Notes, Second Order Transients

ECE 2210

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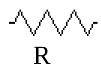
4/6/00,
2/25/16,
10/23/23

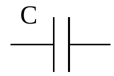
Laplace impedances

Resistor



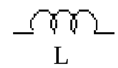
$Z_R = R$

Capacitor



$Z_C = \frac{1}{C \cdot s}$

Inductor



$Z_L = L \cdot s$

Transfer function

Use Laplace impedances, manipulate your circuit equation(s) to find a transfer function:

Rearrange circuit equation to: $\mathbf{H}(s) = \frac{\text{output}}{\text{input}} = \frac{\mathbf{V}_X(s)}{\mathbf{V}_{in}(s)} = \frac{a_1 \cdot s^2 + b_1 \cdot s + k_1}{s^2 + b \cdot s + k}$ = transfer function

May be $\mathbf{I}_X$ or any desired variable

May be $\mathbf{I}_{in}$ or any forcing function

a_1, b_1, k_1 coefficients may be zero

Characteristic equation

To find the poles of the transfer function

$$s^2 + b \cdot s + k = 0$$

characteristic equation

Complete solution

Solutions to the characteristic equation: $s_1 = -\frac{b}{2} + \frac{\sqrt{b^2 - 4 \cdot k}}{2}$ $\quad s_2 = -\frac{b}{2} - \frac{\sqrt{b^2 - 4 \cdot k}}{2}$

Find initial Conditions (v_C and/or i_L)

Find conditions of just before time $t = 0$, $v_C(0-)$ and $i_L(0-)$. These will be the same just after time $t = 0$, $v_C(0+)$ and $i_L(0+)$ and will be your initial conditions.

Use normal circuit analysis to find your desired variable: $v_X(0)$ or $i_X(0)$

Also find: $\frac{d}{dt} v_X(0)$ or $\frac{d}{dt} i_X(0)$ The trick to finding these is to see that: $\frac{d}{dt} v_C(0) = \frac{i_C(0)}{C}$ and $\frac{d}{dt} i_L(0) = \frac{v_L(0)}{L}$

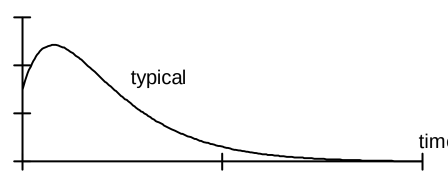
Find final conditions ("steady-state" or "forced" solution)

DC inputs: Inductors are shorts Capacitors are opens Solve by DC analysis $v_X(\infty)$ or $i_X(\infty)$

AC inputs: Solve by AC steady-state analysis using $j\omega$

X(t) may be replaced by $v_X(t)$, $i_X(t)$ or any desired variable in the equations below

<u>Overdamped</u> $b^2 - 4 \cdot k > 0$ $\quad s_1$ and s_2 are real and negative

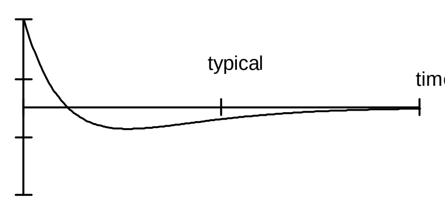


$$X(t) = X(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t}$$

$$X(0) = X(\infty) + B + D \qquad \frac{d}{dt} X(0) = B \cdot s_1 + D \cdot s_2$$

Solve simultaneously for B and D.

<u>Critically damped</u> $b^2 - 4 \cdot k = 0$ $\quad s_1 = s_2 = -\frac{b}{2} = s$ $\quad s_1$ and s_2 are real, equal and negative



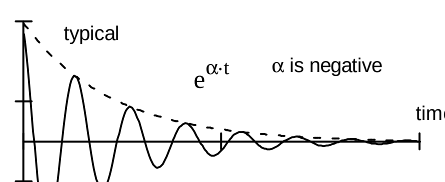
$$X(t) = X(\infty) + B \cdot e^{s \cdot t} + D \cdot t \cdot e^{s \cdot t}$$

$$X(0) = X(\infty) + B$$

so.. $B = X(0) - X(\infty)$ $\qquad \frac{d}{dt} X(0) = B \cdot s + D$ $\qquad$ so.. $D = \frac{d}{dt} X(0) - B \cdot s$

<u>Underdamped</u> $b^2 - 4 \cdot k < 0$ $\quad s_1 = \alpha + j \cdot \omega$ $\quad s_2 = \alpha - j \cdot \omega$ $\quad \alpha$ is negative

complex s_1 and s_2



$f = \frac{\omega}{2 \cdot \pi}$

$$X(t) = X(\infty) + e^{\alpha \cdot t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

$$X(0) = X(\infty) + B \qquad \frac{d}{dt} X(0) = B \cdot \alpha + D \cdot \omega$$

so.. $B = X(0) - X(\infty)$ $\qquad$ so.. $D = \frac{\frac{d}{dt} X(0) - B \cdot \alpha}{\omega}$

Derivation of the Canned Solutions, Second Order Transients ECE 2210

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10/23/23

How do we find B and D ?? You will use the canned solutions, which I will derive here, using initial conditions.

These are worked out within an example, starting on page 1.12 of the main Second-Order Transients handout.

Overdamped

Let's assume we've found that s_1 and s_2 are real and negative, and you're interested in the capacitor voltage.

Then: $v_C(t) = v_C(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$

At time $t = 0$ $v_C(0) = v_C(\infty) + B + D = v_C(0-)$, whatever it was just before time $t = 0$. It CANNOT change instantly
Same for $i_L(0)$

But that's only one equation, and we have two unknowns, B and D.

The trick is to differentiate the solution: $v_C(t) = v_C(\infty) + B\cdot e^{s_1\cdot t} + D\cdot e^{s_2\cdot t}$

$$\frac{d}{dt}v_C(t) = 0 + B\cdot s_1\cdot e^{s_1\cdot t} + D\cdot s_2\cdot e^{s_2\cdot t}$$

At time $t = 0$: $\frac{d}{dt}v_C(0) = B\cdot s_1 + D\cdot s_2$ = initial slope

From initial conditions, above: $\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C} = B\cdot s_1 + D\cdot s_2$ The second equation !

Solve simultaneously for B and D.

But i_C CAN change instantly, so...

We will find $i_C(0)$ from $i_L(0) = i_L(0-)$ because i_L can't change instantly

This will require circuit analysis at time $t = 0+$

Underdamped

Let's assume we've found complex s_1 and s_2 $s_1 = \alpha + j\cdot\omega$ $s_2 = \alpha - j\cdot\omega$ α is negative

Then: $v_C(t) = v_C(\infty) + e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$

At time $t = 0$ $v_C(0) = v_C(\infty) + B = v_C(0-)$ $B = v_C(0) - v_C(\infty)$

Now differentiate the solution: $v_C(t) = v_C(\infty) + e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$

recall: $\frac{d}{dt}(f(t)\cdot g(t)) = \left(\frac{d}{dt}f(t)\right)\cdot g(t) + f(t)\cdot\left(\frac{d}{dt}g(t)\right)$

yields: $\frac{d}{dt}v_C(t) = \alpha\cdot e^{\alpha\cdot t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t)) + e^{\alpha\cdot t}\cdot(-B\cdot\sin(\omega\cdot t)\cdot\omega + D\cdot\cos(\omega\cdot t)\cdot\omega)$

At time $t = 0$: $\frac{d}{dt}v_C(0) = B\cdot\alpha + D\cdot\omega$ Solve for: $D = \dfrac{\frac{d}{dt}v_C(0) - B\cdot\alpha}{\omega}$

Critically damped

Let's assume we've found real $s_1 = s_2 = s$

Then: $v_C(t) = v_C(\infty) + B\cdot e^{s\cdot t} + D\cdot t\cdot e^{s\cdot t}$

At time $t = 0$ $v_C(0) = v_C(\infty) + B = v_C(0-)$ $B = v_C(0) - v_C(\infty)$

Now differentiate the solution: $\frac{d}{dt}v_C(t) = B\cdot s\cdot e^{s\cdot t} + D\cdot e^{s\cdot t} + D\cdot t\cdot s\cdot e^{s\cdot t}$

$\frac{d}{dt}v_C(0) = B\cdot s + D$ Solve for: $D = \frac{d}{dt}v_C(0) - B\cdot s$

Same goes for and variable (like $i_L(t)$, for example). $v_C(0+) = v_C(0-)$ $i_L(0+) = i_L(0-)$

$\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C}$ $\frac{d}{dt}i_L(0) = \frac{v_L(0)}{L}$ And circuit analysis at time $t = 0+$