ECE 2210 Lecture 7 notes Nodal Analysis

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9/10/08,
1/7/13

General Network Analysis

In many cases you have multiple unknowns in a circuit, say the voltages across multiple resistors. Network analysis is a systematic way to generate multiple equations which can be solved to find the multiple unknowns. These equations are based on basic Kirchoff's and Ohm's laws.

Loop or Mesh Analysis You may have used these methods in previous classes, particularly in Physics. The best thing to do now is to forget all that. Loop analysis is rarely the easiest way to analyze a circuit and is inherently confusing. Hopefully I've brought you to a stage where you have some intuitive feeling for how currents flow in circuits. I don't want to ruin that now by screwing around with loop currents that don't really exist.

Nodal analysis This is a much better method. It's just as powerful, usually easier, and helps you develop your intuitive feeling for how circuits work.

Nodal Analysis

Node = all points connected by wire, all at same voltage (potential)

Ground: One node in the circuit which will be our reference node. Ground, by definition, will be the zero voltage node. All other node voltages will be referenced to ground and may be positive or negative. Think of gage pressure in a fluid system. In that case atmospheric pressure is considered zero. If there is no ground in the circuit, define one for yourself. Try to chose a node which is hooked to one side of a voltage source.

Nodal Voltage: The voltage of a node referenced to ground. The objective of nodal analysis is to find all the nodal voltages. If you know the voltage at a node then it's a "known" node. Ground is a known node (duh, it's zero). If one end of a known voltage source hooked to ground, then the node on the other end is also known (another duh).

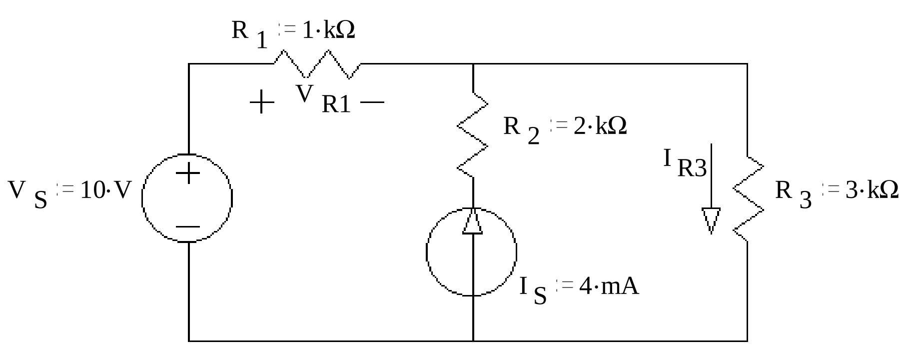
Method: Label all the unknown nodes as; "a", "b", "c", etc. Then the unknown nodal voltages become; V_a, V_b, V_c, etc. Write a KCL equation for each unknown node, defining currents as necessary. Replace each unknown current with an Ohm's law relationship using the nodal voltages. Now you have just as many equations as unknowns. Solve.

Nodal Analysis Steps

1) If the circuit doesn't already have a ground, label one node as ground (zero voltage).
If the ground can be defined as one side of a voltage source, that will make the following steps easier.
Label the remaining node, either with known voltages or with letters, a, b, ...
2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...
3) Write Kirchoff's current equations for each unknown node.
4) Replace the currents in your **KCL** equations with expressions like this. $\frac{V_a - V_b}{R_1}$ Ohm's law relationship using the nodal voltages.
5) Solve the multiple equations for the multiple unknown voltages.

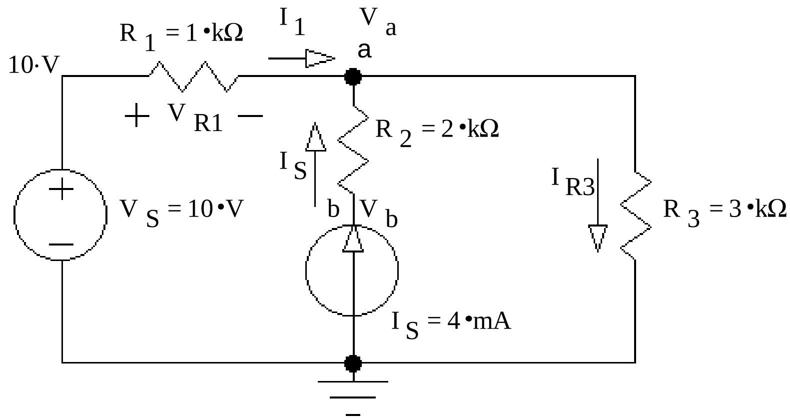
Nodal Analysis Examples

Ex 1 Use nodal analysis to find the voltage across R_1 (V_{R1}).



1) See next page
Label one node as ground (zero voltage). By choosing the negative side of a voltage source as ground, the upper-left node is known (10V). Label the remaining nodes, either with known voltages or with letters, a, b, ...

2) Label unknown node voltages as V_a, V_b, ...
and label the current in each resistor as I_1, I_2, ...



3) Write Kirchoff's current equations for node a.

$$I_1 + I_S = I_{R3}$$

4) Replace the currents in the **KCL** equations with Ohm's law relationships.

$$\frac{V_S - V_a}{R_1} + I_S = \frac{V_a - 0}{R_3}$$

$$\frac{V_S}{R_1} - \frac{V_a}{R_1} + I_S = \frac{V_a}{R_3}$$

5) Solve:

$$\frac{V_S}{R_1} + I_S = \frac{V_a}{R_3} + \frac{V_a}{R_1}$$

$$\frac{V_S}{R_1} + I_S = V_a \cdot \left(\frac{1}{R_1} + \frac{1}{R_3}\right)$$

$$V_a := \frac{\frac{V_S}{R_1} + I_S}{\left(\frac{1}{R_1} + \frac{1}{R_3}\right)} \qquad V_a = 10.5 \cdot V$$

Usually it's easier to put in the numbers at this point

$$\frac{10 \cdot V}{1 \cdot k\Omega} + 4 \cdot mA = \frac{V_a}{3 \cdot k\Omega} + \frac{V_a}{1 \cdot k\Omega}$$

Multiply both sides by a value that will clear the denominators.

$$3 \cdot k\Omega \cdot \left(\frac{10 \cdot V}{1 \cdot k\Omega} + 4 \cdot mA\right) = \left(\frac{V_a}{3 \cdot k\Omega} + \frac{V_a}{1 \cdot k\Omega}\right) \cdot 3 \cdot k\Omega$$

$$30 \cdot V + 3 \cdot k\Omega \cdot 4 \cdot mA = V_a + 3 \cdot V_a$$

$$30 \cdot V + 12 \cdot V = 4 \cdot V_a$$

$$V_a = \frac{42 \cdot V}{4} = 10.5 \cdot V$$

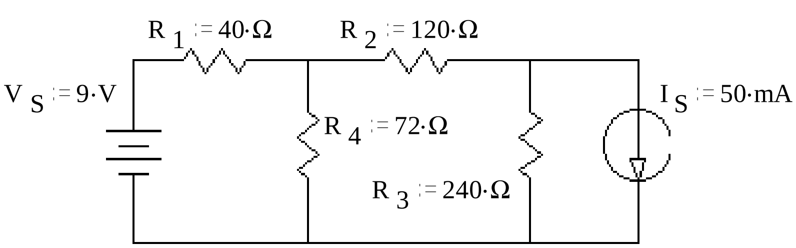
Either way, you still have to find V_{R1} from V_a.

$$V_{R1} := V_S - V_a \qquad V_{R1} = -0.5 \cdot V$$

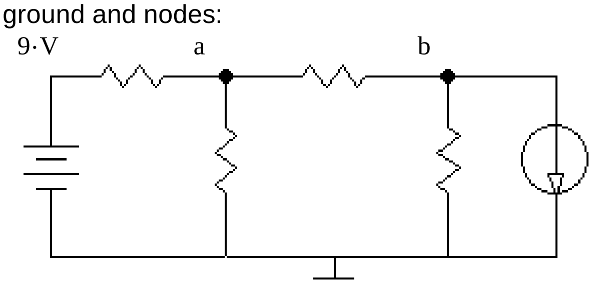
V_b doesn't matter in this case

b) Find the current through R_3 (I_{R3}). $\quad I_{R3} = \frac{V_a}{R_3} = 3.5 \cdot mA$

Ex 2 Same circuit used in Thévenin example, where R_4 was R_L.

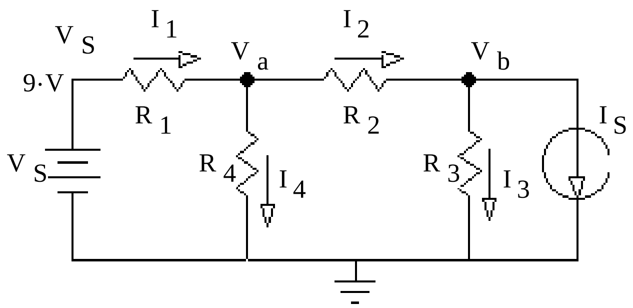


1) Define ground and nodes:



2 unknown nodes ---> will need 2 equations

2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...
It doesn't matter if these currents are in the correct directions.



3) Write Kirchoff's current equations for each unknown node.

node a $\quad I_1 = I_2 + I_4$

node b $\quad I_2 = I_3 + I_S$

4) Replace the currents in your **KCL** equations with expressions like this. $\dfrac{V_a - V_b}{R_1}$

node a $\quad I_1 = I_2 + I_4$

$$\frac{V_S - V_a}{R_1} = \frac{V_a - V_b}{R_2} + \frac{V_a - 0\cdot V}{R_4}$$

node b $\quad I_2 = I_3 + I_S$

$$\frac{V_a - V_b}{R_2} = \frac{V_b - 0\cdot V}{R_3} + I_S$$

Now you have just as many equations as unknowns.

5) Solve the multiple equations for the multiple unknown voltages. Solve by any method you like:

$$\frac{V_S}{R_1} - \frac{V_a}{R_1} = \frac{V_a}{R_2} - \frac{V_b}{R_2} + \frac{V_a}{R_4}$$

$$\frac{V_a}{R_2} - \frac{V_b}{R_2} = \frac{V_b}{R_3} + I_S \qquad V_b = \frac{\dfrac{V_a}{R_2} - I_S}{\dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

$$\frac{V_S}{R_1} - \frac{V_a}{R_1} = \frac{V_a}{R_2} - \frac{\dfrac{V_a}{R_2} - I_S}{R_2\cdot\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)} + \frac{V_a}{R_4}$$

$$V_a := \frac{\left[\dfrac{V_S}{R_1} - \dfrac{1}{\left[R_2\cdot\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)\right]}\cdot I_S\right]}{\left[\dfrac{1}{R_1} + \dfrac{1}{R_2} - \dfrac{1}{R_2^{\,2}\cdot\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)} + \dfrac{1}{R_4}\right]} \qquad V_a = 4.6\,\bullet V$$

$$V_b := \frac{\dfrac{V_a}{R_2} - I_S}{\dfrac{1}{R_2} + \dfrac{1}{R_3}} \qquad V_b = -0.933\,\bullet V$$

Or, with numbers

node a

$$360\cdot\Omega\cdot\left(\frac{9\cdot V - V_a}{40\cdot\Omega}\right) = \left(\frac{V_a - V_b}{120\cdot\Omega} + \frac{V_a}{72\cdot\Omega}\right)\cdot 360\cdot\Omega$$

$$81\cdot V - 9\cdot V_a = 3\cdot V_a - 3\cdot V_b + 5\cdot V_a$$

node b

$$240\cdot\Omega\cdot\frac{V_a - V_b}{120\cdot\Omega} = \left(\frac{V_b - 0\cdot V}{240\cdot\Omega} + 50\cdot mA\right)\cdot 240\cdot\Omega$$

$$2\cdot V_a - 2\cdot V_b = V_b + 48\cdot mA\cdot 240\cdot\Omega$$

$$V_a = \frac{2\cdot V_b + V_b + 12\cdot V}{2} = 1.5\cdot V_b + 6\cdot V$$

<-- substitute for V_a

$$81\cdot V - 9\cdot\left(1.5\cdot V_b + 6\cdot V\right) = 3\cdot\left(1.5\cdot V_b + 6\cdot V\right) - 3\cdot V_b + 5\cdot\left(1.5\cdot V_b + 6\cdot V\right)$$

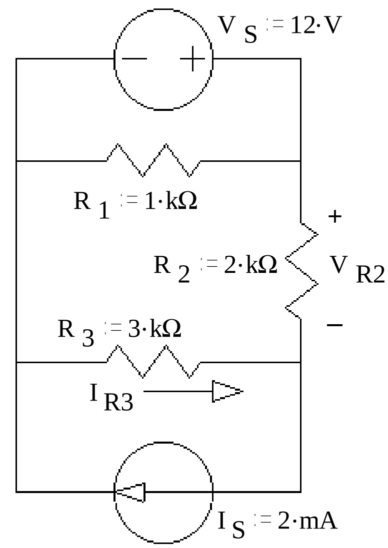
$$81\cdot V - 13.5\cdot V_b - 54\cdot V = 4.5\cdot V_b + 18\cdot V - 3\cdot V_b + 7.5\cdot V_b + 30\cdot V$$

$$81\cdot V - 54\cdot V - 18\cdot V - 30\cdot V = -21\,\bullet V = 4.5\cdot V_b - 3\cdot V_b + 7.5\cdot V_b + 13.5\cdot V_b = 22.5\cdot V_b$$

$$V_b = \frac{-21\cdot V}{22.5} = -0.933\,\bullet V \qquad V_a = 1.5\cdot V_b + 6\cdot V = 4.6\,\bullet V$$

Same as V_L of Ex 4 of Thévenin examples:

Ex 3 Like Superposition Ex.2

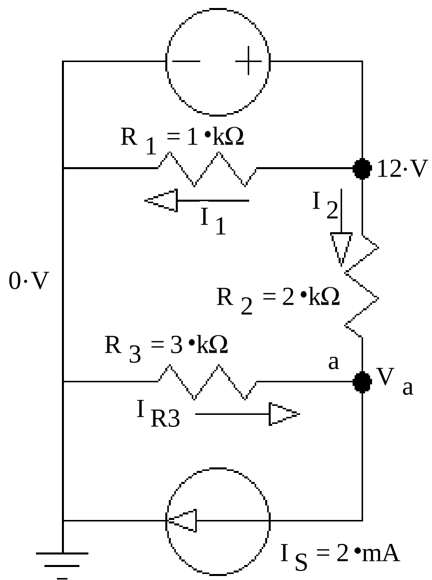


a) Use nodal analysis to find the voltage across R_2 (V_{R2}).

You **MUST** show all the steps of nodal analysis work to get credit, including drawing appropriate symbols and labels on the circuit shown.

1) Define ground and nodes:

2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...



3) Write Kirchoff's current equations for each unknown node.

node a: $I_2 + I_{R3} = I_S$

4) Replace the currents in the **KCL** equations with Ohm's law relationships.

$$\frac{V_S - V_a}{R_2} + \frac{0 - V_a}{R_3} = I_S$$

5) Solve the equation for the unknown voltage.

$$\frac{V_S}{R_2} - \frac{V_a}{R_2} - \frac{V_a}{R_3} = I_S$$

$$\frac{V_S}{R_2} = \frac{V_a}{R_2} + \frac{V_a}{R_3} + I_S$$

$$\frac{V_S}{R_2} - I_S = V_a \cdot \left(\frac{1}{R_2} + \frac{1}{R_3} \right)$$

$$V_a := \frac{\frac{V_S}{R_2} - I_S}{\left(\frac{1}{R_2} + \frac{1}{R_3} \right)} \qquad V_a = 4.8 \cdot V$$

Usually it's easier to put in the numbers at this point

$$\frac{12 \cdot V - V_a}{2 \cdot k\Omega} + \frac{0 - V_a}{3 \cdot k\Omega} = 2 \cdot mA$$

Multiply both sides by a value that will clear the denominators.

$$6 \cdot k\Omega \cdot \left(\frac{12 \cdot V - V_a}{2 \cdot k\Omega} + \frac{0 - V_a}{3 \cdot k\Omega} \right) = 2 \cdot mA \cdot 6 \cdot k\Omega$$

$$36 \cdot V - 3 \cdot V_a - 2 \cdot V_a = 12 \cdot V$$

$$-5 \cdot V_a = -24 \cdot V$$

$$V_a = \frac{-24 \cdot V}{-5} = 4.8 \cdot V$$

Remember, we needed to find the voltage across R_2 (V_{R2}).

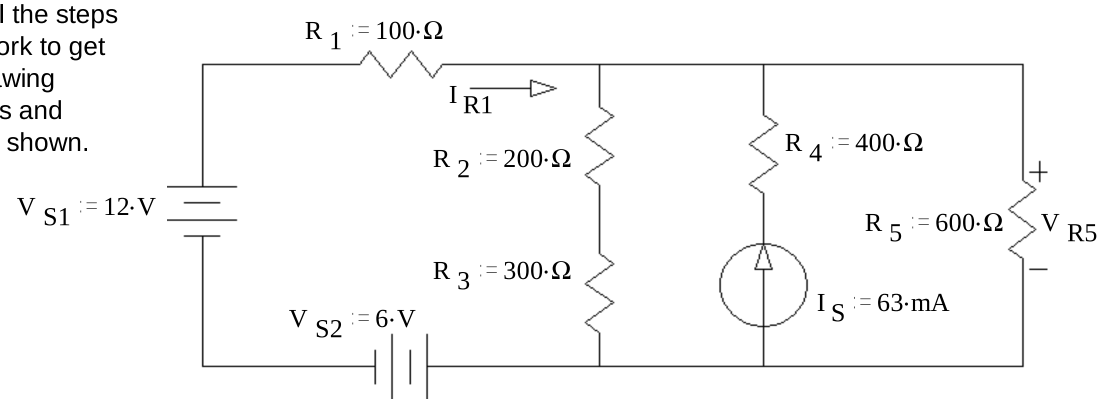
$$V_{R2} = V_S - V_a = 7.2 \cdot V$$

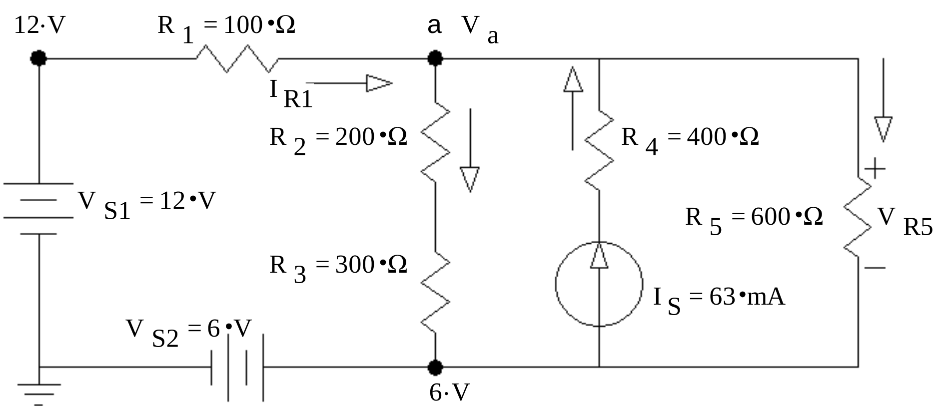
b) Find the current through R_3 (I_{R3}).

$$I_{R3} = \frac{0 - V_a}{R_3} = -1.6 \cdot mA$$ actually flows the other way

Ex 4 Use nodal analysis to find the voltage across R_5 (V_{R5}) and the current through R_1 (I_{R1}). From exam 1, F09

You **MUST** show all the steps of nodal analysis work to get credit, including drawing appropriate symbols and labels on the circuit shown.





node a:

$$I_{R1} + I_S = I_R + I_5$$

$$R_3 = 300\cdot\Omega \qquad \frac{V_{S1} - V_a}{R_1} + I_S = \frac{V_a - V_{S2}}{R_2 + R_3} + \frac{V_a - V_{S2}}{R_5}$$

$$\frac{12\cdot V}{100\cdot\Omega} - \frac{V_a}{100\cdot\Omega} + 63\cdot mA = \frac{V_a}{500\cdot\Omega} - \frac{6\cdot V}{500} + \frac{V_a}{600\cdot\Omega} - \frac{6\cdot V}{600}$$

multiply both sides by 3000Ω

$$3000\cdot\Omega\cdot\left(\frac{12\cdot V}{100\cdot\Omega} - \frac{V_a}{100\cdot\Omega} + 63\cdot mA\right) = \left(\frac{V_a}{500\cdot\Omega} - \frac{6\cdot V}{500} + \frac{V_a}{600\cdot\Omega} - \frac{6\cdot V}{600}\right)\cdot 3000\cdot\Omega$$

$$360\cdot V - 30\cdot V_a + 189\cdot V = 6\cdot V_a - 36\cdot V + 5\cdot V_a - 30\cdot V$$

$$360\cdot V + 189\cdot V + 36\cdot V + 30\cdot V = 6\cdot V_a + 5\cdot V_a + 30\cdot V_a$$

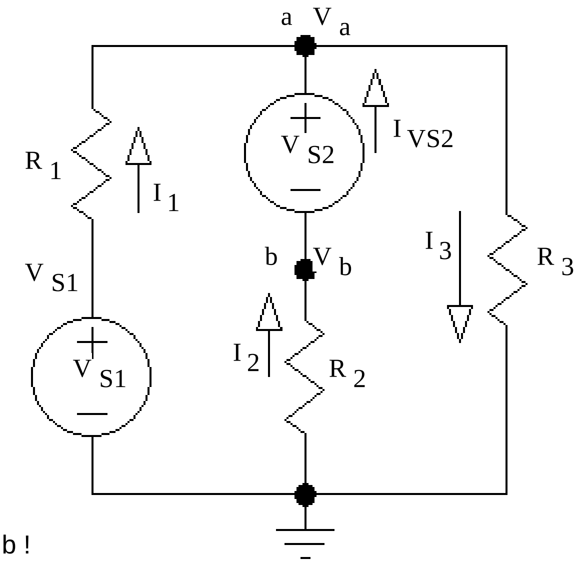
$$615\cdot V = 41\cdot V_a$$

$$V_a := \frac{615\cdot V}{41} \qquad V_a = 15\cdot V$$

$$V_{R5} = V_a - V_{S2} = 9\cdot V$$

$$I_{R1} = \frac{V_{S1} - V_a}{R_1} = -30\cdot mA$$

What if one side of a voltage source isn't ground?



$$I_1 + I_{VS2} = I_3$$

$$\frac{V_{S1} - V_a}{R_1} + \ ? = I_S$$

What do you put in for I_{VS2}??

Go to the other side of V_{S2}.

$$\frac{V_{S1} - V_a}{R_1} + \frac{0 - V_b}{R_2} = I_S$$

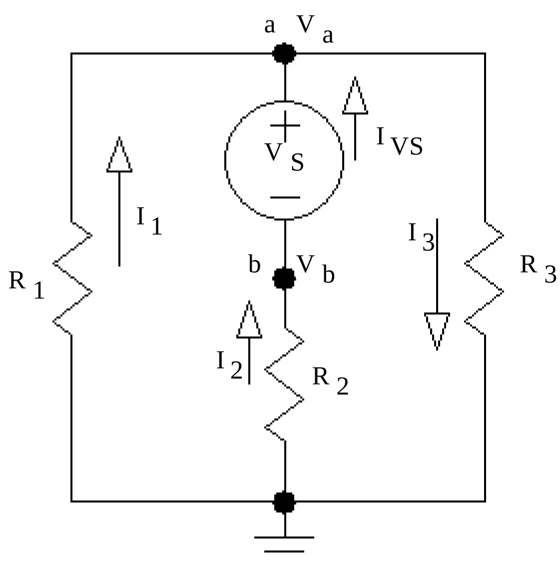
Only problem is that you get the same equation at node b !

Where does the second equation come from?

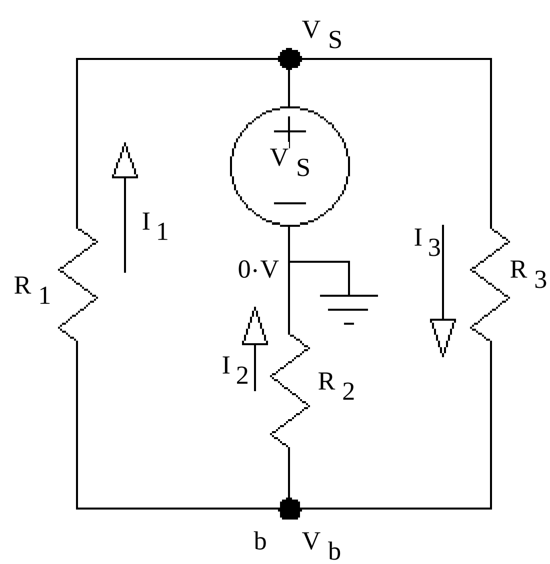
Use something like this: $V_a = V_b + V_{S2}$

Similar Circuit, but no V_{S1}.

If the ground is already at the bottom, use the same method as above.



If you can chose your ground, you can make life a little simpler.



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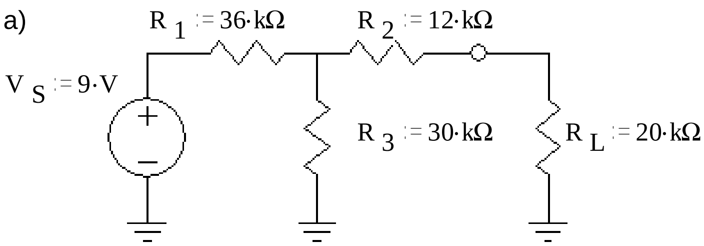
ECE 2210 / 00   homework # 5   Due: Fri, 9/11/20   a

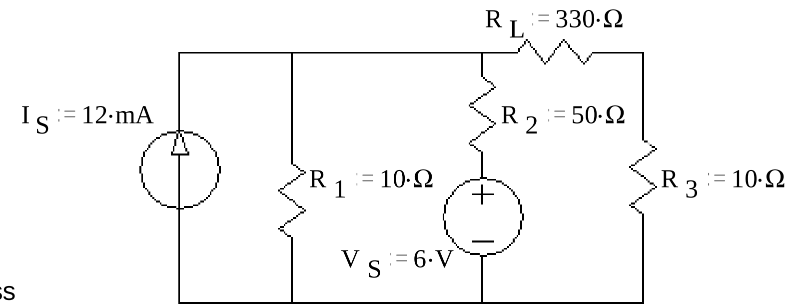
Thevenin & Norton equivalent circuits

1. For each of the circuits below, find and draw the Thevenin equivalent circuit.

a)



b) The load resistor is R_L, and is in a strange place in this circuit.
Hint: use superposition to find V_{Th}.



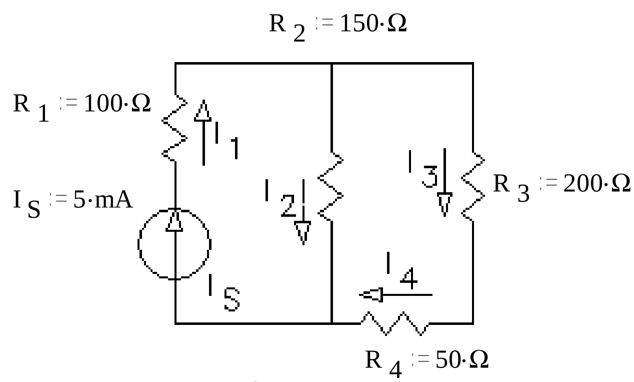
2nd hint: Nodal analysis is even easier.

2. For the circuit of problem 1a, find the voltage across R_L (V_L) and the current through R_L (I_L) using your Thevenin equivalent circuit.

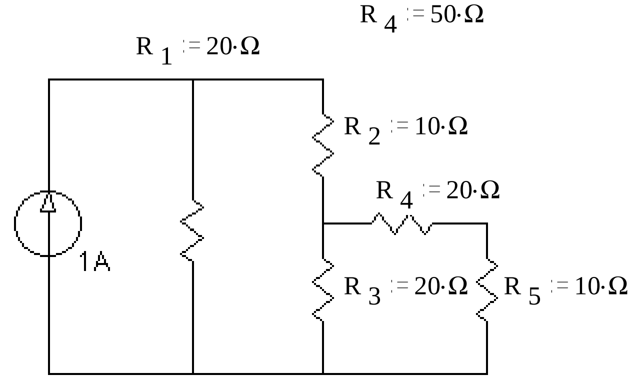
3. For each of the circuits in problem 1, find and draw the Norton equivalent circuit.

4. For the circuit of problem 1b, find V_L and I_L using your Norton equivalent circuit.

5. For the circuit shown at right, use Thevenin's theorem to find the current through the 50 Ω resistor R_4.



6. For the circuit shown, use Norton's theorem to find the value of the current in R_5. Hint: You can find I_N either by calculation of the open circuit voltage (V_{OC}) and R_N or by direct calculation of the short-circuit current (I_{SC}), however, there is something about the values of the resistors which makes the second method easier than it would at first appear.



Source resistance

7. The terminal voltage of a car's battery drops from 12.5 V to 8.5 volts when starting. The starter motor draws 60 A of current.

   a) Draw the voltage-source model (Thevenin equivalent) of this battery. Include the values of V_S and R_S.

   b) Draw the current-source model (Norton equivalent) of this battery. Include the values of I_S and R_S.

   c) Which of these two models is more appropriate for the car battery?

   d) What terminal voltage would you expect if this battery were being charged at 20 A?

Answers

1. a) $4.091\cdot V$, $28.4\cdot k\Omega$ b) $1.1\cdot V$, $18.3\cdot\Omega$ 2. $1.69\cdot V$, $84.6\cdot\mu A$

3. a) $0.144\cdot mA$, $28.4\cdot k\Omega$ b) $60\cdot mA$, $18.3\cdot\Omega$ 4. $3.16\cdot mA$, $1.042\cdot V$ 5. $1.88\cdot mA$

6. $0.19\cdot A$ 7. a) $V_S = 12.5\cdot V$ $R_S := 0.0667\cdot\Omega$ b) $I_S = 187.5\cdot A$ $R_S := 0.0667\cdot\Omega$

c) Thevenin   d) $13.83\cdot V$

ECE 2210 / 00   homework # 5

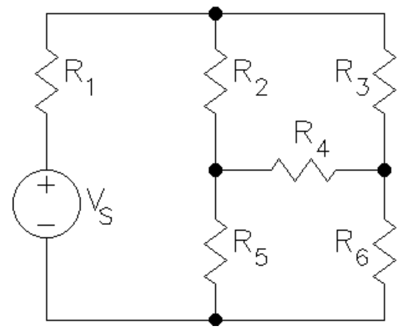
ECE 2210 / 00 homework # 6 Due: Tue, 9/15/20

a

Nodal Analysis

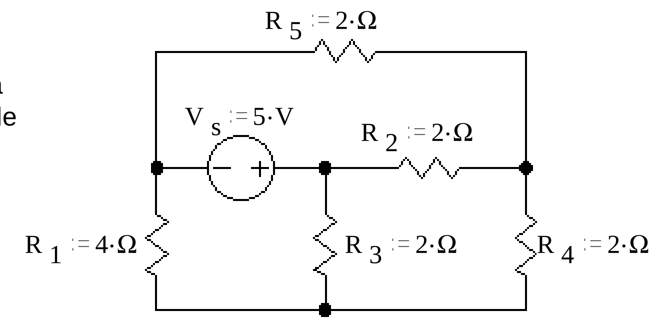
1. a) If you select the bottom node as ground, how many unknown node voltages remain? (Assume V_S is a known quantity.) How many simultaneous equations would you need to solve to analize this circuit?

   b) Use nodal analysis to find all the necessary simultaneous equations.

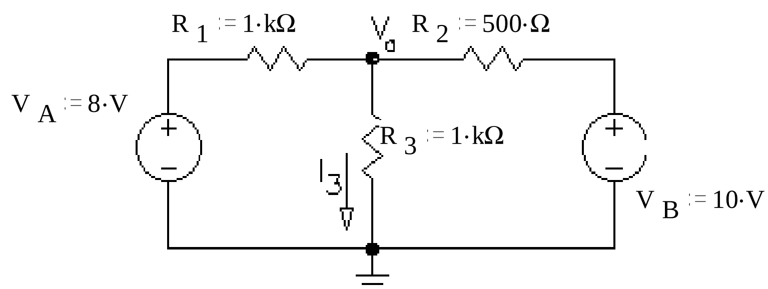


2. a) Use nodal analysis to find all the node voltages.

   b) Your node voltages will depend on your selection of a reference node (ground) as well as your arbitrary node labels, so the grader won't look at these specifically. Use your node voltages to find the potential (voltage) across each resistor. Report the magnitude and polarity of each.

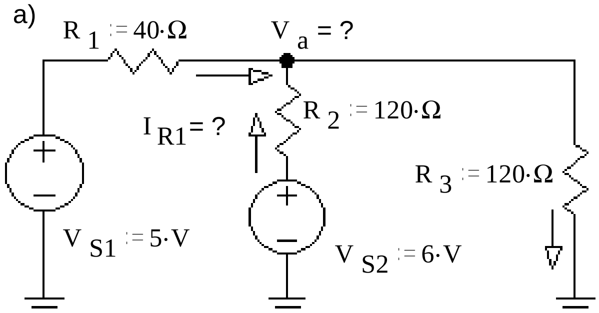


3. Use Nodal analysis to find V_a and use V_a to find I_3.

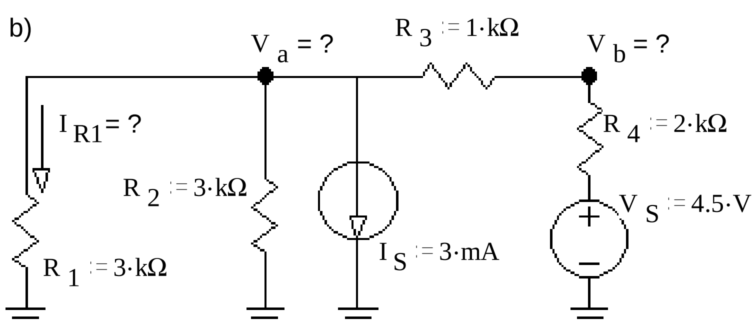


4. Use Nodal analysis to solve following problems: Each problem asks for at least 1 voltage and a current. Use the voltage(s) to find the current.

a)



b)



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hint: you may be able to eliminate one unknown node for the initial calculation.

Answers

1. a) 3,3 b) $V_a \cdot \left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right) - \frac{V_b}{R_2} - \frac{V_c}{R_3} = \frac{V_S}{R_1}$, $\frac{V_a}{R_2} - V_b \cdot \left(\frac{1}{R_2}+\frac{1}{R_5}+\frac{1}{R_4}\right) + \frac{V_c}{R_4} = 0 \cdot A$

$$\frac{V_a}{R_3}+\frac{V_b}{R_4} - V_c \cdot \left(\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_6}\right) = 0 \cdot A$$

2. a) Answer will depend on your choice of ground, so check your answers to part b to see if you did part a right.

   b) $3.077 \cdot V$, + bottom , $2.308 \cdot V$, + left , $1.923 \cdot V$, + top , $0.385 \cdot V$, + bottom , $2.692 \cdot V$, + right

3. $7 \cdot V$, $7 \cdot mA$ 4. a) $4.2 \cdot V$, $20 \cdot mA$ b) $V_a := -1.5 \cdot V$ $V_b := 0.5 \cdot V$ $I_{R1} := -0.5 \cdot mA$

You may not get this homework back before the 1^{st} **exam.** Photocopy it if you want to be sure to have it.

ECE 2210 / 00 homework # 6