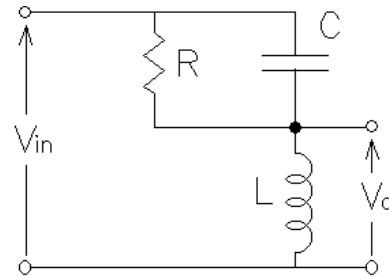


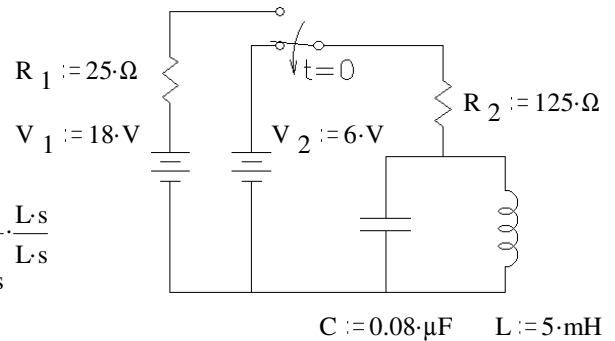
Ex. 1 a) Find the transfer function of the circuit shown. Write your equation in the form of one simple polynomial divided by another



$$\begin{aligned}
 H(s) &= \frac{V_o(s)}{V_{in}(s)} = \frac{L \cdot s}{\frac{1}{\frac{1}{R} + C \cdot s} + L \cdot s} \cdot \left(\frac{1}{R} + C \cdot s \right) \\
 &= \frac{\frac{L \cdot s + L \cdot C \cdot s^2}{R}}{\left[1 + L \cdot s \cdot \left(\frac{1}{R} + C \cdot s \right) \right]} = \frac{\frac{L \cdot s + L \cdot C \cdot s^2}{R}}{1 + \frac{L \cdot s}{R} + L \cdot C \cdot s^2} = \frac{\frac{L \cdot s + L \cdot C \cdot s^2}{R}}{L \cdot C \cdot s^2 + \frac{L \cdot s}{R} + 1} \cdot \frac{1}{L \cdot C} \\
 &= \frac{s^2 + \frac{1}{C \cdot R} \cdot s}{s^2 + \left(\frac{1}{C \cdot R} \right) \cdot s + \frac{1}{L \cdot C}}
 \end{aligned}$$

b) Find the characteristic equation $0 = s^2 + \frac{1}{C \cdot R} \cdot s + \frac{1}{L \cdot C}$

Ex. 2 a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at $t = 0$).



$$\begin{aligned}
 \frac{V_C(s)}{V_{in}(s)} &= H(s) = \frac{\frac{1}{L \cdot s}}{\frac{1}{L \cdot s} + R_2} \cdot \left(\frac{1}{L \cdot s} + C \cdot s \right) = \frac{1}{1 + \frac{R_2}{L \cdot s} + R_2 \cdot C \cdot s} \cdot \frac{L \cdot s}{L \cdot s} \\
 &= \frac{L \cdot s}{L \cdot s + R_2 + R_2 \cdot C \cdot L \cdot s^2} \cdot \frac{R_2 \cdot C \cdot L}{R_2 \cdot C \cdot L} = \frac{1}{\frac{1}{R_2 \cdot C} \cdot s + \frac{1}{L \cdot C} + s^2}
 \end{aligned}$$

Characteristic equation:
 $0 = s^2 + \frac{1}{R_2 \cdot C} \cdot s + \frac{1}{L \cdot C}$

b) Find the solutions of the characteristic equation.

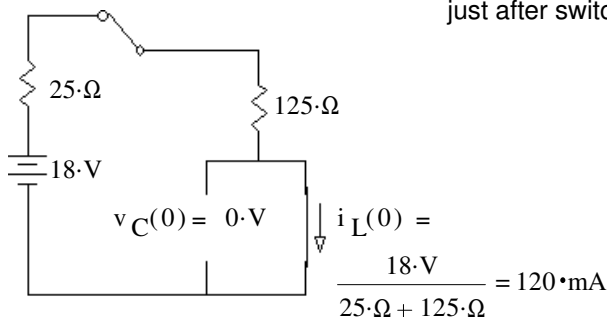
$$\frac{1}{R_2 \cdot C} = 1 \cdot 10^5 \cdot \text{sec}^{-1} \qquad \frac{4}{L \cdot C} = 1 \cdot 10^{10} \cdot \text{sec}^{-2}$$

$$s_1 := \frac{-\frac{1}{R_2 \cdot C} + \sqrt{\left(\frac{1}{R_2 \cdot C} \right)^2 - \frac{4}{L \cdot C}}}{2} \qquad s_1 = -5 \cdot 10^4 \cdot \frac{1}{\text{sec}} \qquad s_2 := \frac{-\frac{1}{R_2 \cdot C} - \sqrt{\left(\frac{1}{R_2 \cdot C} \right)^2 - \frac{4}{L \cdot C}}}{2} \qquad s_2 = -5 \cdot 10^4 \cdot \frac{1}{\text{sec}}$$

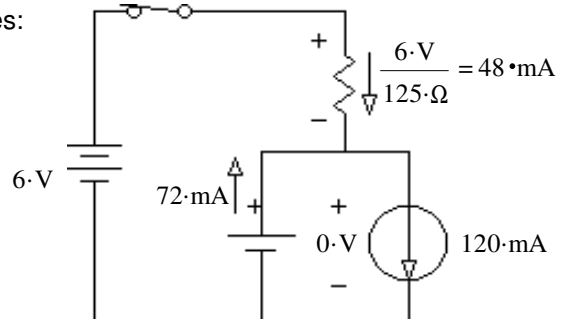
$s_1 = s_2$ so... critically damped

c) Find initial and final conditions for $v_C(t)$

before switch is moved:



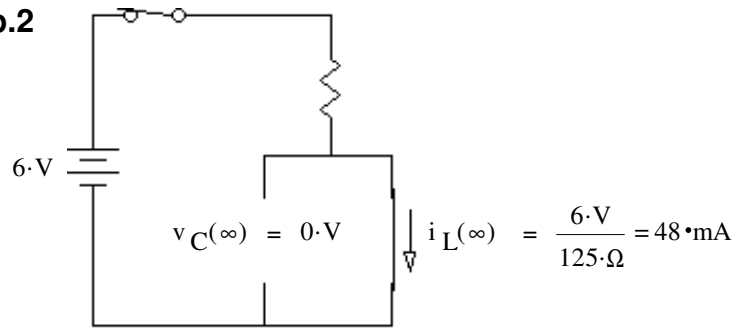
just after switch closes:



$$\frac{d}{dt} v_C(0) = \frac{-72 \text{ mA}}{C} = -9 \cdot 10^5 \cdot \frac{\text{V}}{\text{sec}}$$

Second-Order Transient Examples, p.2

Final conditions:



d) Find the full expression of $v_C(t)$.

Critically damped $v_C(t) = v_C(\infty) + B \cdot e^{s_1 t} + D \cdot t \cdot e^{s_2 t}$

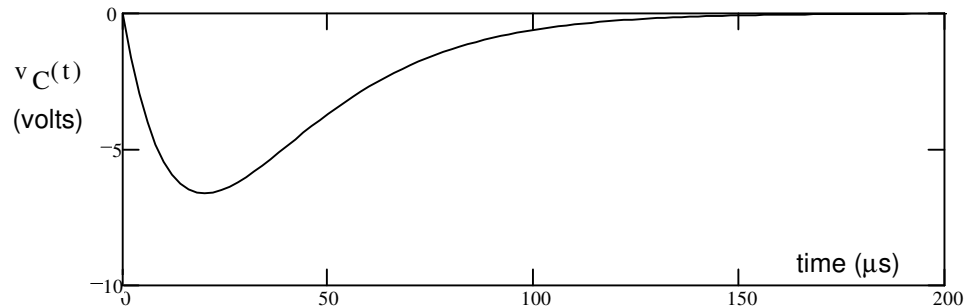
$$v_C(0) = v_C(\infty) + B$$

$$0 = 0 + B \quad B = 0$$

$$\frac{d}{dt}v_C(0) = B \cdot s + D$$

$$-9 \cdot 10^5 \frac{V}{\text{sec}} = 0 \cdot \left(-5 \cdot 10^4 \cdot \frac{1}{\text{sec}}\right) + D \quad D := -9 \cdot 10^5 \frac{V}{\text{sec}}$$

$$v_C(t) := -9 \cdot 10^5 \frac{V}{\text{sec}} \cdot t \cdot e^{-5 \cdot 10^4 t}$$



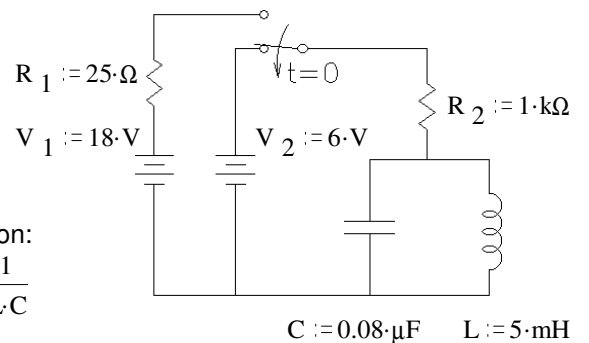
Ex. 2 with bigger R_2

a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at $t = 0$), as before:

$$\frac{V_C(s)}{V_{in}(s)} = H(s) = \frac{\frac{1}{R_2 \cdot C} \cdot s}{\frac{1}{R_2 \cdot C} \cdot s + \frac{1}{L \cdot C} + s^2}$$

Characteristic equation:

$$0 = s^2 + \frac{1}{R_2 \cdot C} \cdot s + \frac{1}{L \cdot C}$$



b) Find the solutions of the characteristic equation.

$$\frac{1}{R_2 \cdot C} = 1.25 \cdot 10^4 \cdot \text{sec}^{-1}$$

$$\frac{4}{L \cdot C} = 1 \cdot 10^{10} \cdot \text{sec}^{-2}$$

$$s_1 := \frac{-\frac{1}{R_2 \cdot C} + \sqrt{\left(\frac{1}{R_2 \cdot C}\right)^2 - \frac{4}{L \cdot C}}}{2}$$

$$s_1 = -6.25 \cdot 10^3 + 4.961 \cdot 10^4 j \quad \cdot \frac{1}{\text{sec}}$$

$$s_2 = \overline{s_1} = -6.25 \cdot 10^3 - 4.961 \cdot 10^4 j \quad \cdot \frac{1}{\text{sec}}$$

s_1 & s_2 complex, so underdamped

$$\alpha := \text{Re}(s_1) \quad \alpha = -6.25 \cdot 10^3 \cdot \text{sec}^{-1}$$

$$\omega := \text{Im}(s_1)$$

$$\omega = 4.961 \cdot 10^4 \cdot \text{sec}^{-1}$$

Second-Order Transient Examples, p.3

- c) Find initial and final conditions for $v_C(t)$
See drawings above

$$v_C(0) = 0 \cdot V$$

$$i_L(0) = \frac{18 \cdot V}{25 \cdot \Omega + 1 \cdot k\Omega} = 17.561 \cdot mA$$

$$i_C(0) = \frac{6 \cdot V}{1 \cdot k\Omega} - 17.561 \cdot mA = -11.561 \cdot mA$$

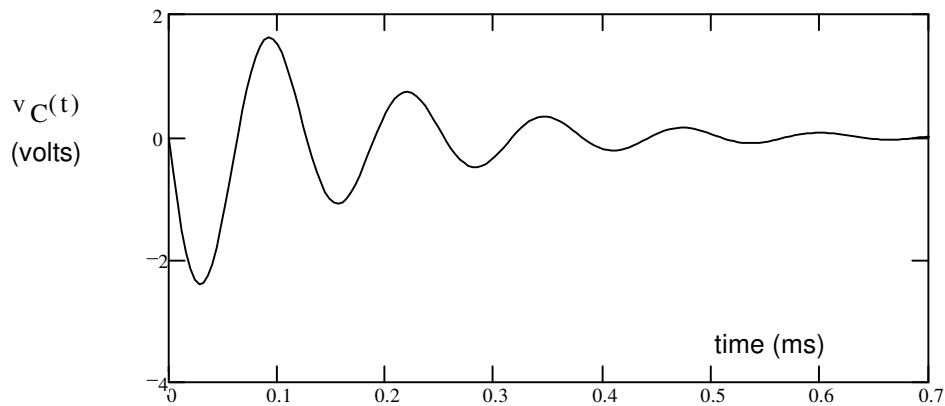
$$\frac{d}{dt}v_C(0) = \frac{-11.561 \cdot mA}{C} = -1.445 \cdot 10^5 \cdot \frac{V}{sec}$$

$$v_C(\infty) = 0 \cdot V$$

- d) Find the full expression of $v_C(t)$. Underdamped $v_C(t) = v_C(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega t) + D \cdot \sin(\omega t))$

$$B = v_C(0) - v_C(\infty) = 0 \cdot V - 0 \cdot V = 0 \cdot V$$

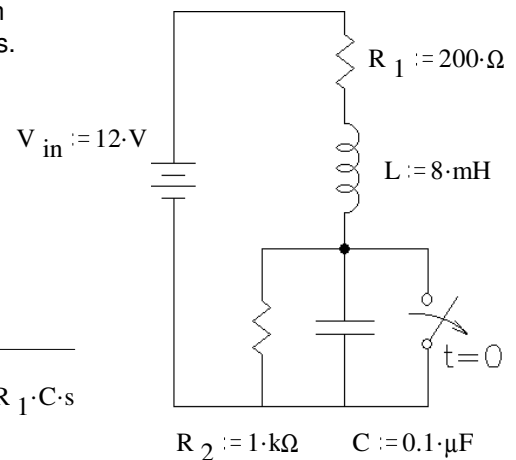
$$D = \frac{\frac{d}{dt}v_C(0) - B \cdot \alpha}{\omega} = \frac{-1.445 \cdot 10^5 \cdot \frac{V}{sec} - 0 \cdot V \cdot \alpha}{\omega} = -2.913 \cdot V \quad v_C(t) := -2.913 \cdot V \cdot e^{-\frac{6250}{sec} t} \cdot \sin\left(\frac{49610}{sec} \cdot t\right)$$



- Ex. 3** a) Find the characteristic equation of the circuit shown. (after the switch opens at $t = 0$). Write your equation in the form of a simple polynomials.

$$H(s) = \frac{I_T(s)}{V_{in}(s)} = \frac{1}{Z(s)} = \frac{1}{\frac{1}{\frac{1}{R_2} + C \cdot s} + L \cdot s + R_1} \cdot \left(\frac{1}{R_2} + C \cdot s\right)$$

$$= \frac{\frac{1}{R_2} + C \cdot s}{1 + L \cdot s \cdot \left(\frac{1}{R_2} + C \cdot s\right) + R_1 \cdot \left(\frac{1}{R_2} + C \cdot s\right)} = \frac{\frac{1}{R_2} + C \cdot s}{1 + \frac{L \cdot s}{R_2} + L \cdot C \cdot s^2 + \frac{R_1}{R_2} + R_1 \cdot C \cdot s}$$



$$= \frac{\frac{1}{R_2} + C \cdot s}{L \cdot C \cdot s^2 + \frac{L \cdot s}{R_2} + R_1 \cdot C \cdot s + 1 + \frac{R_1}{R_2}} \cdot \frac{1}{L \cdot C}$$

$$= \frac{s + \frac{1}{C \cdot R_2}}{s^2 + \left(\frac{L}{L \cdot C \cdot R_2} + \frac{R_1 \cdot C}{L \cdot C}\right) \cdot s + \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C}}$$

Characteristic eq.: $0 = s^2 + \left(\frac{1}{C \cdot R_2} + \frac{R_1}{L}\right) \cdot s + \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C}$

Second-Order Transient Examples, p.4

b) Find the solutions (numbers) of the characteristic equation:

$$b := \frac{1}{C \cdot R_2} + \frac{R_1}{L}$$

$$b = 3.5 \cdot 10^4 \cdot \text{sec}^{-1}$$

$$k := \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C}$$

$$k = 1.5 \cdot 10^9 \cdot \text{sec}^{-2}$$

$$s_1 := \frac{-b + \sqrt{b^2 - 4 \cdot k}}{2}$$

$$s_1 = -1.75 \cdot 10^4 + 3.455 \cdot 10^4 j \cdot \frac{1}{\text{sec}}$$

$$\alpha := -\frac{b}{2}$$

$$\alpha = -1.75 \cdot 10^4 \cdot \text{sec}^{-1}$$

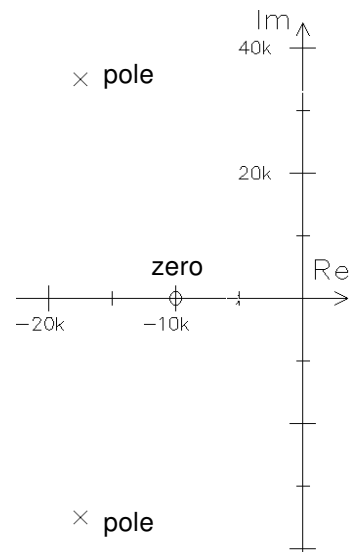
$$s_2 := \frac{-b - \sqrt{b^2 - 4 \cdot k}}{2}$$

$$s_2 = -1.75 \cdot 10^4 - 3.455 \cdot 10^4 j \cdot \frac{1}{\text{sec}}$$

$$\omega := \frac{1}{2} \cdot \sqrt{4 \cdot k - b^2}$$

$$\omega = 3.455 \cdot 10^4 \cdot \text{sec}^{-1}$$

Underdamped



c) Plot the poles and zeroes of the transfer function.

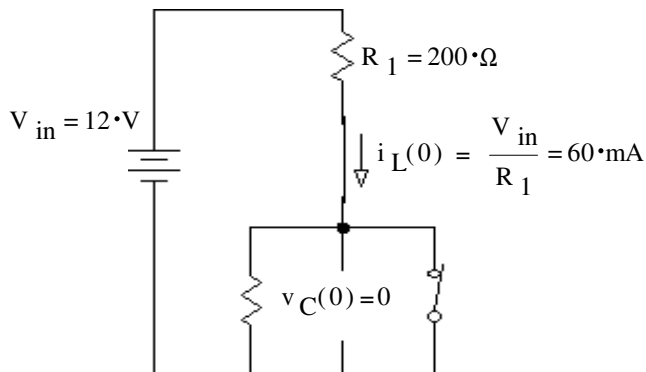
The poles are the s 's where the denominator is zero, that is, the s_1 & s_2 solutions to the characteristic equation.

The zero is the s where the numerator is zero: $0 = \frac{1}{L \cdot C \cdot R_2} + \frac{C}{L \cdot C} \cdot s$

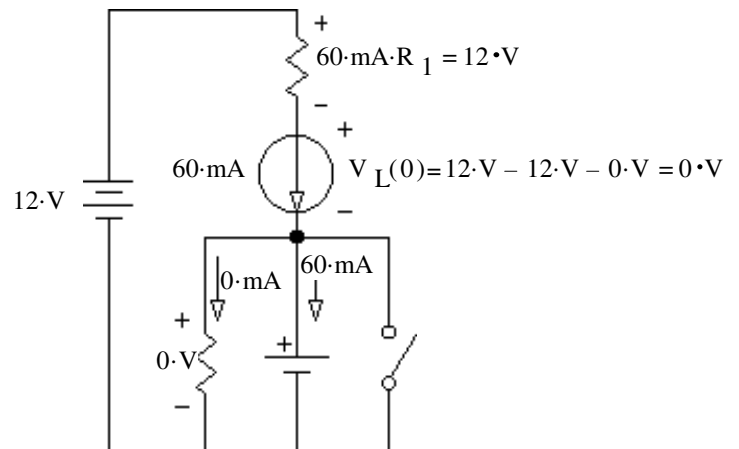
$$s = -\frac{1}{C \cdot R_2} = -1 \cdot 10^4 \cdot \text{sec}^{-1}$$

d) Initial and final conditions for $i_L(t)$ and $v_C(t)$.

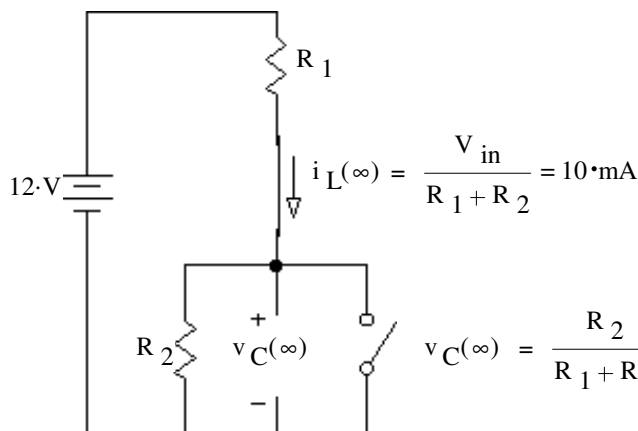
before the switch opens



just after the switch opens



Final condition:



$$\frac{d}{dt} i_L(0) = \frac{V_L(0)}{L} = \frac{0 \cdot \text{V}}{L} = 0 \cdot \frac{\text{A}}{\text{sec}}$$

$$\frac{d}{dt} v_C(0) = \frac{i(0)}{C} = \frac{60 \cdot \text{mA}}{C} = 6 \cdot 10^5 \cdot \frac{\text{V}}{\text{sec}}$$

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} \cdot V_{\text{in}} = \frac{1 \cdot \text{k}\Omega}{200 \cdot \Omega + 1 \cdot \text{k}\Omega} \cdot 12 \cdot \text{V} = 10 \cdot \text{V}$$

Second-Order Transient Examples, p.5

e) Find the full expression of $i_L(t)$.

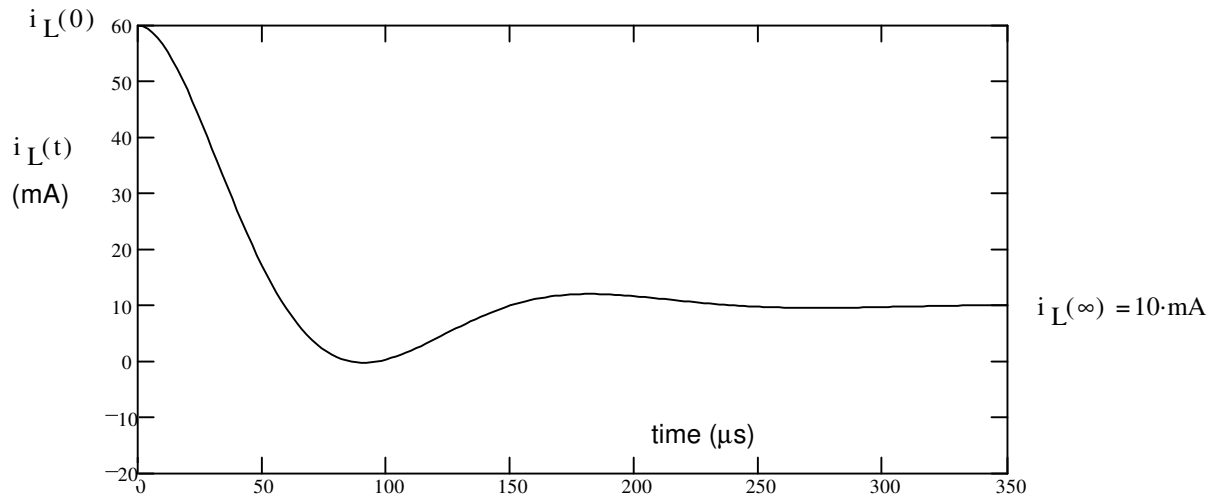
Underdamped $X(t) = X(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$

$$i_L(t) = i_L(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

$$i_L(0) = i_L(\infty) + B \quad \text{so..} \quad B = i_L(0) - i_L(\infty) \quad B := 60 \cdot \text{mA} - 10 \cdot \text{mA} \quad B = 50 \cdot \text{mA}$$

$$\frac{d}{dt}i_L(0) = B \cdot \alpha + D \cdot \omega \quad \text{so..} \quad D = \frac{\frac{d}{dt}i_L(0) - B \cdot \alpha}{\omega} \quad D := \frac{0 \cdot \frac{\text{A}}{\text{sec}} - B \cdot \alpha}{\omega} \quad D = 25.325 \cdot \text{mA}$$

$$i_L(t) := 10 \cdot \text{mA} + e^{-\frac{17500}{\text{sec}} \cdot t} \cdot \left(50 \cdot \text{mA} \cdot \cos\left(\frac{34550}{\text{sec}} \cdot t\right) + 25.325 \cdot \text{mA} \cdot \sin\left(\frac{34550}{\text{sec}} \cdot t\right) \right)$$

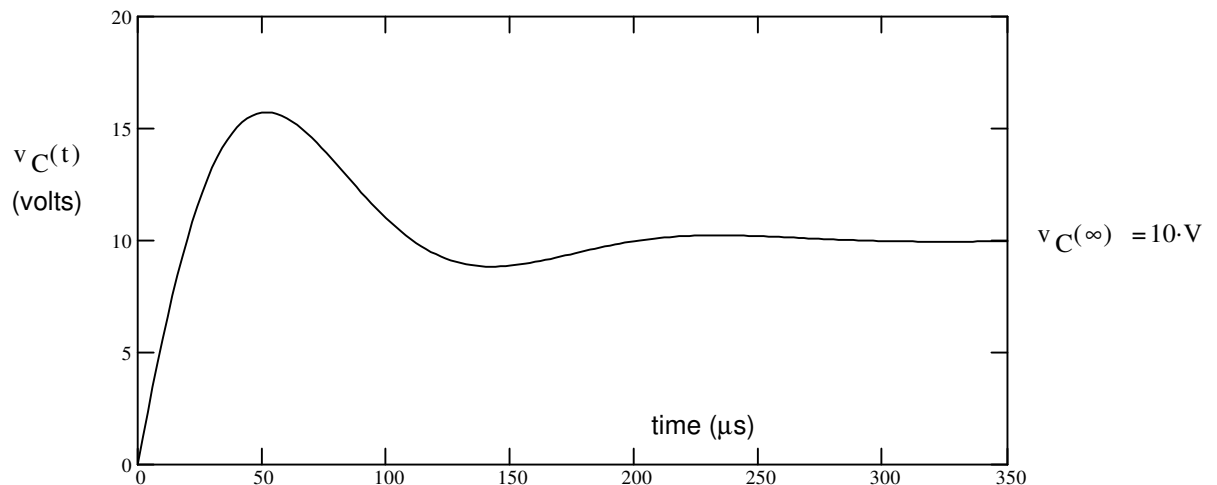


f) Find the full expression of $v_C(t)$.

$$B := 0 \cdot \text{V} - 10 \cdot \text{V} \quad B = -10 \cdot \text{V} \quad D := \frac{6 \cdot 10^5 \cdot \frac{\text{V}}{\text{sec}} - B \cdot \alpha}{\omega} \quad D = 12.301 \cdot \text{V}$$

$$v_C(t) = v_C(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

$$v_C(t) := 10 \cdot \text{V} + e^{-\frac{17500}{\text{sec}} \cdot t} \cdot \left(-10 \cdot \text{V} \cdot \cos\left(\frac{34550}{\text{sec}} \cdot t\right) + 12.301 \cdot \text{V} \cdot \sin\left(\frac{34550}{\text{sec}} \cdot t\right) \right)$$



Second-Order Transient Examples, p.6

h) What value of R_1 would make this system critically damped?

$$\left(\frac{1}{C \cdot R_2} + \frac{R_1}{L}\right)^2 = 4 \cdot \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C}$$

$$\frac{1}{C^2 \cdot R_2^2} + \frac{2}{C \cdot R_2} \cdot \frac{R_1}{L} + \frac{R_1^2}{L^2} = \frac{4}{L \cdot C} + \frac{4}{C \cdot R_2} \cdot \frac{R_1}{L}$$

$$0 = \left(\frac{1}{L^2} \cdot R_1^2 - \frac{2}{L \cdot C \cdot R_2} \cdot R_1\right) + \left(\frac{1}{C^2 \cdot R_2^2} - \frac{4}{L \cdot C}\right)$$

$$0 = \left(R_1^2 - \frac{2 \cdot L}{C \cdot R_2} \cdot R_1\right) + \left(\frac{L^2}{C^2 \cdot R_2^2} - \frac{4 \cdot L}{C}\right)$$

Solve for R_1 with quadratic equation:

$$R_1 = \frac{\frac{2 \cdot L}{C \cdot R_2} + \sqrt{\left(\frac{2 \cdot L}{C \cdot R_2}\right)^2 - 4 \cdot \left(\frac{L^2}{C^2 \cdot R_2^2} - \frac{4 \cdot L}{C}\right)}}{2}$$

Quadratic equation can be reduced to:

$$= \frac{L}{C \cdot R_2} - \frac{4}{2} \cdot \sqrt{\frac{L}{C}} = -485.7 \cdot \Omega \quad \text{this solution can't be}$$

$$= \frac{L}{C \cdot R_2} + \frac{4}{2} \cdot \sqrt{\frac{L}{C}} = 645.7 \cdot \Omega \quad \text{this must be the solution}$$

Ex. 2 with bigger R_1

$R_1 := 1 \cdot k\Omega$ This should make the system overdamped

$$b := \frac{1}{C \cdot R_2} + \frac{R_1}{L} \quad b = 1.35 \cdot 10^5 \cdot \text{sec}^{-1} \quad k := \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C} \quad k = 2.5 \cdot 10^9 \cdot \text{sec}^{-2}$$

$$s_1 := \frac{-b + \sqrt{b^2 - 4 \cdot k}}{2} \quad s_1 = -2.215 \cdot 10^4 \cdot \frac{1}{\text{sec}} \quad s_2 := \frac{-b - \sqrt{b^2 - 4 \cdot k}}{2} \quad s_2 = -1.128 \cdot 10^5 \cdot \frac{1}{\text{sec}}$$

Overdamped

$$v_C(0) = 0 \quad i_L(0) = \frac{V_{in}}{R_1} = 12 \cdot \text{mA} = i_C(0) \quad \frac{d}{dt} v_C(0) = \frac{i(0)}{C} = \frac{12 \cdot \text{mA}}{C} = 1.2 \cdot 10^5 \cdot \frac{V}{\text{sec}}$$

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} \cdot V_{in} = 6 \cdot V \quad i_L(\infty) = \frac{V_{in}}{R_1 + R_2} = 6 \cdot \text{mA} \quad 1.2 \cdot 10^5 \cdot \frac{1}{\text{sec}} =$$

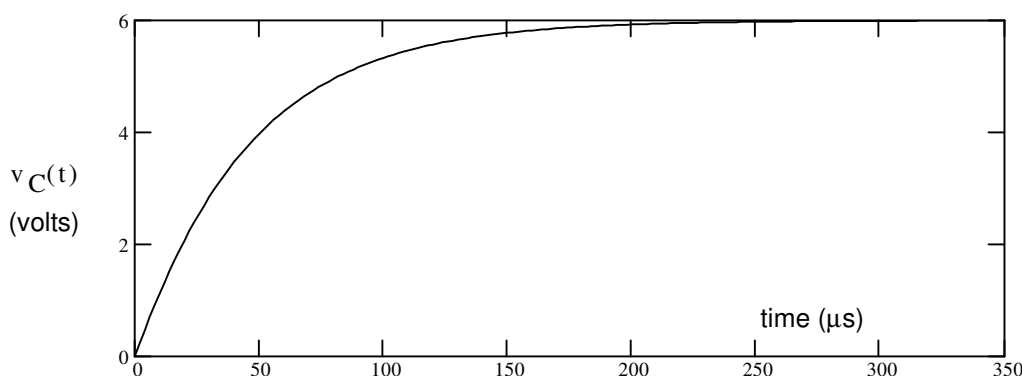
$$v_C(0) = v_C(\infty) + B + D$$

$$0 \cdot V = 6 \cdot V + B + D \quad B = -(6 \cdot V + D)$$

$$\frac{d}{dt} v_C(0) = B \cdot s_1 + D \cdot s_2 = -6 \cdot V \cdot s_1 - D \cdot s_1 + D \cdot s_2 \quad D := \frac{1.2 \cdot 10^5 \cdot \frac{V}{\text{sec}} + 6 \cdot V \cdot s_1}{s_2 - s_1} \quad D = 0.143 \cdot V$$

$$B = -(6 \cdot V + D) = -6.143 \cdot V$$

$$v_C(t) = v_C(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot e^{s_2 \cdot t} \quad v_C(t) := 6 \cdot V - 6.143 \cdot V \cdot e^{-\frac{2.215 \cdot 10^4}{\text{sec}} \cdot t} + 0.143 \cdot V \cdot e^{-\frac{1.128 \cdot 10^5}{\text{sec}} \cdot t}$$



$$v_C(\infty) = 6 \cdot V$$