# ECE 2210 Second-Order Transient Examples

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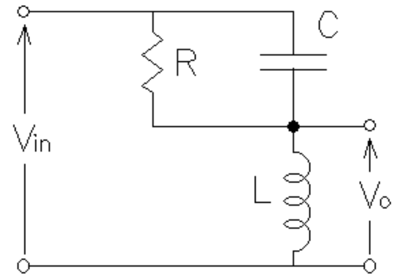
**Ex. 1** a) Find the transfer function of the circuit shown. Write your equation in the form of one simple polynomial divided by another




$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{L\cdot s}{\dfrac{1}{\dfrac{1}{R}+C\cdot s}+L\cdot s}\cdot\frac{\left(\dfrac{1}{R}+C\cdot s\right)}{\left(\dfrac{1}{R}+C\cdot s\right)}$$

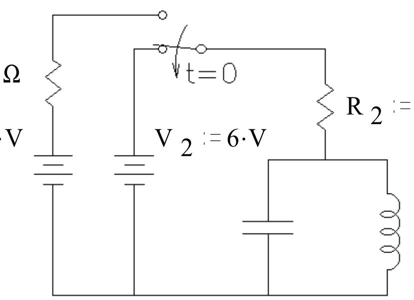
$$= \frac{\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2}{\left[1+L\cdot s\cdot\left(\dfrac{1}{R}+C\cdot s\right)\right]} = \frac{\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2}{1+\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2} = \frac{\dfrac{L\cdot s}{R}+L\cdot C\cdot s^2}{L\cdot C\cdot s^2+\dfrac{L\cdot s}{R}+1}\cdot\frac{\dfrac{1}{L\cdot C}}{\dfrac{1}{L\cdot C}} = \frac{s^2+\dfrac{1}{C\cdot R}\cdot s}{s^2+\left(\dfrac{1}{C\cdot R}\right)\cdot s+\dfrac{1}{L\cdot C}}$$

b) Find the characteristic equation $\quad 0 = s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{L\cdot C}$

**Ex. 2**

a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at t = 0).

$R_1 := 25\cdot\Omega$ $\quad V_1 := 18\cdot V$ $\quad V_2 := 6\cdot V$ $\quad R_2 := 125\cdot\Omega$




$C := 0.08\cdot\mu F \qquad L := 5\cdot mH$

$$\frac{V_C(s)}{V_{in}(s)} = H(s) = \frac{\dfrac{1}{\dfrac{1}{L\cdot s}+C\cdot s}}{\dfrac{1}{\dfrac{1}{L\cdot s}+C\cdot s}+R_2}\cdot\frac{\left(\dfrac{1}{L\cdot s}+C\cdot s\right)}{\left(\dfrac{1}{L\cdot s}+C\cdot s\right)} = \frac{1}{1+\dfrac{R_2}{L\cdot s}+R_2\cdot C\cdot s}\cdot\frac{L\cdot s}{L\cdot s}$$

$$= \frac{L\cdot s}{L\cdot s+R_2+R_2\cdot C\cdot L\cdot s^2}\cdot\frac{\dfrac{1}{R_2\cdot C\cdot L}}{\dfrac{1}{R_2\cdot C\cdot L}} = \frac{\dfrac{1}{R_2\cdot C}\cdot s}{\dfrac{1}{R_2\cdot C}\cdot s+\dfrac{1}{L\cdot C}+s^2}$$

Characteristic equation:

$$0 = s^2+\frac{1}{R_2\cdot C}\cdot s+\frac{1}{L\cdot C}$$

b) Find the solutions of the characteristic equation.

$$\frac{1}{R_2\cdot C} = 1\cdot 10^5\ \cdot sec^{-1} \qquad \frac{4}{L\cdot C} = 1\cdot 10^{10}\ \cdot sec^{-2}$$

$$s_1 := \frac{-\dfrac{1}{R_2\cdot C}+\sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2-\dfrac{4}{L\cdot C}}}{2} \qquad s_2 := \frac{-\dfrac{1}{R_2\cdot C}-\sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2-\dfrac{4}{L\cdot C}}}{2}$$

$$s_1 = -5\cdot 10^4\ \cdot\frac{1}{sec} \qquad s_2 = -5\cdot 10^4\ \cdot\frac{1}{sec}$$

c) Find initial and final conditions for $v_C(t)$

$s_1 = s_2$ so... critically damped

before switch is moved: $\quad v_L(\infty) = 0 = v_L(0\text{-}) = v_C(0\text{-}) = 0 = v_C(0+)$

$$\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C}$$

before switch is moved: $\quad i_{L0} := \dfrac{V_1 - 0\cdot V}{R_1+R_2}$ $\quad$ ($v_C(0)$) $\quad$ so... $\quad i_{L0} = 120\ \cdot mA$

after switch is moved: $\quad i_{R0} := \dfrac{6\cdot V - 0\cdot V}{R_2}$ $\quad i_{R0} = 48\ \cdot mA$

not all of the inductor's current comes through the resistor anymore, so the difference must come from the capacitor

$$i_{C0} := i_{R0} - i_{L0} \qquad i_{C0} = -72\ \cdot mA \qquad \frac{d}{dt}v_C(0) = \frac{i_{C0}}{C} = -9\cdot 10^5\ \cdot\frac{V}{sec}$$

d) Find the full expression of $v_C(t)$.

Critically damped $b^2 - 4 \cdot k = 0$ $\qquad s_1 = s_2 = -\frac{b}{2} = s$ $\qquad$ $s_1$ and $s_2$ are real, equal and negative

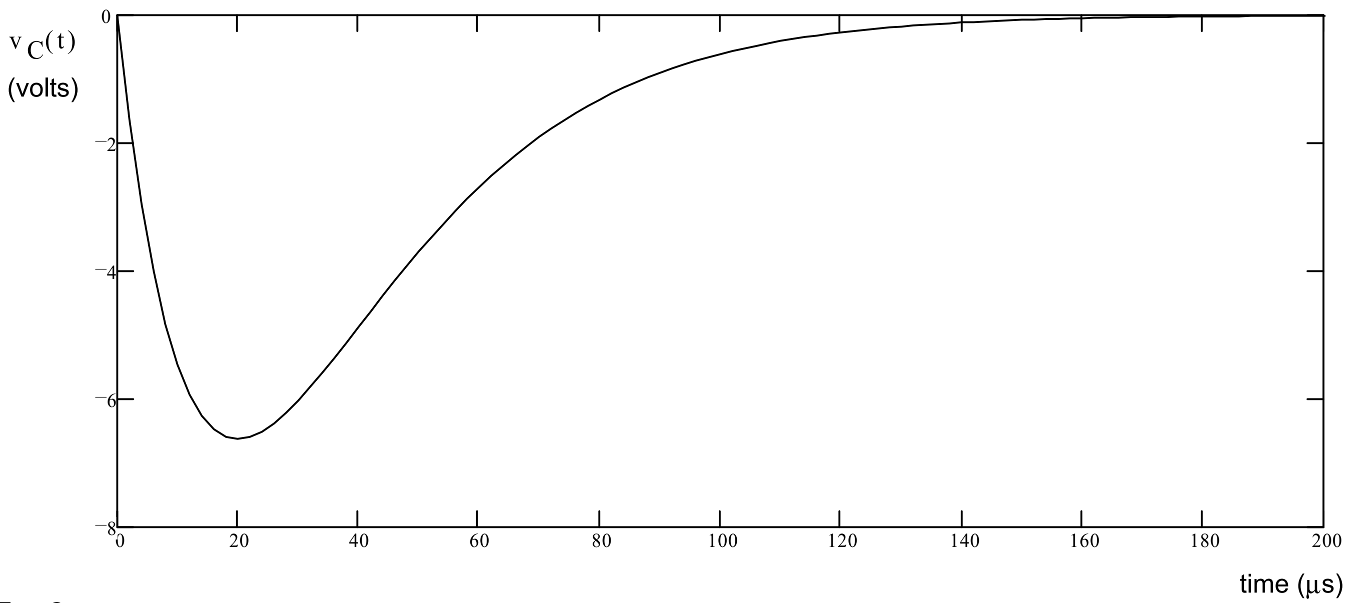
$$v_X(t) = v_X(\infty) + B \cdot e^{s_1 \cdot t} + D \cdot t \cdot e^{s_2 \cdot t}$$

$$v_X(0) = v_X(\infty) + B \qquad \frac{d}{dt} v_X(0) = B \cdot s + D \qquad \text{Solve simultaneously}$$

$$v_C(0) = v_C(\infty) + B \qquad \frac{d}{dt} v_C(0) = B \cdot s + D$$

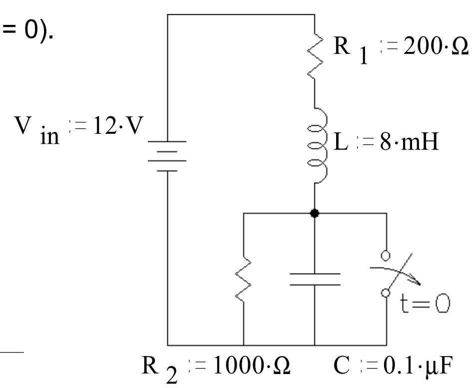
$$0 = 0 + B \qquad B = 0 \qquad -9 \cdot 10^5 \cdot \frac{V}{sec} = 0 \cdot \left(-5 \cdot 10^4 \cdot \frac{1}{sec}\right) + D \qquad D := -9 \cdot 10^5 \cdot \frac{V}{sec}$$

$$v_C(t) := -9 \cdot 10^5 \cdot \frac{V}{sec} \cdot t \cdot e^{s_1 \cdot t}$$




## Ex. 3

a) Find the characteristic equation of the circuit shown. (after the switch opens at t = 0). Write your equation in the form of a simple polynomials.




$$H(s) = \frac{I_T(s)}{V_{in}(s)} = \frac{1}{Z(s)} = \frac{1}{\dfrac{1}{\dfrac{1}{R_2} + C \cdot s} + L \cdot s + R_1} \cdot \frac{\left(\dfrac{1}{R_2} + C \cdot s\right)}{\left(\dfrac{1}{R_2} + C \cdot s\right)}$$

$$= \frac{\dfrac{1}{R_2} + C \cdot s}{1 + L \cdot s \cdot \left(\dfrac{1}{R_2} + C \cdot s\right) + R_1 \cdot \left(\dfrac{1}{R_2} + C \cdot s\right)} = \frac{\dfrac{1}{R_2} + C \cdot s}{1 + \dfrac{L \cdot s}{R_2} + L \cdot C \cdot s^2 + \dfrac{R_1}{R_2} + R_1 \cdot C \cdot s}$$

$$= \frac{\dfrac{1}{R_2} + C \cdot s}{L \cdot C \cdot s^2 + \dfrac{L \cdot s}{R_2} + R_1 \cdot C \cdot s + 1 + \dfrac{R_1}{R_2}} \cdot \frac{\dfrac{1}{L \cdot C}}{\dfrac{1}{L \cdot C}} = \frac{\dfrac{1}{L \cdot C \cdot R_2} + \dfrac{C}{L \cdot C} \cdot s}{s^2 + \left(\dfrac{L}{L \cdot C \cdot R_2} + \dfrac{R_1 \cdot C}{L \cdot C}\right) \cdot s + \left(1 + \dfrac{R_1}{R_2}\right) \cdot \dfrac{1}{L \cdot C}}$$

$$= \frac{1}{L} \cdot \frac{s + \dfrac{1}{C \cdot R_2}}{s^2 + \left(\dfrac{1}{C \cdot R_2} + \dfrac{R_1}{L}\right) \cdot s + \left(1 + \dfrac{R_1}{R_2}\right) \cdot \dfrac{1}{L \cdot C}}$$

Characteristic eq.: $0 = s^2 + \left(\dfrac{1}{C \cdot R_2} + \dfrac{R_1}{L}\right) \cdot s + \left(1 + \dfrac{R_1}{R_2}\right) \cdot \dfrac{1}{L \cdot C}$

b) Find the solutions (numbers) of the characteristic equation:

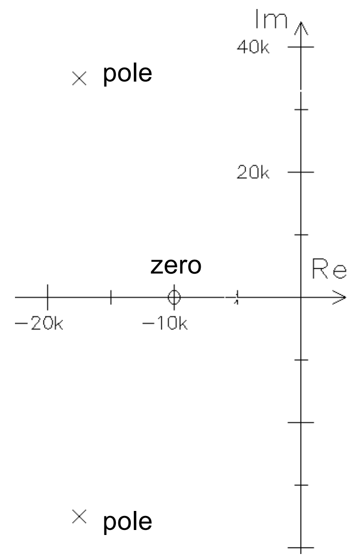
$$b := \frac{1}{C \cdot R_2} + \frac{R_1}{L} \qquad b = 3.5 \cdot 10^4 \cdot \mathrm{sec}^{-1} \qquad k := \left(1 + \frac{R_1}{R_2}\right) \cdot \frac{1}{L \cdot C} \qquad k = 1.5 \cdot 10^9 \cdot \mathrm{sec}^{-2}$$

Underdamped

$$s_1 := \frac{-b + \sqrt{b^2 - 4 \cdot k}}{2} \qquad s_1 = -1.75 \cdot 10^4 + 3.455 \cdot 10^4 j \cdot \frac{1}{\mathrm{sec}} \qquad \alpha := -\frac{b}{2} \qquad \alpha = -1.75 \cdot 10^4 \cdot \mathrm{sec}^{-1}$$

$$s_2 := \frac{-b - \sqrt{b^2 - 4 \cdot k}}{2} \qquad s_2 = -1.75 \cdot 10^4 - 3.455 \cdot 10^4 j \cdot \frac{1}{\mathrm{sec}} \qquad \omega := \frac{1}{2} \cdot \sqrt{4 \cdot k - b^2} \qquad \omega = 3.455 \cdot 10^4 \cdot \mathrm{sec}^{-1}$$

c) Plot the poles and zeroes of the transfer function.




The poles are the s's where the denominator is zero, $s_1$ & $s_2$

The zero is the s where the numerator is zero: $-\frac{1}{C \cdot R_2} = -1 \cdot 10^4 \cdot \mathrm{sec}^{-1}$

d) Initial and final conditions for $i_L(t)$ $\qquad i_{L0} := \frac{V_{in}}{R_1} \qquad i_{L0} = 60 \cdot \mathrm{mA}$

initial voltage across $R_1$: $\quad v_{R10} := i_{L0} \cdot R_1 \qquad v_{R10} = 12 \cdot V$

capacitor was shorted before switch opened, so... $\quad v_{C0} := 0 \cdot V$

initial voltage across the inductor: $\quad v_{L0} := V_{in} - v_{R10} - v_{C0} \qquad v_{L0} = 0 \cdot V$

$$\frac{d}{dt} i_L(0) = \frac{v_{L0}}{L} = 0 \cdot \frac{A}{\mathrm{sec}}$$

e) Find the full expression of $i_L(t)$.

Underdamped

$$v_X(t) = v_X(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

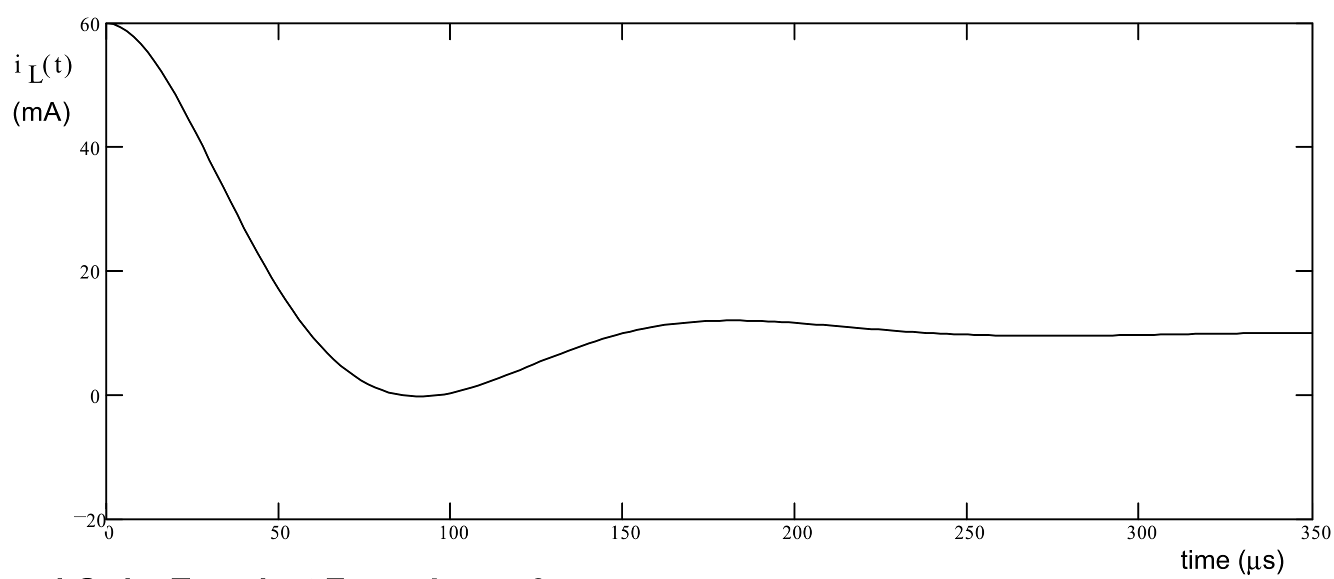
$$i_L(t) = i_L(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

$$i_L(\infty) = i_{Lfin} := \frac{V_{in}}{R_1 + R_2} \qquad i_{Lfin} = 10 \cdot \mathrm{mA}$$

$$i_L(0) = i_L(\infty) + B \quad \text{so..} \quad B = i_L(0) - i_L(\infty) \qquad B := i_{L0} - i_{Lfin} \qquad B = 50 \cdot \mathrm{mA}$$

$$\frac{d}{dt} i_L(0) = B \cdot \alpha + D \cdot \omega \quad \text{so..} \quad D = \frac{\frac{d}{dt} i_L(0) - B \cdot \alpha}{\omega} \qquad D := \frac{0 \cdot \frac{A}{\mathrm{sec}} - B \cdot \alpha}{\omega} \qquad D = 25.325 \cdot \mathrm{mA}$$

$$i_L(t) := 10 \cdot \mathrm{mA} + e^{-\frac{17500}{\mathrm{sec}} \cdot t} \cdot \left(50 \cdot \mathrm{mA} \cdot \cos\left(\frac{34550}{\mathrm{sec}} \cdot t\right) + 25.325 \cdot \mathrm{mA} \cdot \sin\left(\frac{34550}{\mathrm{sec}} \cdot t\right)\right)$$



f) Initial and final conditions for $v_C(t)$

$v_{C0} = 0 \cdot V$ $\qquad$ $i_{R20} := \dfrac{0 \cdot V}{R_2}$ $\qquad$ $i_{R20} = 0 \cdot mA$

$i_{L0} = 60 \cdot mA$ $\qquad$ $i_{C0} := i_{L0} - i_{R20}$ $\qquad$ $i_{C0} = 60 \cdot mA$

$$\frac{d}{dt} v_C(0) = \frac{i(0)}{C} = \frac{i_{C0}}{C} = 6 \cdot 10^5 \cdot \frac{V}{sec}$$

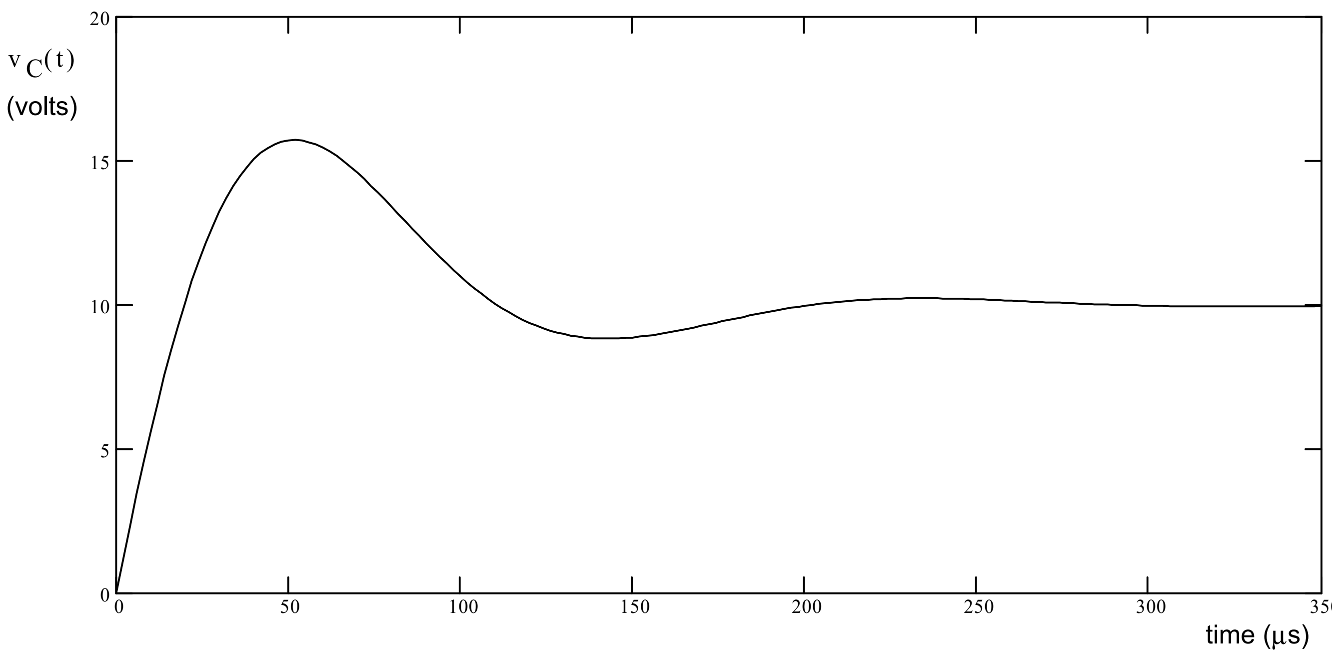
$V_{Cfin} := \dfrac{R_2}{R_1 + R_2} \cdot V_{in}$ $\qquad$ $V_{Cfin} = 10 \cdot V$

g) Find the full expression of $v_C(t)$.

$B := v_{C0} - V_{Cfin}$ $\qquad$ $B = -10 \cdot V$ $\qquad$ $D := \dfrac{6 \cdot 10^5 \cdot \frac{V}{sec} - B \cdot \alpha}{\omega}$ $\qquad$ $D = 12.301 \cdot V$

$$v_C(t) = v_C(\infty) + e^{\alpha t} \cdot (B \cdot \cos(\omega \cdot t) + D \cdot \sin(\omega \cdot t))$$

$$v_C(t) := 10 \cdot V + e^{-\frac{17500}{sec} \cdot t} \cdot \left( -10 \cdot V \cdot \cos\left(\frac{34550}{sec} \cdot t\right) + 12.301 \cdot V \cdot \sin\left(\frac{34550}{sec} \cdot t\right)\right)$$




**Second-Order Transient Examples, p.4**

h) What value of $R_1$ would make this system critically damped?

$$\left(\frac{1}{C\cdot R_2}+\frac{R_1}{L}\right)^2 = 4\cdot\left(1+\frac{R_1}{R_2}\right)\cdot\frac{1}{L\cdot C}$$

$$\frac{1}{C^2\cdot R_2^{\,2}}+\frac{2}{C\cdot R_2}\cdot\frac{R_1}{L}+\frac{R_1^{\,2}}{L^2} = \frac{4}{L\cdot C}+\frac{4}{C\cdot R_2}\cdot\frac{R_1}{L}$$

$$\frac{1}{L^2}\cdot R_1^{\,2}-\frac{2}{L\cdot C\cdot R_2}\cdot R_1+\left(\frac{1}{C^2\cdot R_2^{\,2}}-\frac{4}{L\cdot C}\right) = 0$$

$$R_1^{\,2}-\frac{2\cdot L}{C\cdot R_2}\cdot R_1+\left(\frac{L^2}{C^2\cdot R_2^{\,2}}-\frac{4\cdot L}{C}\right) = 0$$

Solve for $R_1$ with quadradic equation:

$$R_1 = \frac{\dfrac{2\cdot L}{C\cdot R_2}+\sqrt{\left(\dfrac{2\cdot L}{C\cdot R_2}\right)^2-4\cdot\left(\dfrac{L^2}{C^2\cdot R_2^{\,2}}-\dfrac{4\cdot L}{C}\right)}}{2} = 645.685\,\Omega$$

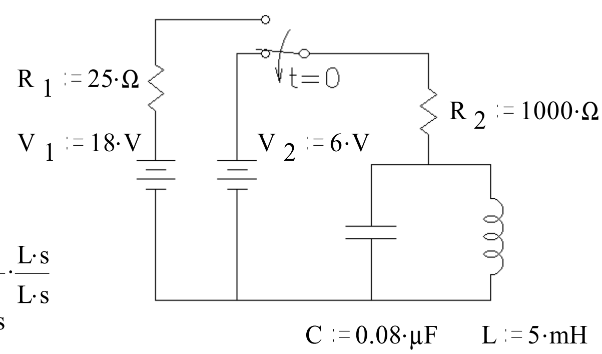
this must be the only solution

Quadradic equation can be reduced to:

$$\frac{L}{C\cdot R_2}-\frac{4}{2}\cdot\sqrt{\frac{L}{C}} = -485.685\,\Omega$$ this solution can't be

## Ex. 2 with bigger $R_2$

a) Find the characteristic equation of the circuit shown (after the switch moves to the lower position at t = 0).




$$\frac{V_C(s)}{V_{in}(s)} = H(s) = \frac{\dfrac{1}{\dfrac{1}{L\cdot s}+C\cdot s}}{\dfrac{1}{\dfrac{1}{L\cdot s}+C\cdot s}+R_2}\cdot\frac{\left(\dfrac{1}{L\cdot s}+C\cdot s\right)}{\left(\dfrac{1}{L\cdot s}+C\cdot s\right)} = \frac{1}{1+\dfrac{R_2}{L\cdot s}+R_2\cdot C\cdot s}\cdot\frac{L\cdot s}{L\cdot s}$$

$$= \frac{L\cdot s}{L\cdot s+R_2+R_2\cdot C\cdot L\cdot s^2}\cdot\frac{\dfrac{1}{R_2\cdot C\cdot L}}{\dfrac{1}{R_2\cdot C\cdot L}} = \frac{\dfrac{1}{R_2\cdot C}\cdot s}{\dfrac{1}{R_2\cdot C}\cdot s+\dfrac{1}{L\cdot C}+s^2}$$

Characteristic equation:

$$0 = s^2+\frac{1}{R_2\cdot C}\cdot s+\frac{1}{L\cdot C}$$

b) Find the solutions of the characteristic equation.

$$\frac{1}{R_2\cdot C} = 1.25\cdot 10^4\ \mathrm{sec}^{-1} \qquad \frac{4}{L\cdot C} = 1\cdot 10^{10}\ \mathrm{sec}^{-2}$$

$$s_1 := \frac{-\dfrac{1}{R_2\cdot C}+\sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2-\dfrac{4}{L\cdot C}}}{2} \qquad s_2 := \frac{-\dfrac{1}{R_2\cdot C}-\sqrt{\left(\dfrac{1}{R_2\cdot C}\right)^2-\dfrac{4}{L\cdot C}}}{2}$$

$$s_1 = -6.25\cdot 10^3+4.961\cdot 10^4 j\ \frac{1}{\mathrm{sec}} \qquad s_2 = -6.25\cdot 10^3-4.961\cdot 10^4 j\ \frac{1}{\mathrm{sec}}$$

$s_1$ & $s_2$ complex, so underdamped

$$\alpha := \mathrm{Re}(s_1) \qquad \alpha = -6.25\cdot 10^3\ \mathrm{sec}^{-1} \qquad \omega := \mathrm{Im}(s_1) \qquad \omega = 4.961\cdot 10^4\ \mathrm{sec}^{-1}$$

c) Find initial and final conditions for $v_C(t)$

before switch is moved: $v_L(\infty) = 0 = v_L(0-) = v_C(0-) = 0 = v_C(0+)$

$$\frac{d}{dt}v_C(0) = \frac{i_C(0)}{C}$$

before switch is moved: $i_{L0} := \dfrac{V_1 - \overset{v_C(0)}{0\cdot V}}{R_1 + R_2}$ so... $i_{L0} = 17.561 \cdot mA$

after switch is moved: $i_{R0} := \dfrac{6\cdot V - 0\cdot V}{R_2}$ $i_{R0} = 6 \cdot mA$

not all of the inductor's current comes through the resistor anymore, so the difference must come from the capacitor

$i_{C0} := i_{R0} - i_{L0}$ $\quad i_{C0} = -11.561 \cdot mA$ $\quad \dfrac{d}{dt}v_C(0) = \dfrac{i_{C0}}{C} = -1.445\cdot 10^5 \cdot \dfrac{V}{sec}$

d) Find the full expression of $v_C(t)$.

<u>Underd</u>amped $b^2 - 4\cdot k < 0$ $\quad s_1 = \alpha + j\cdot\omega$ $\quad s_2 = \alpha - j\cdot\omega$ $\quad \alpha$ is negative

$$v_X(t) = v_X(\infty) + e^{\alpha t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$$

$v_X(0) = v_X(\infty) + B$

so.. $B = v_X(0) - v_X(\infty)$

$B := 0\cdot V - 0\cdot V$ $\quad B = 0 \cdot V$

$\dfrac{d}{dt}v_X(0) = B\cdot\alpha + D\cdot\omega$ so.. $D = \dfrac{\frac{d}{dt}v_X(0) - B\cdot\alpha}{\omega}$

$$D := \frac{-1.445\cdot 10^5\cdot\frac{V}{sec} - B\cdot\alpha}{\omega}$$

$D = -2.913 \cdot V$

$$v_C(t) = v_C(\infty) + e^{\alpha t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t))$$

$$v_C(t) := 0\cdot V + e^{\alpha t}\cdot(B\cdot\cos(\omega\cdot t) + D\cdot\sin(\omega\cdot t)) = -2.913\cdot V\cdot e^{-\frac{6250}{sec}\cdot t}\cdot\sin\left(\frac{49610}{sec}\cdot t\right)$$