# Lecture 12 Introduction to AC Phasors

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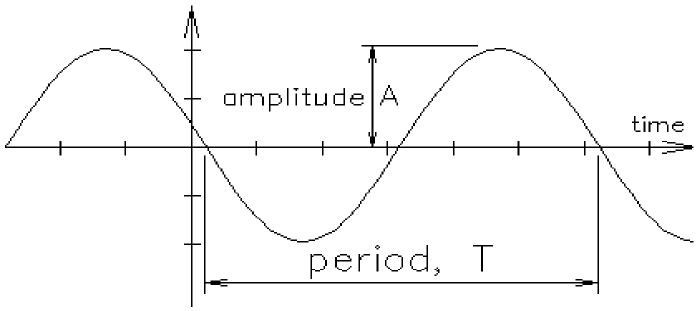
rev,

**Phasor analysis with impedances**, For steady-state sinusoidal response ONLY

## Sinusoidal AC




T = Period = repeat time

f = frequency, cycles / second $\quad f = \frac{1}{T} = \frac{\omega}{2\cdot\pi}$

ω = radian frequency, radians/sec $\quad \omega = 2\cdot\pi\cdot f$

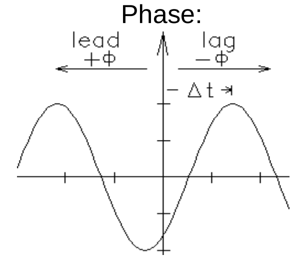
A = amplitude

Phase: $\quad \phi = -\frac{\Delta t}{T}\cdot 360\cdot deg$

or: $\quad \phi = -\frac{\Delta t}{T}\cdot 2\cdot\pi\cdot rad$

$y(t) = A\cdot\cos(\omega\cdot t + \theta)$

Phase:




## Phasor analysis

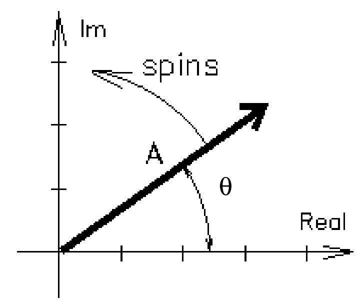
The math is all based on the Euler's equation

**Euler's equation** $\quad e^{j\cdot\alpha} = \cos(\alpha) + j\cdot\sin(\alpha)$

OR:

$$\cos(\alpha) = \frac{e^{j\cdot\alpha} + e^{-j\cdot\alpha}}{2}$$

$$\sin(\theta) = \frac{e^{j\cdot\alpha} - e^{-j\cdot\alpha}}{2\cdot j}$$




$$e^{j\cdot(\omega\cdot t+\theta)} = \cos(\omega\cdot t + \theta) + j\cdot\sin(\omega\cdot t + \theta)$$

$$Re\left[e^{j\cdot(\omega\cdot t+\theta)}\right] = \cos(\omega\cdot t + \theta)$$

If we freeze this at time t=0, then we can represent $\cos(\omega\cdot t + \theta)$ by $e^{j\cdot\theta}$ That's the phasor

<u>Phasor</u>

voltage: $\quad v(t) = V_p\cdot\cos(\omega\cdot t + \phi) \qquad \mathbf{V}(\omega) = V_p\cdot e^{j\cdot\phi}$

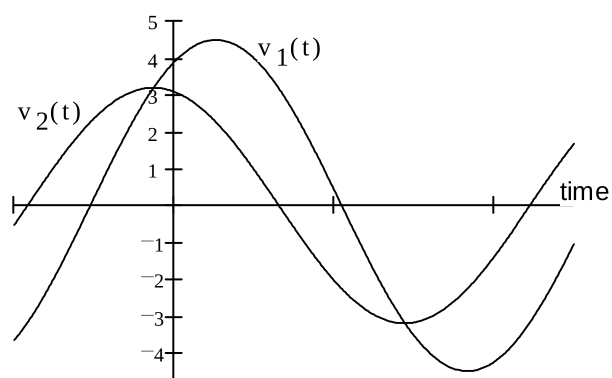
current: $\quad i(t) = I_p\cdot\cos(\omega\cdot t + \phi) \qquad \mathbf{I}(\omega) = I_p\cdot e^{j\cdot\phi}$

Phasors are drawn on a complex plane.

Phasors are used for adding and subtracting sinusoidal waveforms.

Ex1. Add the sinusoidal voltages $\quad v_1(t) = 4.5\cdot V\cdot\cos(\omega\cdot t - 30\cdot deg)$

and $\quad v_2(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 15\cdot deg)$




using phasor notation, draw a phasor diagram of the three phasors, then convert back to time domain form.

$v_1(t) = 4.5\cdot V\cdot\cos(\omega\cdot t - 30\cdot deg)$

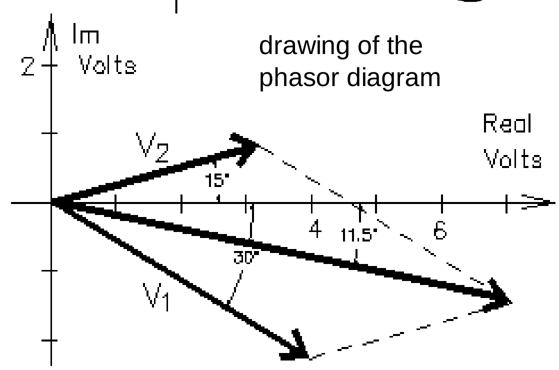
$\mathbf{V}_1(\omega) = 4.5V\ \underline{/-30^o}$ or: $\mathbf{V}_1(\omega) = 4.5\cdot V\cdot e^{-j\cdot 30\cdot deg}$

and

$v_2(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 15\cdot deg)$

$\mathbf{V}_2(\omega) = 3.2V\ \underline{/15^o}$ or: $\mathbf{V}_2(\omega) = 3.2\cdot V\cdot e^{j\cdot 15\cdot deg}$

drawing of the phasor diagram




I'm going to drop the (ω) notation from the phasor notation, it gets cumbersome, but remember that phasors are in the frequency domain..

$\mathbf{V}_1 = 4.5V\ \underline{/-30^o}$ or: $\mathbf{V}_1 := 4.5\cdot V\cdot e^{-j\cdot 30\cdot deg}$

$\mathbf{V}_2 = 3.2V\ \underline{/15^o}$ or: $\mathbf{V}_2 := 3.2\cdot V\cdot e^{j\cdot 15\cdot deg}$

Add like vectors, first change to the rectangular form

| | | | |
|---|---|---|---|
| $\mathbf{V_1} = 4.5V\ \underline{/-30^o}$ | $4.5\cdot V\cdot\cos(-30\cdot deg) = 3.897\cdot V$ | $4.5\cdot V\cdot\sin(-30\cdot deg) = -2.25\cdot V$ | $\mathbf{V_1} = 3.897 - 2.25j\ \cdot V$ \ |
| | | | } add |
| $\mathbf{V_2} = 3.2V\ \underline{/15^o}$ | $3.2\cdot V\cdot\cos(15\cdot deg) = 3.091\cdot V$ | $3.2\cdot V\cdot\sin(15\cdot deg) = 0.828\cdot V$ | $\mathbf{V_2} = 3.091 + 0.828j\ \cdot V$ / |
| | Add real parts: | $3.897 + 3.091 = 6.988$ | $\mathbf{V_3} := \mathbf{V_1} + \mathbf{V_2}$ |
| | Add imaginary parts: | $-2.25 + 0.828 = -1.422$ | $\mathbf{V_3} = 6.988 - 1.422j\ \cdot V$ sum |

Change $\mathbf{V_3}$ back to polar coordinates:

$$\sqrt{6.988^2 + 1.422^2} = 7.131 \qquad \operatorname{atan}\left(\frac{-1.422}{6.988}\right) = -11.502\cdot deg$$

OR, in Mathcad notation (you'll see these in future solutions):

$$\left|\mathbf{V_3}\right| = 7.131\cdot V \qquad \arg\left(\mathbf{V_3}\right) = -11.5\cdot deg$$

Change $\mathbf{V_3}$ back to the time domain:

$$v_3(t) = v_1(t) + v_2(t) = 7.13\cdot\cos(\omega\cdot t - 11.5\cdot deg)\cdot V$$

Ex 2. Two sinusoidal voltages: $v_1(t) = 5\cdot V\cdot\cos(\omega\cdot t + 36.87\cdot deg)$ and $v_2(t) = 3.162\cdot V\cdot\cos(\omega\cdot t - 18.44\cdot deg)$

a) using phasor notation, find $v_3 = v_1 - v_2$

$\mathbf{V_1} := 5\cdot V\cdot e^{j\cdot(36.87\cdot deg)}$ $\qquad 5\cdot V\cdot\cos(36.87\cdot deg) = 4\cdot V$

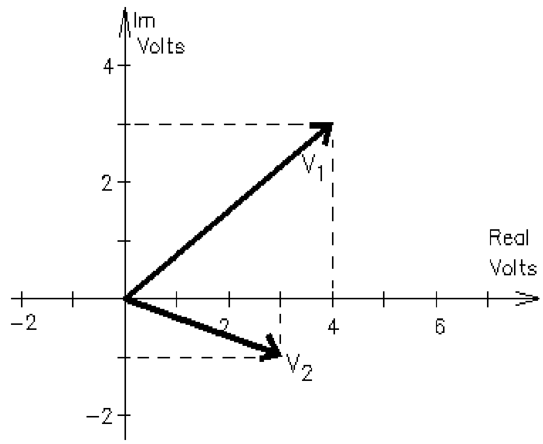
$\qquad 5\cdot V\cdot\sin(36.87\cdot deg) = 3\cdot V$

$\mathbf{V_1} = 4 + 3j\ \cdot V$

$\mathbf{V_2} := 3.162\cdot V\cdot e^{j\cdot(-18.44\cdot deg)}$ $\qquad 3.162\cdot V\cdot\cos(-18.44\cdot deg) = 3\cdot V$

$\qquad 3.162\cdot V\cdot\sin(-18.44\cdot deg) = -1\cdot V$

$\mathbf{V_2} = 3 - j\ \cdot V$




Subtract real parts: $4\cdot V - 3\cdot V = 1\cdot V$

Subtract imaginary parts: $3\cdot V - -1\cdot V = 4\cdot V$

$\mathbf{V_3} := \mathbf{V_1} - \mathbf{V_2}$ $\qquad \mathbf{V_3} = 1 + 4j\ \cdot V$

Magnitude: $\sqrt{(1\cdot V)^2 + (4\cdot V)^2} = 4.123\cdot V$

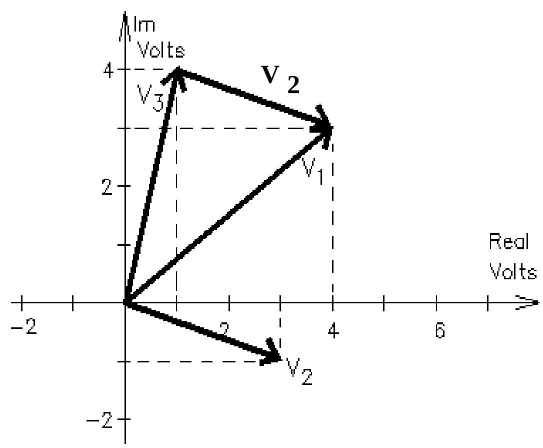
Angle: $\operatorname{atan}\left(\frac{4\cdot V}{1\cdot V}\right) = 75.96\cdot deg$

OR:

$\left|\mathbf{V_3}\right| = 4.123\cdot V$

$\arg\left(\mathbf{V_3}\right) = 75.96\cdot deg$

So: $v_3(t) = v_1(t) - v_2(t) = 4.123\cdot V\cdot\cos(\omega\cdot t + 75.96\cdot deg)\cdot V$




**What about Capacitors and Inductors?**

Capacitors and Inductors in AC circuits cause 90º phase shifts between voltages and currents because they integrate and differentiate. But... integration and differentiation is a piece-of-cake in phasors.

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**Calculus**

$$\frac{d}{dt}\left[A\cdot e^{j\cdot(\omega\cdot t+\theta)}\right] = j\cdot\omega\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta)} = \omega\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta+90\cdot deg)} = \omega\cdot A\cdot e^{j\cdot(\theta+90\cdot deg)}$$

Drop the ωt (t=0) to get:

$$\int A\cdot e^{j\cdot(\omega\cdot t+\theta)}\,dt = \frac{1}{j\cdot\omega}\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta)} = \frac{1}{\omega}\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta-90\cdot deg)} = \frac{1}{\omega}\cdot A\cdot e^{j\cdot(\theta-90\cdot deg)}$$

# Impedance (like resistance)

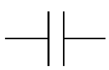
**AC impedance**

Inductor

$$v_L = L\cdot\frac{d}{dt}i_L = L\cdot\frac{d}{dt}I_p\cdot e^{j\cdot(\omega\cdot t+\theta)} = j\cdot\omega\cdot L\cdot\left[I_p\cdot e^{j\cdot(\omega\cdot t+\theta)}\right]$$

\ in phasor notation ----> $\mathbf{V_L}(\omega) = j\cdot\omega\cdot L\cdot\mathbf{I}(\omega)$ $\qquad \mathbf{Z_L} = j\cdot\omega\cdot L$

Capacitor



$$i_C = C\cdot\frac{d}{dt}v_C = C\cdot\frac{d}{dt}V_p\cdot e^{j\cdot(\omega\cdot t+\theta)} = j\cdot\omega\cdot C\cdot\left[V_p\cdot e^{j\cdot(\omega\cdot t+\theta)}\right]$$

\ in phasor notation ----> $\mathbf{I_C}(\omega) = j\cdot\omega\cdot C\cdot\mathbf{V}(\omega)$

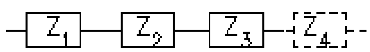
$$\mathbf{V_C}(\omega) = \frac{1}{j\cdot\omega\cdot C}\cdot\mathbf{I}(\omega) \qquad \mathbf{Z_C} = \frac{1}{j\cdot\omega\cdot C} = \frac{-j}{\omega\cdot C}$$

Resistor

$$v_R = i_R\cdot R \qquad \mathbf{V_R}(\omega) = R\cdot\mathbf{I}(\omega) \qquad \mathbf{Z_R} = R$$

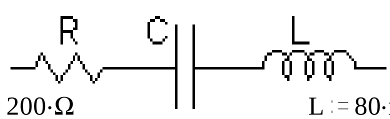
You can use impedances just like resistances as long as you deal with the complex arithmetic.
ALL the DC circuit analysis techniques will work with AC.

**series:**




$$\mathbf{Z_{eq}} = \mathbf{Z_1} + \mathbf{Z_2} + \mathbf{Z_3} + \ldots$$

Example:




$f := 500\cdot Hz$ $\qquad R := 200\cdot\Omega$ $\qquad C := 0.6\cdot\mu F$ $\qquad L := 80\cdot mH$

$\omega := 2\cdot\pi\cdot f = \omega = 3141.6\cdot\frac{rad}{sec}$ $\qquad j\cdot\omega\cdot L = 251.327j\cdot\Omega$

$$\frac{1}{j\cdot\omega\cdot C} = -530.516j\cdot\Omega$$

$$\mathbf{Z_{eq}} := R + \frac{1}{j\cdot\omega\cdot C} + j\cdot\omega\cdot L = 200\cdot\Omega - 530.5\cdot j\cdot\Omega + 251.3\cdot j\cdot\Omega = 200 - 279.2j\ \cdot\Omega$$ rectangular form

$$\sqrt{(200\cdot\Omega)^2 + (279.2\cdot\Omega)^2} = 343.4\cdot\Omega \qquad \operatorname{atan}\left(\frac{-279.2\cdot\Omega}{200\cdot\Omega}\right) = -54.38\cdot deg$$

$\mathbf{Z_{eq}} = 343.4\Omega\ \underline{/-54.4}^{o}$ polar form

If: $\mathbf{V} := 12\cdot V\cdot e^{j\cdot 0\cdot deg}$ $\qquad \mathbf{I} := \frac{\mathbf{V}}{\mathbf{Z_{eq}}} = \frac{12\cdot V}{343.4\cdot\Omega} = 34.945\cdot mA$ $\quad \underline{/}\ \ 0 - -54.4 = 54.4$ deg

$\mathbf{I} = 34.95mA\ \underline{/54.4}^{o} = \mathbf{I} = 20.348 + 28.405j\ \cdot mA$

**Voltage divider:**

$$\mathbf{V_{Zn}} = \mathbf{V_{total}}\cdot\frac{\mathbf{Z_n}}{\mathbf{Z_1} + \mathbf{Z_2} + \mathbf{Z_3} + \ldots}$$

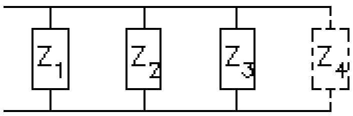
Note: $\frac{1}{j} = -j = 1\ \underline{/-90}^{o}$

Eg: $\mathbf{V_C} := \mathbf{V}\cdot\frac{\frac{1}{j\cdot\omega\cdot C}}{\mathbf{Z_{eq}}} = 12\cdot V\cdot e^{j\cdot 0\cdot deg}\cdot\frac{530.516\cdot e^{-j\cdot 90\cdot deg}\cdot\Omega}{343.4\cdot e^{-j\cdot 54.38\cdot deg}\cdot\Omega}$

$12\cdot V\cdot\frac{530.516\cdot\Omega}{343.4\cdot\Omega} = 18.539\cdot V$ $\quad \underline{/}\ \ 0 + -90 - -54.4 = -35.6$ deg

$\mathbf{V_C} = 18.54V\ \underline{/-35.6}^{o} = \mathbf{V_C} = 15.069 - 10.795j\ \cdot V$

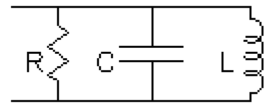
**parallel:**




$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots}$$

Example:

$f := 500 \cdot Hz \qquad \omega := 2 \cdot \pi \cdot f \qquad \omega = 3141.6 \cdot \frac{rad}{sec}$




$L := 80 \cdot mH$

$R := 200 \cdot \Omega \qquad C := 0.6 \cdot \mu F$

$j \cdot \omega \cdot L = 251.327j \cdot \Omega$

$\frac{1}{\omega \cdot L} = 3.979 \cdot 10^{-3} \cdot \frac{1}{\Omega}$

$\frac{1}{j \cdot \omega \cdot C} = -530.516j \cdot \Omega$

$\omega \cdot C = 1.885 \cdot 10^{-3} \cdot \frac{1}{\Omega}$

$$\mathbf{Z}_{eq} := \frac{1}{\frac{1}{R} + \frac{1}{\left(\frac{1}{j \cdot \omega \cdot C}\right)} + \frac{1}{j \cdot \omega \cdot L}} = \frac{1}{\frac{1}{R} + j \cdot \omega \cdot C - \frac{j}{\omega \cdot L}} = \frac{1}{\frac{1}{200 \cdot \Omega} + 1.885 \cdot 10^{-3} \cdot j \cdot \frac{1}{\Omega} - 3.979 \cdot 10^{-3} \cdot j \cdot \frac{1}{\Omega}} = \frac{1}{\left(5 \cdot 10^{-3} - 2.094 \cdot 10^{-3} \cdot j\right) \cdot \frac{1}{\Omega}}$$

$$= \frac{1}{\left(5 \cdot 10^{-3} - 2.094 \cdot 10^{-3} \cdot j\right) \cdot \frac{1}{\Omega}} \cdot \frac{5 \cdot 10^{-3} + 2.094 \cdot 10^{-3} \cdot j}{\left(5 \cdot 10^{-3} + 2.094 \cdot 10^{-3} \cdot j\right)} = 170.156 + 71.261j \cdot \Omega$$

$2.93848 \cdot 10^{-5}$

If you want the answer in polar form, it's easier to convert the denominator first.

$$\sqrt{\left(5 \cdot 10^{-3} \cdot \frac{1}{\Omega}\right)^2 + \left(2.094 \cdot 10^{-3} \cdot \frac{1}{\Omega}\right)^2} = 5.4 \cdot 10^{-3} \cdot \frac{1}{\Omega} \qquad \operatorname{atan}\left(\frac{-2.094 \cdot 10^{-3} \cdot \Omega}{5 \cdot 10^{-3} \cdot \Omega}\right) = -22.72 \cdot deg$$

$$\frac{1}{5.4 \cdot 10^{-3} \cdot \frac{1}{\Omega}} = 185.185 \cdot \Omega \ \angle \ 0 - -22.7 = 22.7 \ deg$$

$\mathbf{Z}_{eq} = 185.2 \ \angle 22.7^o$

If: $\mathbf{V} := 12 \cdot V \cdot e^{j \cdot 0 \cdot deg}$

$$\mathbf{I} := \frac{\mathbf{V}}{\mathbf{Z}_{eq}} = \frac{12 \cdot V}{185.2 \cdot \Omega} = 64.795 \cdot mA \ \angle \ 0 - 22.7 = -22.7 \ deg$$

$\mathbf{I} = 60 - 25.127j \cdot mA$

**Current divider:**

$$\mathbf{I}_{Zn} = \mathbf{I}_{total} \cdot \frac{\frac{1}{\mathbf{Z}_n}}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots}$$

Eg: $$\mathbf{I}_L := \mathbf{I} \cdot \frac{\frac{1}{j \cdot \omega \cdot L}}{\frac{1}{R} + j \cdot \omega \cdot C + \frac{1}{j \cdot \omega \cdot L}} = \mathbf{I} \cdot \frac{\left(\frac{1}{j \cdot \omega \cdot L}\right)}{\left(\frac{1}{\mathbf{Z}_{eq}}\right)} = \mathbf{I} \cdot \frac{\mathbf{Z}_{eq}}{j \cdot \omega \cdot L}$$

$$= 64.795 \cdot mA \cdot e^{-j \cdot 22.7 \cdot deg} \cdot \frac{185.2 \cdot e^{j \cdot 22.7 \cdot deg} \cdot \Omega}{251.327 \cdot e^{j \cdot 90 \cdot deg} \cdot \Omega}$$

$$= 64.795 \cdot mA \cdot \frac{185.2 \cdot \Omega}{251.327 \cdot \Omega} = 47.747 \cdot mA \ \angle \ -22.7 + 22.7 - 90 = -90 \ deg$$

$\mathbf{I}_L = -47.746j \cdot mA$

Duh... $\frac{\mathbf{V}}{j \cdot \omega \cdot L} = -47.746j \cdot mA$