + B, \quad B = 0$$

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$$\frac{d}{dt}i(t) = \alpha \cdot e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t)) + e^{\alpha \cdot t} \cdot (-B \cdot \sin(\omega \cdot t) \cdot \omega + D \cdot \cos(\omega \cdot t) \cdot \omega)$$

$$\frac{d}{dt}i(0) = B \cdot \alpha + D \cdot \omega = \frac{12 \cdot V}{L} \quad \text{Since all initial voltage will all be across inductor.}$$

Solve simultaneously for B and D $\quad D = \dfrac{12 \cdot V}{\omega \cdot L} = 0.27 \cdot A$

Substitute back in: $\quad i(t) := e^{\alpha \cdot t} \cdot (0.27 \cdot \sin(\omega \cdot t)) \qquad\qquad\qquad t := 0, .0001 \ldots .02 \cdot \sec \text{(for plotting)}$

time (ms)

Now this is much more interesting.

Critically Damped Example

First we have to figure out how to get this case

Change R's value to create critical damping: $\quad \left(\dfrac{R}{L}\right)^2 - \dfrac{4}{L \cdot C} = 0 \quad R := 2 \cdot \sqrt{\dfrac{L}{C}} \quad R = 89.44271909999159 \cdot \Omega \quad$ (exactly)

$$s_1 := -\frac{R}{2 \cdot L} + \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \quad s_1 = -2236 \cdot \sec^{-1} \qquad s_2 := -\frac{R}{2 \cdot L} - \frac{1}{2} \cdot \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L \cdot C}} \quad s_2 = -2236 \cdot \sec^{-1}$$

$$i(t) = i(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t} \qquad i(\infty) = \text{final condition} = 0 \cdot A \qquad \text{Capacitor will charge up and current will stop.}$$
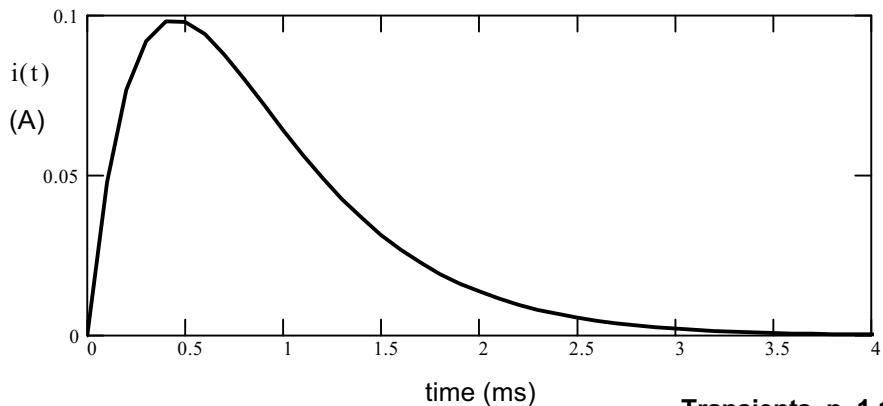
$$i(0) = i(\infty) + B = 0 \quad , \quad B = 0$$

$$\frac{d}{dt}i(0) = B \cdot s_1 \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t} + D \cdot t \cdot s_2 \cdot e^{s_2 \cdot t} = B \cdot s_1 + D = \frac{12 \cdot V}{L} \qquad \text{Since all initial voltage will be across inductor.}$$

Solve for D: $\quad D = \dfrac{12 \cdot V}{L} = 600 \cdot \dfrac{A}{\sec} \qquad\qquad$ Substitute back in: $\quad i(t) := 600 \cdot \dfrac{A}{\sec} \cdot t \cdot e^{-\frac{2236}{\sec} \cdot t}$

if you notice a remarkable similarity with the overdamped case, that's common for critical damping.

time (ms)

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1.7 The Easy Way to get the Characteristic Equation

Recall from your Ordinary Differential Equations class, the Laplace transform method of solving differential equations. The Laplace transform allowed you to change time-domain functions to frequency-domain functions. We've already done this for steady-state AC circuits. We changed functions of t into functions of ω. That was the frequency domain. Laplace let's us do the same sort of thing for transients. The general procedure is as follows.

  1) Transform your forcing functions into the frequency domain with the Laplace transform.

  2) Solve your differential equations with plain old algebra, where:

$$\frac{d}{dt} \text{ operation can be replaced with s,} \qquad \text{and} \qquad \int \blacksquare \, dt \text{ can be replaced by } \frac{1}{s}$$

  3) Transform your result back to the time domain with the inverse Laplace transform.

Step 1 isn't too bad, but step 3 can be a total pain without a good computer program to do the job. However, step 2 sounds great. It turns out that we can use step 2 alone and still learn a great deal about our circuits and other systems without ever bothering with steps 1 and 3.

First remember from your study of Laplace that differentiation in the time domain was the same as multiplication by s in the frequency domain. That's really all we need and we're off and running.

$$v_L(t) = L\frac{d}{dt}i_L(t) \; ---> \; V_L(s) = L \cdot s \cdot I_L(s) \qquad \text{and} \qquad i_C(t) = C\frac{d}{dt}v_C(t) \; ---> \; I_C(s) = C \cdot s \cdot V_C(s)$$

Leading to the Laplace impedances: Ls for an inductor and $\dfrac{1}{Cs}$ for a capacitor.

That's it, now we can use these impedances just like the $j\omega$ impedances, and we can use all the tools developed for DC. And with Laplace we don't even have to mess with complex numbers.

Look what happens to the RLC circuit now.

Pick any dependent variable ($I(s)$, $V_R(s)$, $V_L(s)$, or $V_C(s)$) and write a transfer function, which is a ratio of the dependent variable to the input ($V_{in}(s)$), like this:

$$V_{in}(s) = I(s) \cdot \left(\frac{1}{C \cdot s} + R + L \cdot s \right)$$

Transfer function = $H(s) = \dfrac{I(s)}{V_{in}(s)} = \dfrac{1}{\left(\dfrac{1}{C \cdot s} + R + L \cdot s \right)}$

Manipulate this transfer function into this form: $\dfrac{a_1 \cdot s^2 + b_1 \cdot s + k_1}{s^2 + b \cdot s + k}$

One polynomial divided by another.

$$\frac{I(s)}{V_{in}(s)} = \frac{1 \cdot (C \cdot s)}{(1 + R + L \cdot s \cdot (C \cdot s))} = \frac{\dfrac{1}{L} \cdot s}{\left(s^2 + \dfrac{R}{L} \cdot s + \dfrac{1}{L \cdot C} \right)} \qquad \text{in the correct form.}$$

Set the denominator to 0 and you get the characteristic equation: $\qquad s^2 + \dfrac{R}{L} \cdot s + \dfrac{1}{L \cdot C} = 0$

At this point you just proceed with the solution like you did before; Solve the characteristic equation to find s_1 and s_2. Decide which case you have (over-, under-, or critically damped). Use the two initial conditions, $i_L(0)$ and $v_C(0)$ to find the initial condition and the initial slope of your variable of interest, then use those to find the constants B and D.

Differential equation from the transfer function
You can also use the transfer function to go back and find the differential equation, just replace each s with a

$\dfrac{d}{dt}\blacksquare$ and go back to functions of t. $\qquad \dfrac{1}{L} \cdot \dfrac{d}{dt} V_{in}(t) = \left(\dfrac{d^2}{dt^2} i(t) + \dfrac{R}{L} \dfrac{d}{dt} i(t) + \dfrac{1}{L \cdot C} \cdot i(t) \right)$ Actually this is a pretty useless thing to do.

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