ECE 2200/10 Lecture 1 Introduction to Electrical Engineering for non-majors

A. Stolp
12/30/11
8/24/15

2200 = 1/2 semester (Mining, Mat. Sci.)

ECE 2200 Without the Physics is hard, Plan on it!

2200, Decide today when you want to take the **final**:
Bad option: In your last lab session, Start labs Today
2nd option: With 2210 Exam 2 on 3/4. Start labs next week.

If you don't take the later final you will have to start labs **THIS WEEK**.

2210 = Full semester (Mechanical, Chemical, etc.)

Labs start **next week** Possible new labs, Th, 10:45 & F, 11:50

2210 Final Friday, April 24, 8:00am
Subject to change, listen in class

Make sure you are registered for the right class (2200 or 2210) and that you have the right syllabus.

BOTH

Bring a lab notebook and a U-card with $20 to 1st lab.

Homeworks are due by 5:00 pm in locker ________ (see map for location of lockers)

WARNING: HWs are often due on non-class days.

How to survive

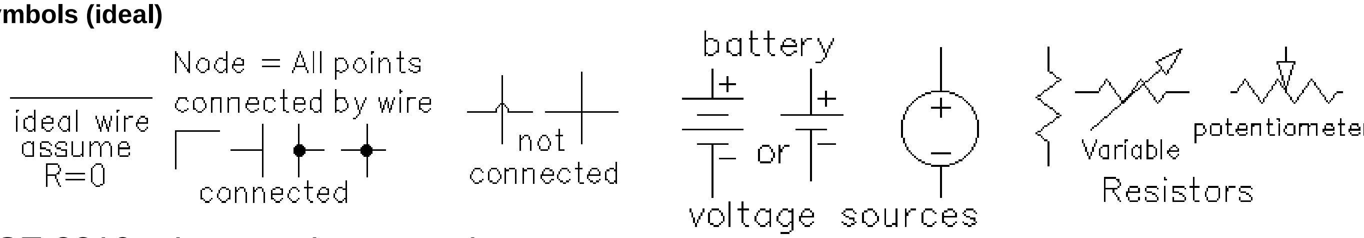
1. Easiest way to get through school is to actually learn and retain what you are asked to learn.
   Even if you're too busy, don't lose your good study practices.
   What you "just get by" on today will cost you later.

   Don't fall for the "I'll never need to know this" trap. Sure, much of what you learn you may not use, but you will need some of it, some day, either in the current class, future classes, or maybe sometime in your career. Don't waste time second-guessing the curriculum, It'll still be easier to just do your best to learn and retain what is covered.
2. Don't fall for the "traps".
   Homework answers, Problem session solutions, Posted solutions, Lecture notes.
3. KEEP UP! Use calendar.
4. Make "permanent notes" after you've finished a subject or section and feel that you know it.

Lecture

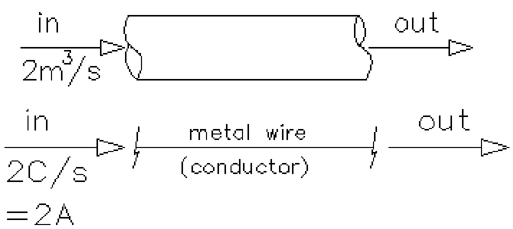
Basic electrical quantities	Letter used	Units	Fluid Analogy
Charge, actually moves	Q	Coulomb (C)	m^3
Current, like fluid flow	$I = \frac{Q}{sec}$	Amp (A, mA, µA,...)	$\frac{m^3}{sec}$
Voltage, like pressure	V or E	volt (V, mV, kV,...)	$Pa = 1 \cdot \frac{N}{m^2}$
Resistance	$R = \frac{V}{I}$	Ohm (Ω, kΩ, MΩ,...)	
Conductance	$G = \frac{1}{R}$	Siemens (S, also mho, old unit)	
Power = energy/time	$P = V \cdot I$	Watt (W, mW, kW, MW,...)	W

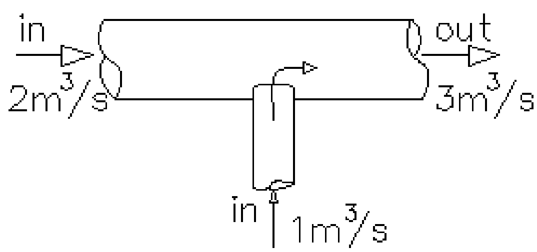
Symbols (ideal)

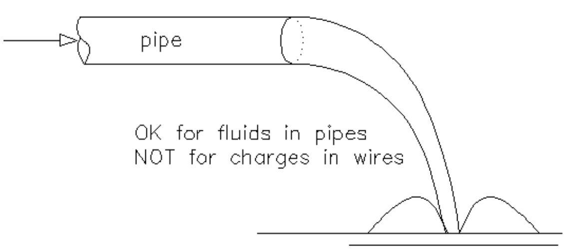


KCL, **K**irchhoff's **C**urrent **L**aw

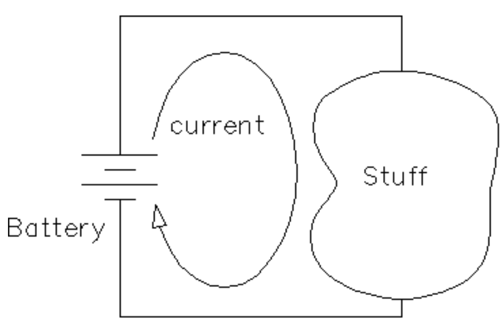
$I_{in} = I_{out}$ of any point, part, or section







Battery also obeys KCL
No accumulation of charge anywhere,
so it must circulate around.
Leads to the concept of a "Circuit"

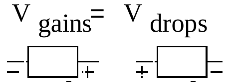


A circuit

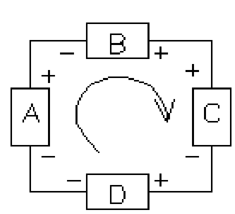
Voltage is like pressure

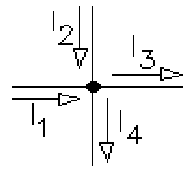
KVL, **K**irchhoff's **V**oltage **L**aw

$V_{gains} = V_{drops}$



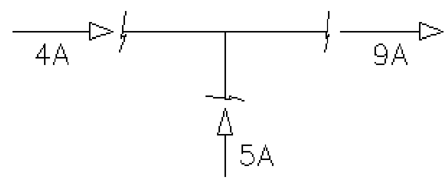
around any loop

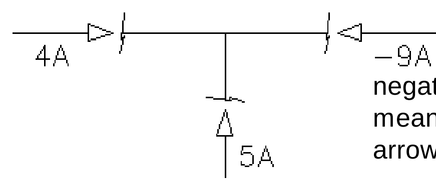




$I_{in} = I_{out}$

$I_1 + I_2 = I_3 + I_4$





negative current
means the direction
arrow is wrong

Conductors Nonconductors

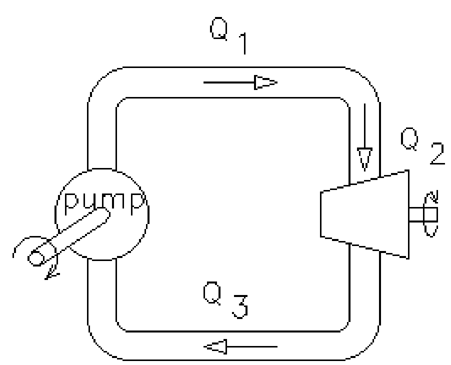
Massless fluid in our analogy
No gravity effects No Bernoulli effects

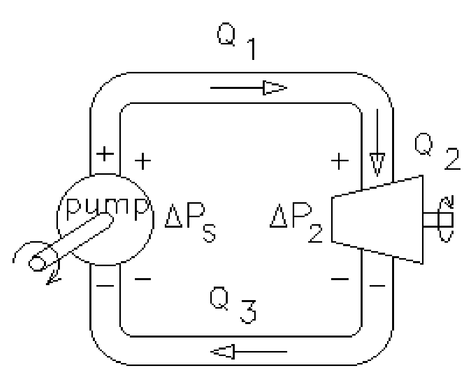
Reasonable because:

Electron mass is $9.11 \cdot 10^{-31} \cdot kg$

Election charge is $-1.6 \cdot 10^{-16} \cdot C$

Negative charge flows
in negative direction





ECE 2210 Lecture 1 notes p2

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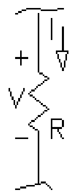
1/28/06,
9/5/08

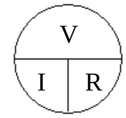
Ohm's law (resistors)

$V = I \cdot R$



$I = \dfrac{V}{R}$

$R = \dfrac{V}{I}$ definition of resistance and the unit "Ω"



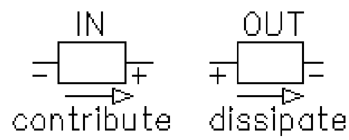
Power

flow $\dfrac{m^3}{sec}$ pressure $\dfrac{N}{m^3}$ flow x pressure: $\dfrac{m^3}{sec} \cdot \dfrac{N}{m^2} = \dfrac{m}{sec} \cdot \dfrac{N}{1} = \dfrac{N \cdot m}{sec} = \dfrac{Joule}{sec} = W$ = power

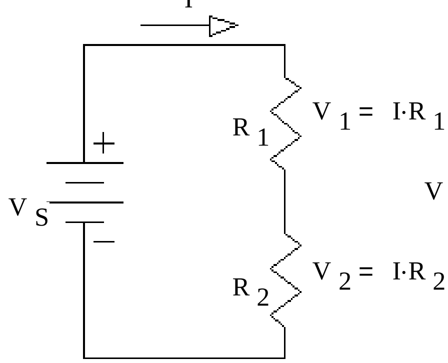
same for electricity power $P = I \cdot V$

Power dissipated by resistors: $P = V \cdot I = \dfrac{V^2}{R} = I^2 \cdot R$

$P_{IN} = P_{OUT}$

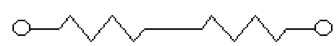


Series Resistors



$V_T = V_1 + V_2$

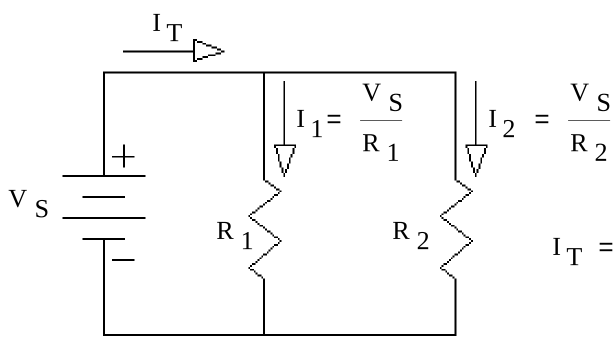
$R_{eq} = \dfrac{V_T}{I} = \dfrac{V_1 + V_2}{I} = \dfrac{V_1}{I} + \dfrac{V_2}{I} = R_1 + R_2$



$R_{eq} = R_1 + R_2$

Resistors are in series if and only if exactly the **same current** flows through each resistor.

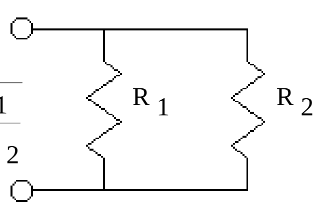
Parallel Resistors



$I_T = \dfrac{V_S}{R_1} + \dfrac{V_S}{R_2}$

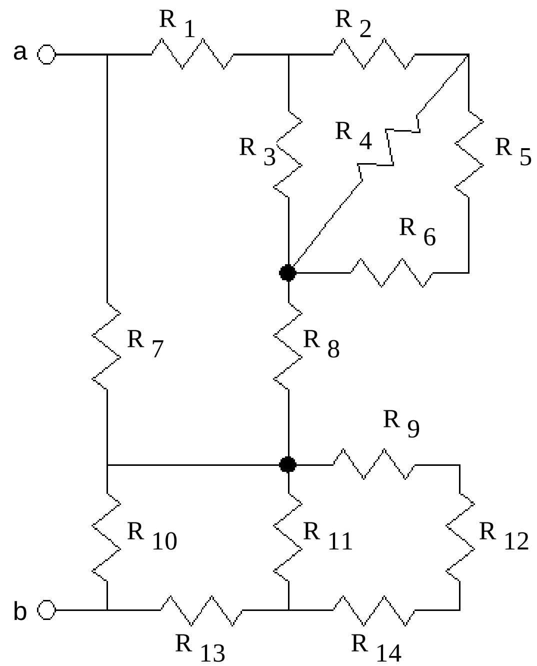
$R_{eq} = \dfrac{V_S}{I_T} = \dfrac{V_S}{\dfrac{V_S}{R_1} + \dfrac{V_S}{R_1}} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$

$R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$



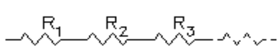
Resistors are in parallel if and only if the **same voltage** is across each resistor.

Series and Parallel



All resistor-only networks can be reduced to a single equivalent, but not always by means of series and parallel concepts.

Voltage Divider



series: $R_{eq} = R_1 + R_2 + R_3 + \ldots$

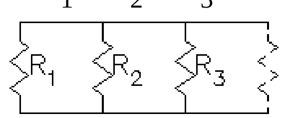
Exactly the **same current** through each resistor

Voltage divider:

$$V_{Rn} = V_{total} \cdot \frac{R_n}{R_1 + R_2 + R_3 + \ldots}$$

Current Divider

parallel: $R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$

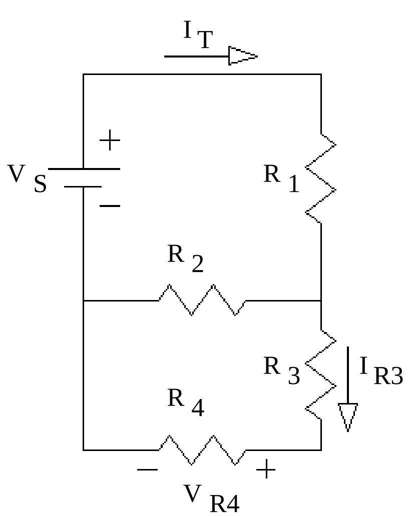


Exactly the **same voltage** across each resistor

current divider:

$$I_{Rn} = I_{total} \cdot \frac{\dfrac{1}{R_n}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$$

May have to combine some resistors first to get series and parallel resistors to use with divider expressions.

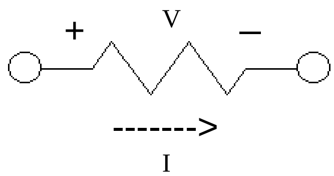


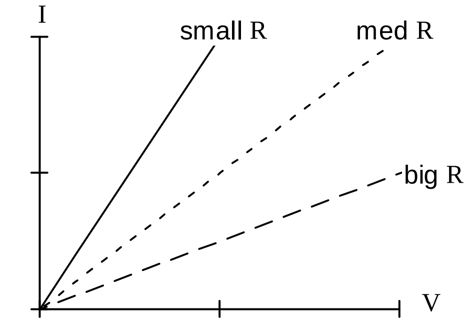
$V_{R4} =$

$I_T =$

$I_{R3} =$

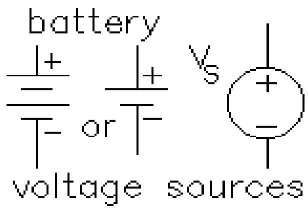
Resistors



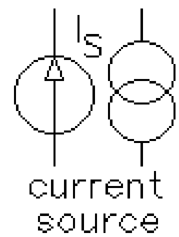


$$R = \frac{1}{\text{slope}} = \frac{\Delta V}{\Delta I}$$

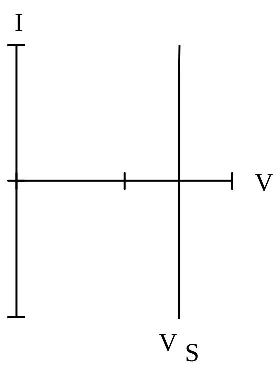
Sources

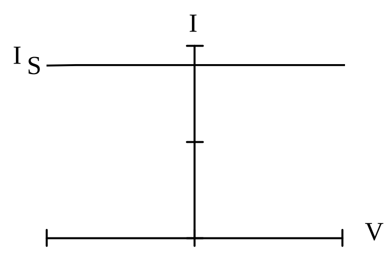


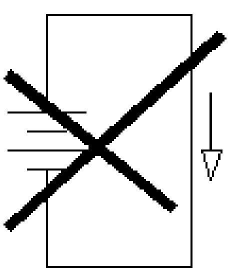
Battery Cell



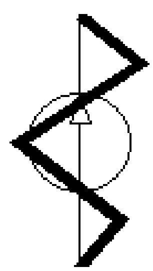
Less intuitive, less like sources we are used to seeing.



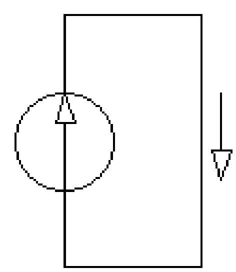




Doesn't make sense with for ideal voltage sources and ideal wires

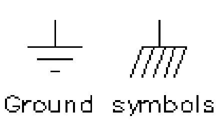


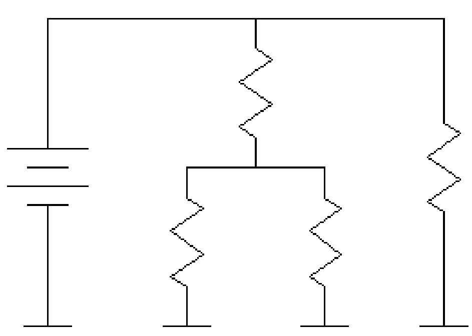
Doesn't make sense for ideal current sources

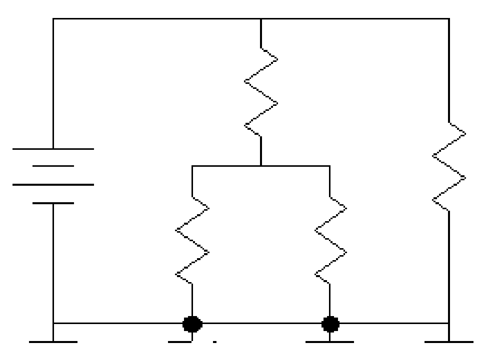


Must have a path for the current to flow

Ground



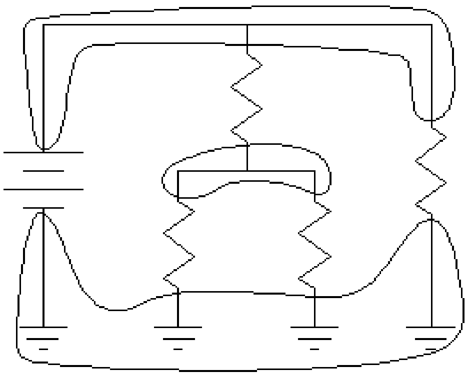




Ground is considered zero volts and is a reference for other voltages.

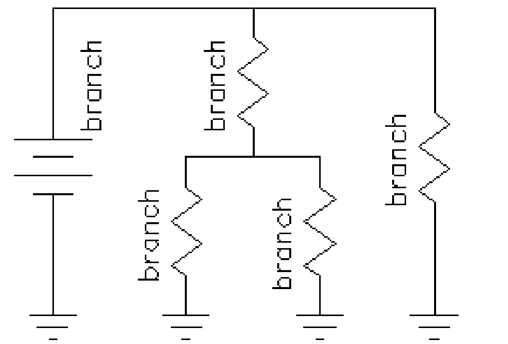
Nodes & Branches

Node = all points connected by wire, all at same voltage (potential)

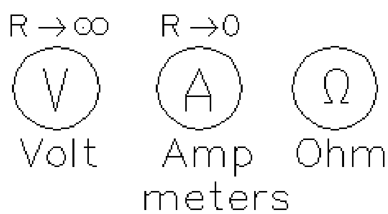


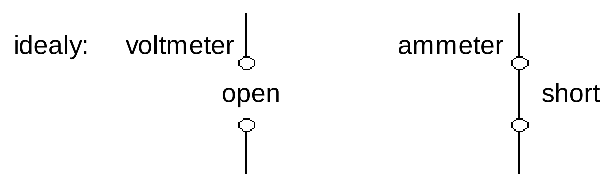
ground is a node

Branch = all parts with the same current

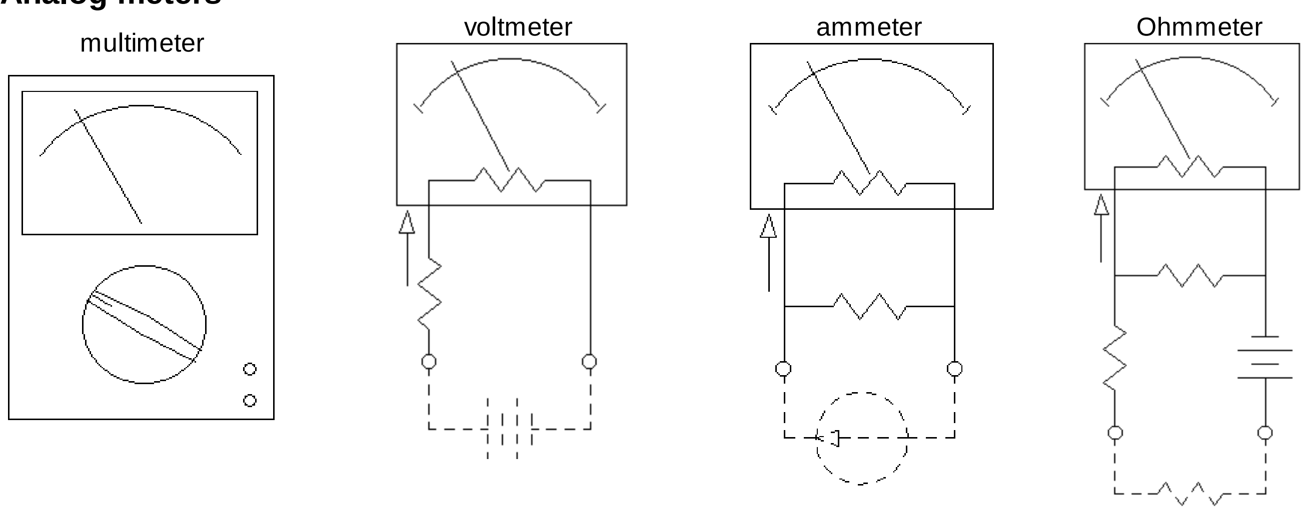


Meters

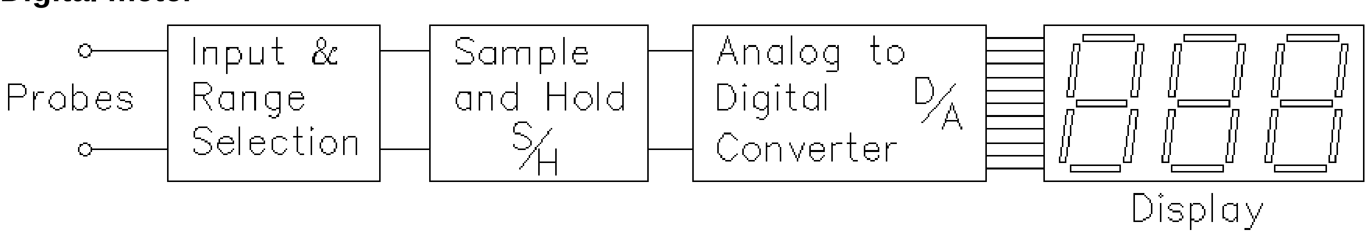




Analog meters



Digital meter



ECE 2210 Lecture 4 notes Superposition

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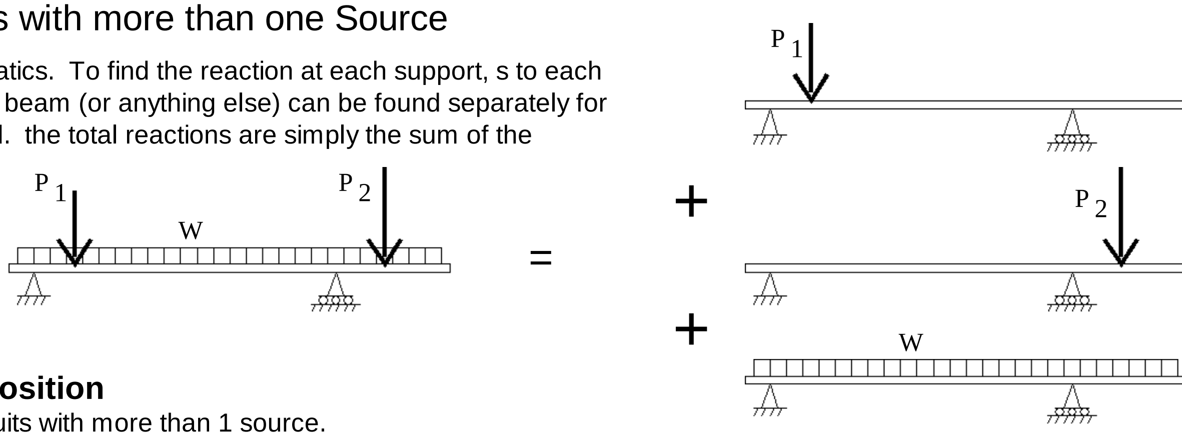
9/3/08,
?/31/09

Circuits with more than one Source

Recall Statics. To find the reaction at each support, s to each load on a beam (or anything else) can be found separately for each load. the total reactions are simply the sum of the

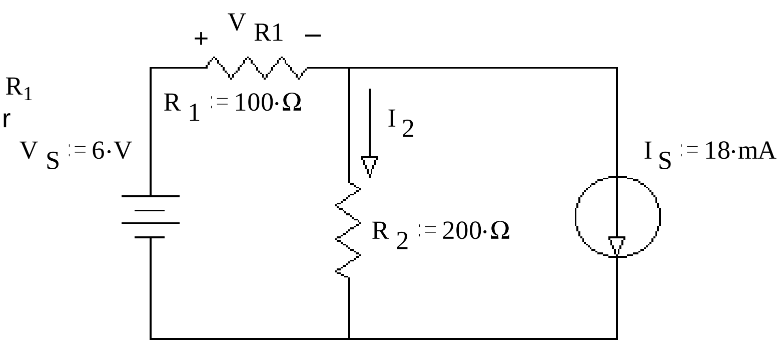


Superposition

For circuits with more than 1 source.

1) Zero all but one source.
   (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
2) Compute your wanted voltage or current due to the remaining source. Careful, some may be negative.
3) Repeat the first two steps for all the sources.
4) Sum all the contributions from all the sources to find the actual voltage or current. **Watch your signs!**

Ex1. Use the method of superposition to find the current I_2 (through R_2) and the voltage across R_1 (V_{R1}). Be sure to clearly show and **circle** your intermediate results.

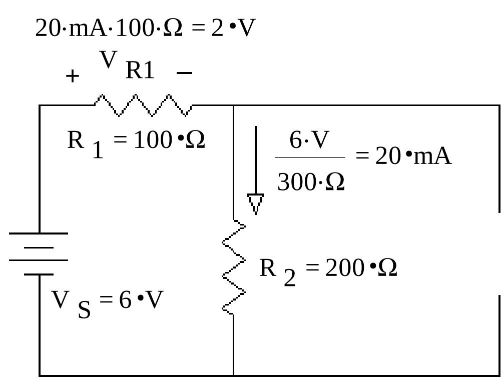


superposition:

Eliminate current source

$I_{2.Vs} := \dfrac{V_S}{R_1 + R_2}$ $\qquad$ $I_{2.Vs} = 20\cdot mA$

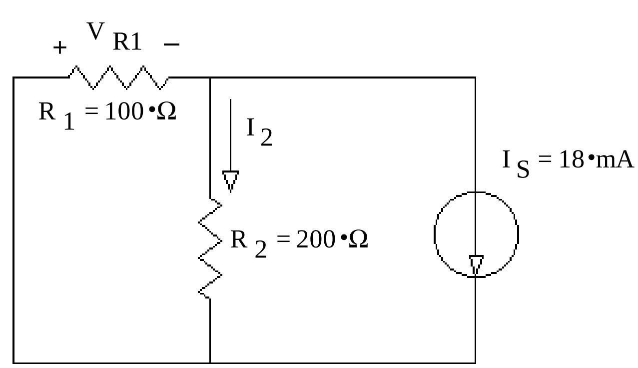
$V_{R1.Vs} := \dfrac{R_1}{R_1 + R_2}\cdot V_S$ $\qquad$ $V_{R1.Vs} = 2\cdot V$



Eliminate voltage source

$I_{2.Is} := -\dfrac{\dfrac{1}{R_2}}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}\cdot I_S$ $\qquad$ $I_{2.Is} = -6\cdot mA$

$V_{R1.Is} := -I_{2.Is}\cdot R_2$ $\qquad$ $V_{R1.Is} = 1.2\cdot V$

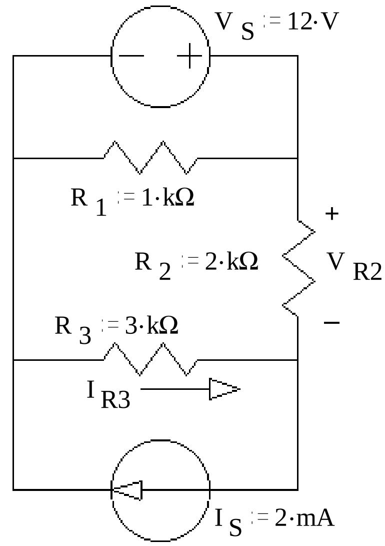


Add results

$I_2 := I_{2.Vs} + I_{2.Is}$ $\qquad$ $I_2 = 14\cdot mA$

$V_{R1} := V_{R1.Vs} + V_{R1.Is}$ $\qquad$ $V_{R1} = 3.2\cdot V$

Ex2. Use the method of superposition to find the voltage accross through R_2 and the current through R_3. Be sure to clearly show and **circle** your intermediate results.

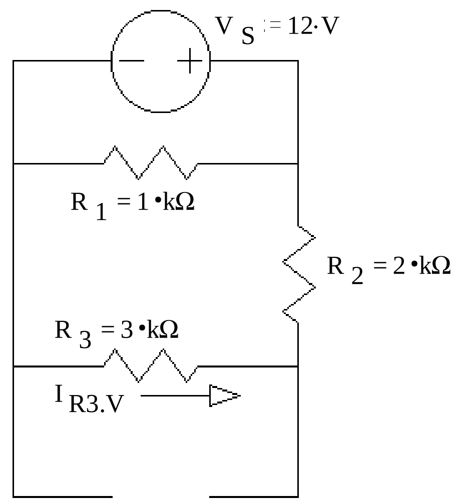


Eliminate current source

R_1 is a separate path and doesn't matter.

$$V_{R2.Vs} := \frac{R_2}{R_2 + R_3} \cdot V_S \qquad V_{R2.Vs} = 4.8 \cdot V$$

$$I_{R3.Vs} := -\frac{V_S}{R_2 + R_3} \qquad I_{R3.Vs} = -2.4 \cdot mA$$

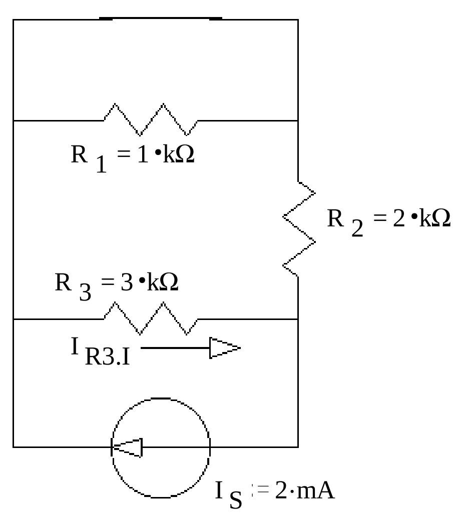


Eliminate voltage source

R_1 is shorted and doesn't matter.

$$V_{R2.Is} := I_S \cdot \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}} \qquad V_{R2.Is} = 2.4 \cdot V$$

$$I_{R3.Is} := \frac{\frac{1}{R_3}}{\frac{1}{R_2} + \frac{1}{R_3}} \cdot I_S \qquad I_{R3.Is} = 0.8 \cdot mA$$



Add results

$$V_{R2} := V_{R2.Vs} + V_{R2.Is} \qquad V_{R2} = 7.2 \cdot V$$

$$I_{R3} := I_{R3.Vs} + I_{R3.Is} \qquad I_{R3} = -1.6 \cdot mA$$

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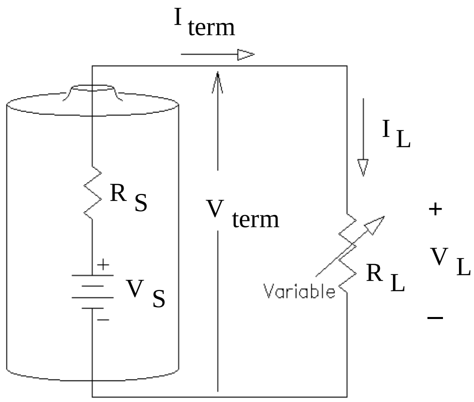
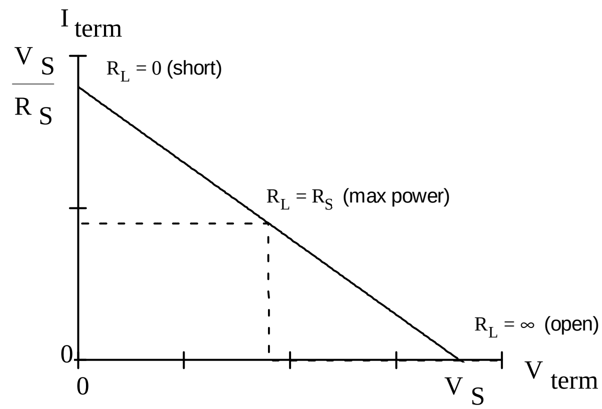
1/28/06
9/2/09

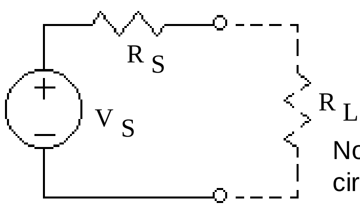
ECE 2210 Lectures 5 & 6 notes Thévenin & Norton Equivalent Circuits

Model of a Real Source

Real sources are not ideal, but we will model them with two ideal components.



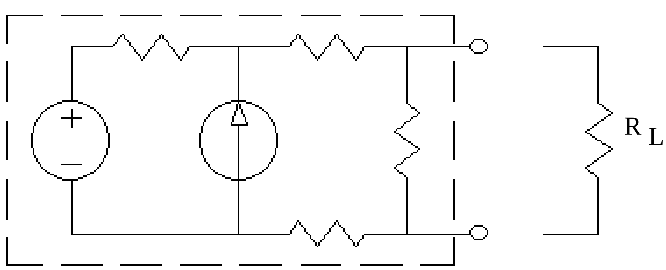


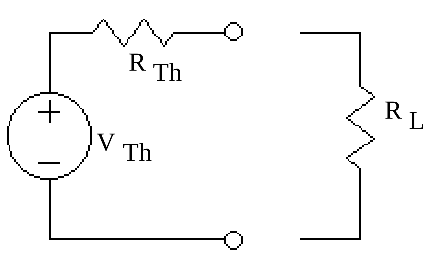


Note: R_L is NOT part of the Thévenin equivalent circuit and does not need to be shown.

Thévevin Equivalent Circuit

The same model can be used for any combination of sources and resistors.

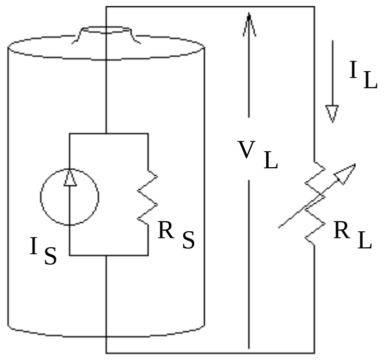


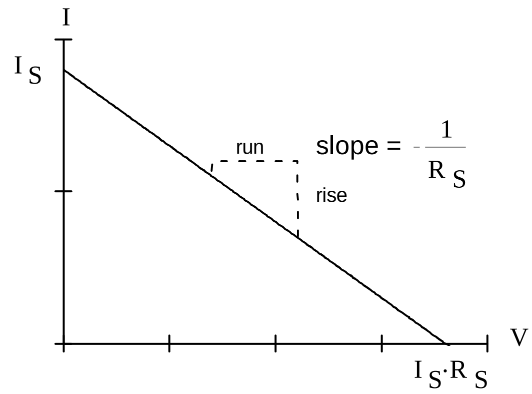


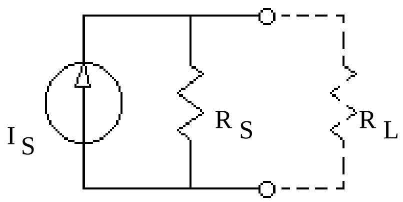
Thévenin equivalent

To calculate a circuit's Thévenin equivalent:

1) Remove the load and calculate the open-circuit voltage where the load used to be.
   This is the Thévenin voltage (V_{Th}).
2) Zero all the sources.
   (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
3) Compute the total resistance between the load terminals.
   (DO NOT include the load in this resistance.) This is the Thévenin source resistance (R_{Th}).
4) Draw the Thévenin equivalent circuit and add your values.





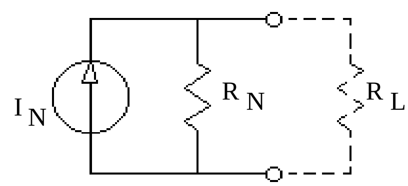


Note: R_L is not part of the Norton equivalent and does not need to be shown.

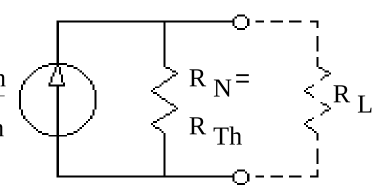
Norton equivalent

To calculate a circuit's Norton equivalent:

1) Replace the load with a short (a wire) and calculate the short-circuit current in this wire. This is the Norton current (I_N). Remove the short.
2) Zero all the sources. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
3) Compute the total resistance between the load terminals. (DO NOT include the load in this resistance.) This is the Norton source resistance (R_N). (Exactly the same as the Thévenin source resistance (R_{Th})).
4) Draw the Norton equivalent circuit and add your values.



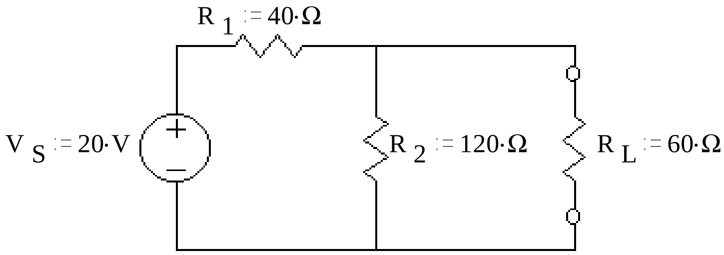
OR (the more common way)...
1) Find the Thévenin equivalent circuit.
2) Convert to Norton circuit, then >>> $R_N = R_{Th}$ and $I_N = \frac{V_{Th}}{R_{Th}}$



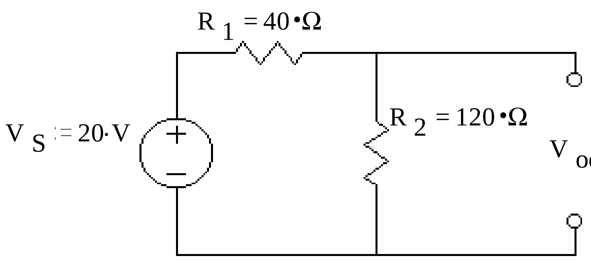
 Thévevin & Norton Examples

A.Stolp
1/23/03,
1/6/13

Ex 1 Find the Thévenin equivalent:



To calculate a circuit's Thévenin equivalent:
1) Remove the load and calculate the open-circuit voltage where the load used to be.
 This is the Thévenin voltage (V_{Th}).

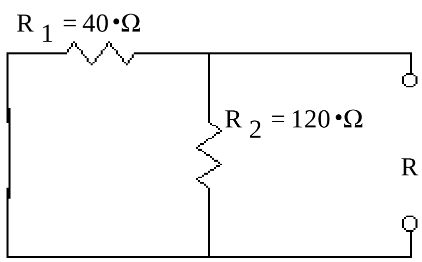


Find the open circuit voltage:

$$V_{oc} = V_{Th} := V_S\cdot\frac{R_2}{R_1 + R_2} \qquad V_{Th} = 15\cdot V$$

2) Zero all the sources.
 (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)

Zero the source



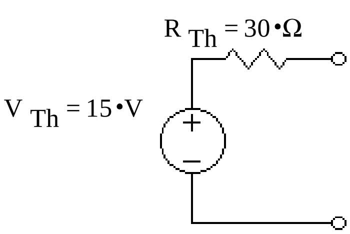
3) Compute the total resistance between the load terminals.
 (DO NOT include the load in this resistance.)
 This is the Thévenin source resistance (R_{Th}).

Find the Thevenin resistance:

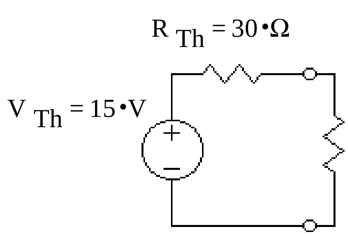
$$R_{Th} := \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \qquad R_{Th} = 30\cdot\Omega$$

4) Draw the Thévenin equivalent circuit and add your values.

Thevenin equivalent circuit:



If the load were reconnected:



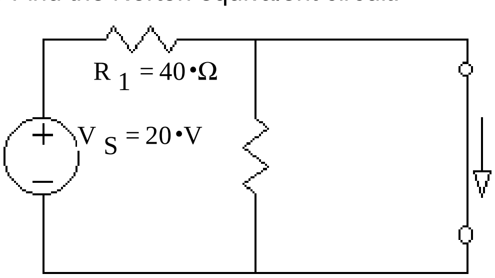
$R_L = 60\cdot\Omega$

$$V_L = V_{Th}\cdot\frac{R_L}{R_{Th} + R_L} = 10\cdot V$$

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = 166.7\cdot mA$$

$$P_L = 10\cdot V\cdot 166.7\cdot mA = 1.667\cdot W$$

b) Find the Norton equivalent circuit:



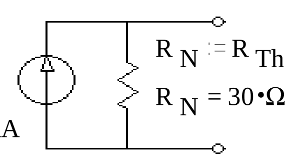
$$I_N := \frac{V_S}{R_1}$$

$$I_N = 500\cdot mA$$

Norton equivalent circuit:

$$I_N := \frac{V_{Th}}{R_{Th}}$$

$$I_N = 500\cdot mA$$



c) Show that the Thévenin circuit is indeed equivalent to the original at several values of R_L.

	Original Circuit		Thévenin Circuit	
R_L	V_L	I_L	I_L	V_L
$R_L := 0\cdot\Omega$	$0\cdot V$	$\frac{V_S}{R_1} = 500\cdot mA$	$\frac{V_{Th}}{R_{Th}+R_L} = 500\cdot mA$	$500\cdot mA\cdot 0\cdot\Omega = 0\cdot V$

Using either numbers: $P_L = V_L \cdot I_L = 0\cdot W$

$R_L := 10\cdot\Omega$ $\quad R_o := \dfrac{1}{\dfrac{1}{R_2}+\dfrac{1}{R_L}}$ $\quad R_o = 9.231\cdot\Omega$ $\quad I_L := \dfrac{V_{Th}}{R_{Th}+R_L}$ $\quad V_L := I_L \cdot R_L$

$I_L = 375\cdot mA$ $\quad V_L = 3.75\cdot V$

$$V_L = V_S \cdot \frac{R_o}{R_1+R_o} = 3.75\cdot V$$

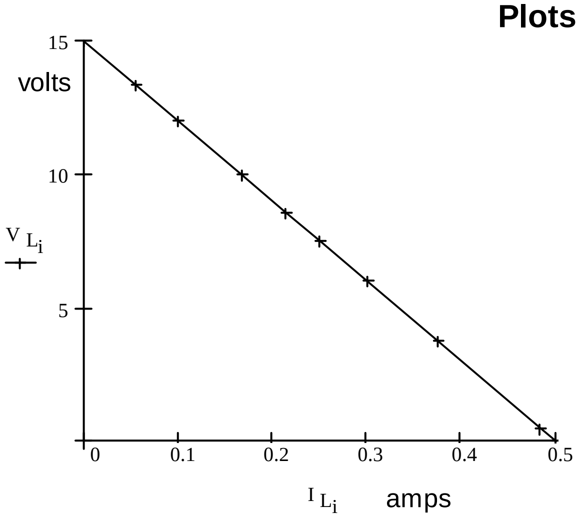
$$I_L = \frac{V_L}{R_L} = 375\cdot mA$$

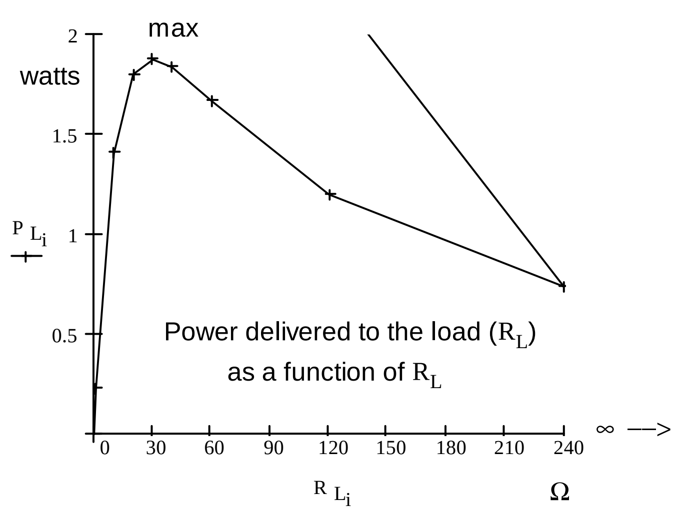
Using either numbers: $P_L = V_L \cdot I_L = 1.406\cdot W$

Repeat these calculations for a number of load resistors

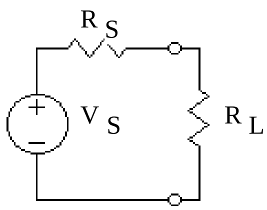
$R_{L_i} :=$	R_{o_i} (Ω)	$V_L = V_S \cdot \frac{R_{o_i}}{R_1+R_{o_i}}$ (V)	$I_L = \frac{V_{L_i}}{R_{L_i}}$ (mA)	$I_L = \frac{V_{Th}}{R_{Th}+R_{L_i}}$ (mA)	$V_L = I_{L_i}\cdot R_{L_i}$ (V)	P_{L_i} (W)	
$0\cdot\Omega$	0	0	0	500	0	0	
$1\cdot\Omega$	0.992	0.484	483.871	483.871	0.484	0.234	
$10\cdot\Omega$	9.231	3.75	375	375	3.75	1.406	
$20\cdot\Omega$	17.143	6	300	300	6	1.8	
$30\cdot\Omega$	24	7.5	250	250	7.5	1.875	max
$40\cdot\Omega$	30	8.571	214.286	214.286	8.571	1.837	
$60\cdot\Omega$	40	10	166.667	166.667	10	1.667	
$120\cdot\Omega$	60	12	100	100	12	1.2	
$240\cdot\Omega$	80	13.333	55.556	55.556	13.333	0.741	
$\infty\cdot\Omega$	120	15	0	0	15	0	

Plots





Maximum power transfer If I wanted to maximize the power dissipated by the load, what R_L would I choose?



$$P_L = \frac{V_L^2}{R_L} = \left(\frac{R_L}{R_S + R_L}\cdot V_S\right)^2 \cdot \frac{1}{R_L} = \frac{R_L^2}{\left(R_S + R_L\right)^2}\cdot V_S^2 \cdot \frac{1}{R_L}$$

$$= \frac{R_L^2}{R_S^2 + 2\cdot R_S \cdot R_L + R_L^2}\cdot V_S^2 \cdot \frac{1}{R_L} = \frac{R_L}{R_S^2 + 2\cdot R_S \cdot R_L + R_L^2}\cdot V_S^2$$

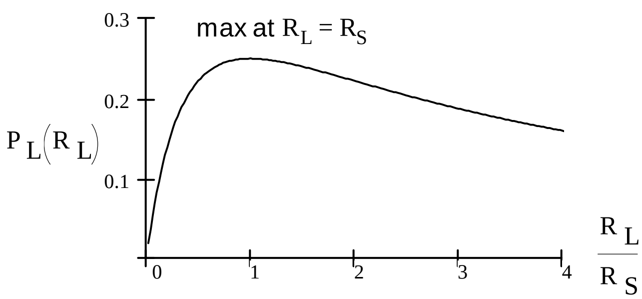
$$= \frac{1}{\frac{R_S^2}{R_L} + 2\cdot R_S + R_L}\cdot V_S^2$$

Next step would be to differentiate $\frac{d}{dR_L} P_L\left(R_L\right)$,

set this equal to 0 and solve for R_L to find the maximum

Unfortunately this function is a pain to differentiate. What if we just differentiate the denominator and find its minimum, wouldn't that work just as well?

$$\frac{d}{dR_L}\left(\frac{R_S^2}{R_L} + 2\cdot R_S + R_L\right) = -1\cdot\frac{R_S^2}{R_L^2} + 0 + 1 = 0$$



Maximum power transfer happens when: $R_L = R_S$

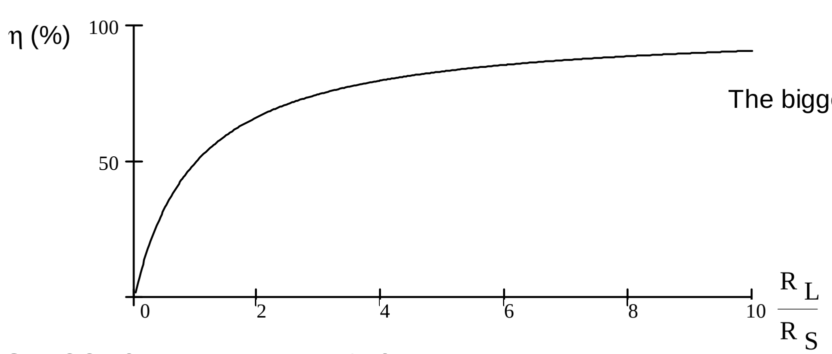
Just what we saw in Example 1

This is rarely important in power circuitry, where there should be plenty of power and R_S should be small. It is much more likely to be important in signal circuitry where the voltages can be very small and the source resistance may be significant -- say a microphone or a radio antenna.

All you need to remember is: $R_L = R_S$ to maximize the power dissipation in R_L

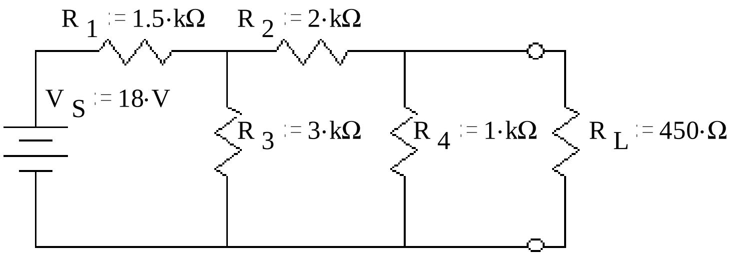
What about efficiency?

$$\frac{P_L\left(R_L\right)}{P_S\left(R_L\right)} = \frac{I^2\cdot R_L}{I^2\cdot\left(R_S + R_L\right)} = \frac{R_L}{R_S + R_L}$$

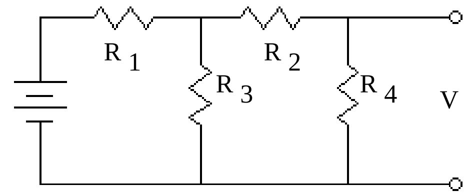


The bigger R_L is, the higher the efficiency.

Ex 2 a) Find and draw the Thévenin equivalent circuit.

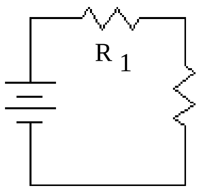


Find the open circuit voltage:



$V_{oc} = V_{Th}$

First do some simplification:



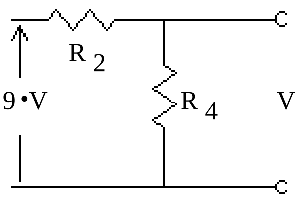
$$R_{eq234} := \frac{1}{\dfrac{1}{R_3} + \dfrac{1}{R_2 + R_4}} \qquad R_{eq234} = 1.5\cdot k\Omega$$

$$V_{234} := \frac{R_{eq234}}{R_1 + R_{eq234}} \cdot V_S$$

$$V_{234} = 9\cdot V$$

Divide this voltage between R_2 and R_4:

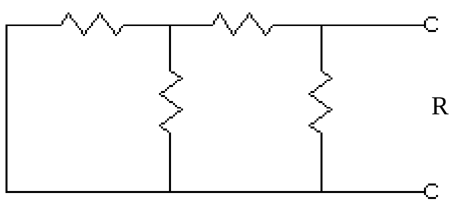
$V_{234} = 9\cdot V$



$$V_{Th} := \frac{R_4}{R_2 + R_4} \cdot V_{234} \qquad V_{Th} = 3\cdot V$$

Find the Thévenin resistance:

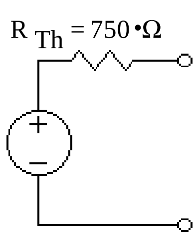
Zero the source



$$R_{Th} := \frac{1}{\dfrac{1}{R_4} + \dfrac{1}{R_2 + \dfrac{1}{\left(\dfrac{1}{R_1} + \dfrac{1}{R_3}\right)}}} \qquad R_{Th} = 750\cdot\Omega$$

Thévenin equivalent circuit:

$V_{Th} = 3\cdot V$



If the load were reconnected:

$$V_L := V_{Th} \cdot \frac{R_L}{R_{Th} + R_L} \qquad V_L = 1.125\cdot V$$

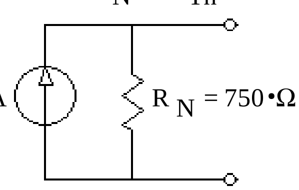
$$I_L := \frac{V_{Th}}{R_{Th} + R_L} \qquad I_L = 2.5\cdot mA$$

b) Find and draw the Norton equivalent circuit.

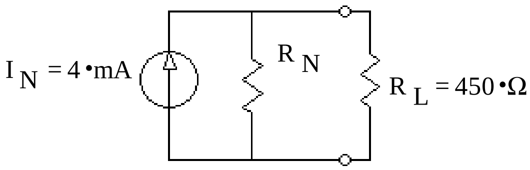
$R_N := R_{Th}$

$$I_N := \frac{V_{Th}}{R_{Th}}$$

$I_N = 4\cdot mA$



c) Use your Norton equivalent circuit to find the current through the load.



$$I_L := \frac{\frac{1}{R_L}}{\left(\frac{1}{R_N}+\frac{1}{R_L}\right)} \cdot I_N \qquad I_L = 2.5 \cdot mA$$

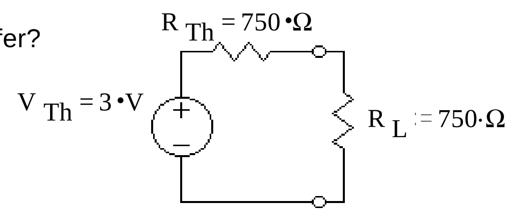
$$V_L := I_L \cdot R_L \qquad V_L = 1.125 \cdot V$$

same as above

d) What value of R_L would result in the maximum power delivery to R_L?

For maximum power transfer $R_L = R_{Th} = 750 \cdot \Omega$

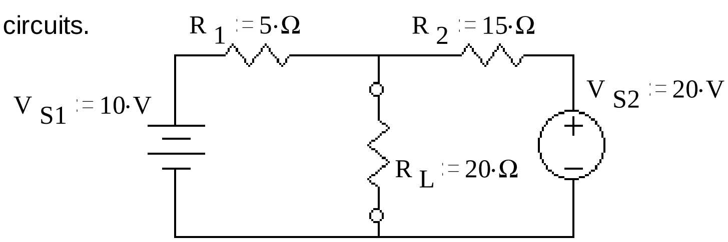
e) What is the maximum power transfer?



$$V_L := \frac{V_{Th}}{2}$$

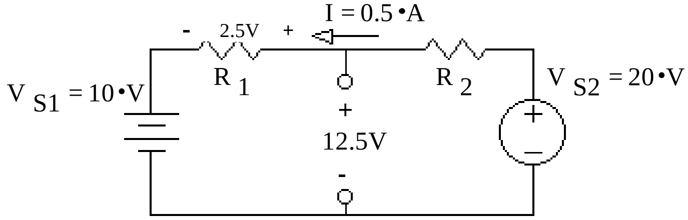
$$P_L = \frac{V_L^{\,2}}{R_L} = 3 \cdot mW$$

Ex 3 a) Find and draw the Thévenin & Norton equivalent circuits.

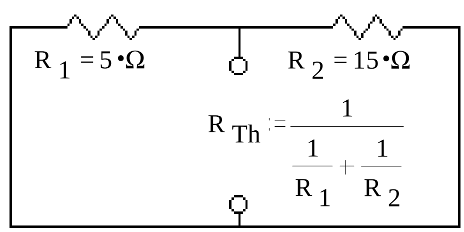


$$I := \frac{V_{S2} - V_{S1}}{R_1 + R_2}$$

$$I = 0.5 \cdot A$$



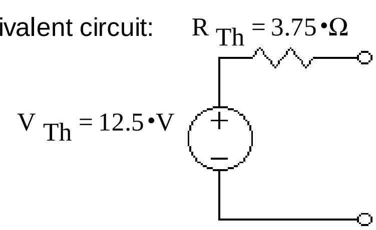
$$V_{Th} := 10 \cdot V + 2.5 \cdot V$$



$$R_{Th} := \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$$

$$R_{Th} = 3.75 \cdot \Omega$$

Thévenin equivalent circuit:

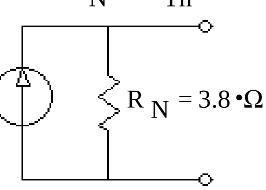


Norton equivalent circuit:

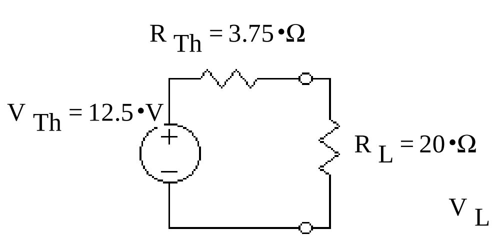
$$R_N := R_{Th}$$

$$I_N := \frac{V_{Th}}{R_{Th}}$$

$$I_N = 3.333 \cdot A$$

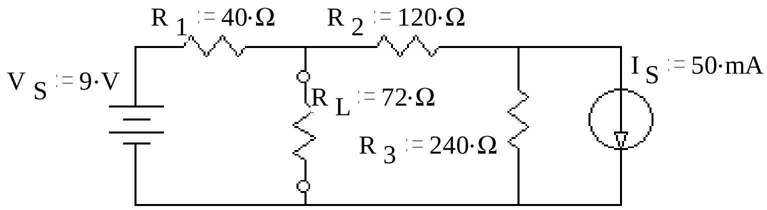


b) Use your Thévenin equivalent circuit to find the voltage across the load.

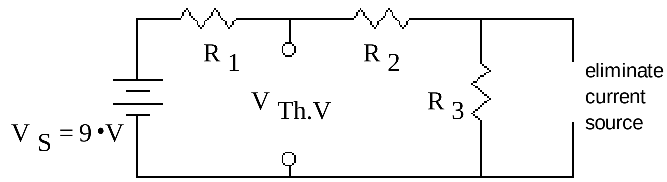


$$V_L = \frac{R_L}{R_{Th} + R_L} \cdot V_{Th} = 10.526 \cdot V$$

Ex 4 a) Find and draw the Thévenin & Norton equivalent circuits.

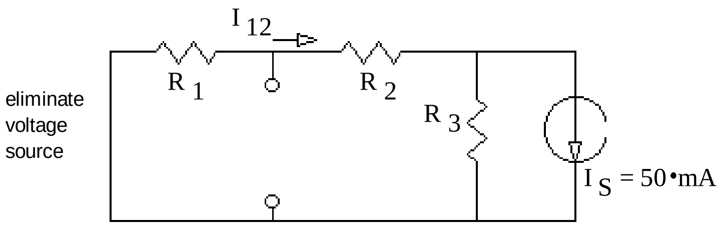


Use superposition to find V_{Th}.



$$V_{Th.V} := \frac{R_2 + R_3}{R_1 + R_2 + R_3} \cdot V_S$$

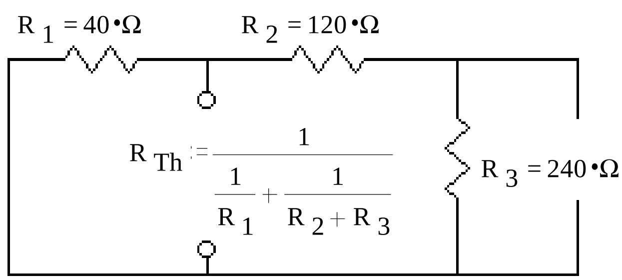
$$V_{Th.V} = 8.1 \cdot V$$



current divider: $I_{12} := \dfrac{\dfrac{1}{R_1 + R_2}}{\dfrac{1}{R_1 + R_2} + \dfrac{1}{R_3}} \cdot I_S$ $\quad I_{12} = 30 \cdot mA$ $\quad V_{Th.I} := -I_{12} \cdot R_1$ $\quad V_{Th.I} = -1.2 \cdot V$

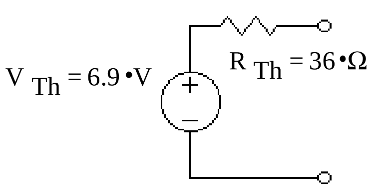
$V_{Th} := V_{Th.V} + V_{Th.I}$ $\quad V_{Th} = 6.9 \cdot V$

Find the Thévenin resistance

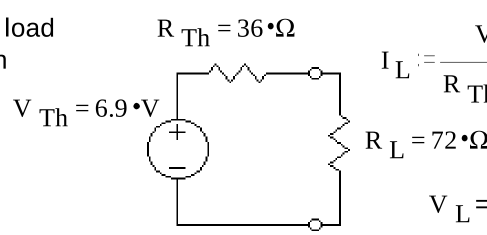


$$R_{Th} := \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2 + R_3}}$$

Thévenin equivalent circuit:



Put the load back on

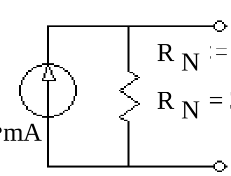


$I_L := \dfrac{V_{Th}}{R_{Th} + R_L}$ $\quad I_L = 63.889 \cdot mA$

$V_L = I_L \cdot R_L = 4.6 \cdot V$

Norton equivalent circuit: $I_N := \dfrac{V_{Th}}{R_{Th}}$

$I_N = 191.7 \cdot mA$



$R_N := R_{Th}$

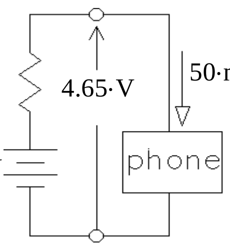
$R_N = 36 \cdot \Omega$

Ex 5 A NiCad Battery pack is used to power a cell phone. When the phone is switched on the battery pack voltage drops from 4.80 V to 4.65 V and the cell phone draws 50 mA. $V_S := 4.80 \cdot V$ $V_{50} := 4.65 \cdot V$

a) Draw a simple, reasonable model of the battery pack using ideal parts.
Find the value of each part.

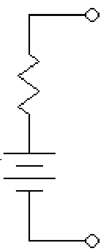
$R_S := \dfrac{V_S - V_{50}}{50 \cdot mA}$

$V_S = 4.8 \cdot V$



$R_S = 3 \cdot \Omega$

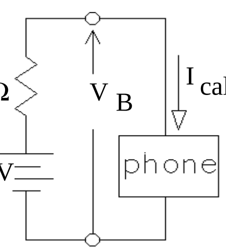
$V_S = 4.8 \cdot V$



b) The cell phone is used to make a call. Now it draws 300 mA.
What is the battery pack voltage now?

$R_S = 3 \cdot \Omega$

$V_S = 4.8 \cdot V$



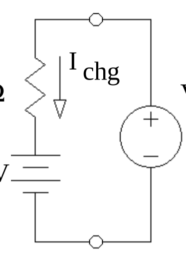
$I_{call} := 300 \cdot mA$

$V_B = V_S - I_{call} \cdot R_S = 3.9 \cdot V$

c) The battery pack is placed in a charger. The charger supplies 5.10 V. How much current flows into the battery pack?

$R_S = 3 \cdot \Omega$

$V_S = 4.8 \cdot V$



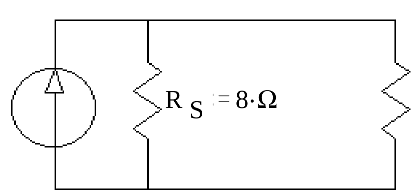
$V_{chg} := 5.10 \cdot V$

$I_{chg} = \dfrac{V_{chg} - V_S}{R_S} = 100 \cdot mA$

Ex 6 Consider the circuit at right.

a) What value of load resistor (R_L) would you choose if you wanted to maximize the power dissipation in that load resistor.

$I_S := 1 \cdot A$

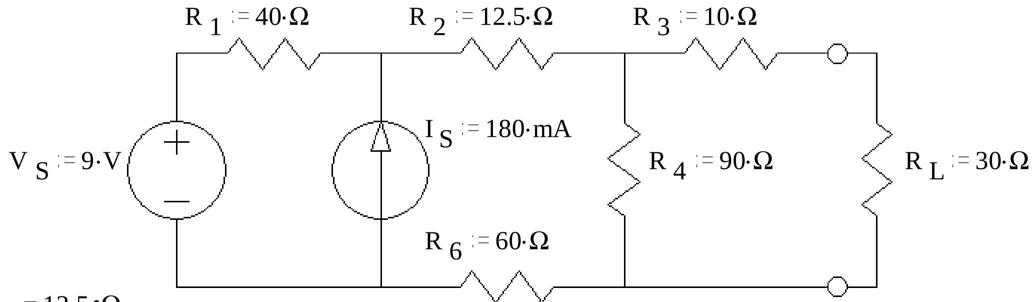


$R_L := R_S$ $R_L = 8 \cdot \Omega$

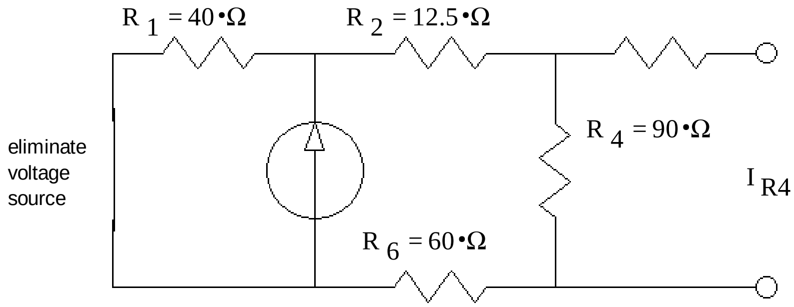
b) With that load resistor (R_L) find the power dissipation in the load.

$I_L := \dfrac{I_S}{2}$ $P_L = I_L^2 \cdot R_L = 2 \cdot W$

Ex 7



Use superposition to find V_{Th}.



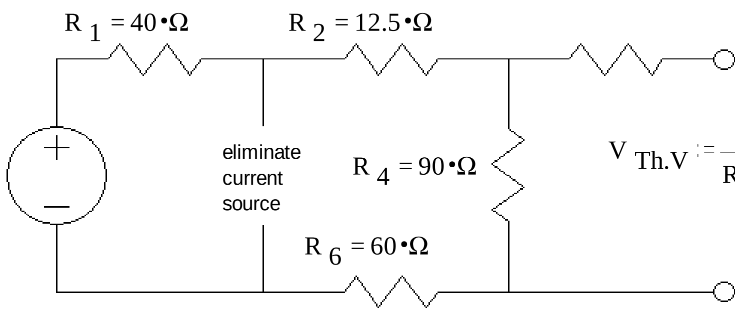
current divider:

$$I_{R4} := \frac{\dfrac{1}{R_2 + R_4 + R_6}}{\dfrac{1}{R_2 + R_4 + R_6} + \dfrac{1}{R_1}} \cdot I_S \qquad I_{R4} = 35.556 \cdot mA$$

$$V_{Th.I} := I_{R4} \cdot R_4$$

$$V_{Th.I} = 3.2 \cdot V$$

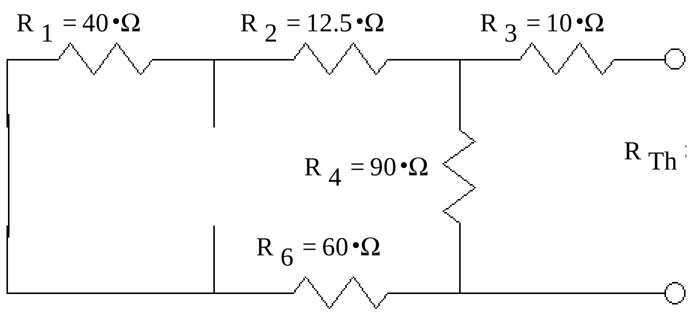
$$R_{246} := R_2 + R_4 + R_6 \qquad R_{246} = 162.5 \cdot \Omega$$



$$V_{Th.V} := \frac{R_4}{R_1 + R_2 + R_4 + R_6} \cdot V_S \qquad V_{Th.V} = 4 \cdot V$$

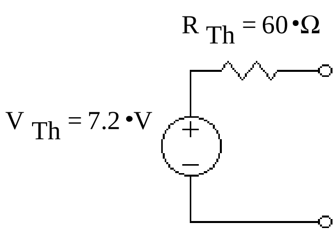
$$V_{Th} := V_{Th.V} + V_{Th.I}$$

$$V_{Th} = 7.2 \cdot V$$

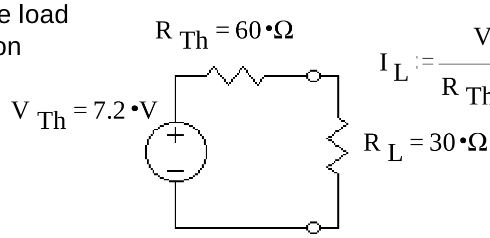


$$R_{Th} := \frac{1}{\dfrac{1}{R_4} + \dfrac{1}{R_1 + R_2 + R_6}} + R_3 \qquad R_{Th} = 60 \cdot \Omega$$

Thévenin equivalent circuit:



Put the load back on



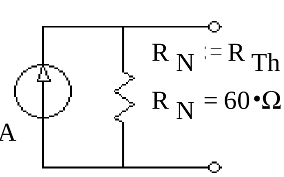
$$I_L := \frac{V_{Th}}{R_{Th} + R_L} \qquad I_L = 80 \cdot mA$$

$$V_L = \; I_L \cdot R_L = 2.4 \cdot V$$

Norton equivalent circuit:

$$I_N := \frac{V_{Th}}{R_{Th}}$$

$$I_N = 120 \cdot mA$$



ECE 2210 Lecture 7 notes Nodal Analysis

A. Stolp
9/10/08,
1/7/13

General Network Analysis

In many cases you have multiple unknowns in a circuit, say the voltages across multiple resistors. Network analysis is a systematic way to generate multiple equations which can be solved to find the multiple unknowns. These equations are based on basic Kirchoff's and Ohm's laws.

Loop or Mesh Analysis You may have used these methods in previous classes, particularly in Physics. The best thing to do now is to forget all that. Loop analysis is rarely the easiest way to analyze a circuit and is inherently confusing. Hopefully I've brought you to a stage where you have some intuitive feeling for how currents flow in circuits. I don't want to ruin that now by screwing around with loop currents that don't really exist.

Nodal analysis This is a much better method. It's just as powerful, usually easier, and helps you develop your intuitive feeling for how circuits work.

Nodal Analysis

Node = all points connected by wire, all at same voltage (potential)

Ground: One node in the circuit which will be our reference node. Ground, by definition, will be the zero voltage node. All other node voltages will be referenced to ground and may be positive or negative. Think of gage pressure in a fluid system. In that case atmospheric pressure is considered zero. If there is no ground in the circuit, define one for yourself. Try to chose a node which is hooked to one side of a voltage source.

Nodal Voltage: The voltage of a node referenced to ground. The objective of nodal analysis is to find all the nodal voltages. If you know the voltage at a node then it's a "known" node. Ground is a known node (duh, it's zero). If one end of a known voltage source hooked to ground, then the node on the other end is also known (another duh).

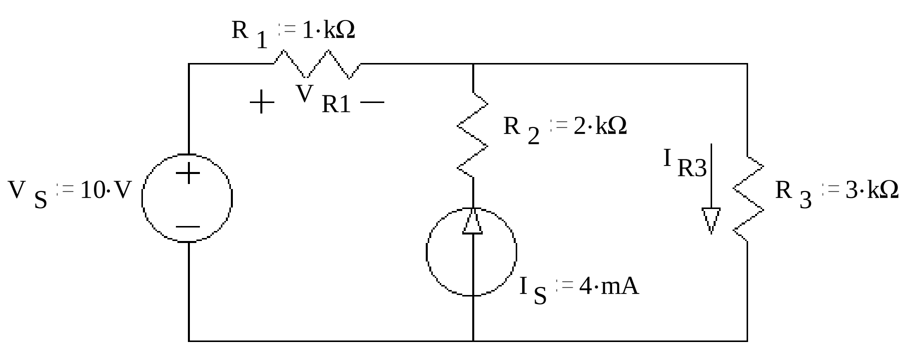
Method: Label all the unknown nodes as; "a", "b", "c", etc. Then the unknown nodal voltages become; V_a, V_b, V_c, etc. Write a KCL equation for each unknown node, defining currents as necessary. Replace each unknown current with an Ohm's law relationship using the nodal voltages. Now you have just as many equations as unknowns. Solve.

Nodal Analysis Steps

1) If the circuit doesn't already have a ground, label one node as ground (zero voltage).
   If the ground can be defined as one side of a voltage source, that will make the following steps easier.
   Label the remaining node, either with known voltages or with letters, a, b, ...
2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...
3) Write Kirchoff's current equations for each unknown node.
4) Replace the currents in your **KCL** equations with expressions like this. $\frac{V_a - V_b}{R_1}$ Ohm's law relationship using the nodal voltages.
5) Solve the multiple equations for the multiple unknown voltages.

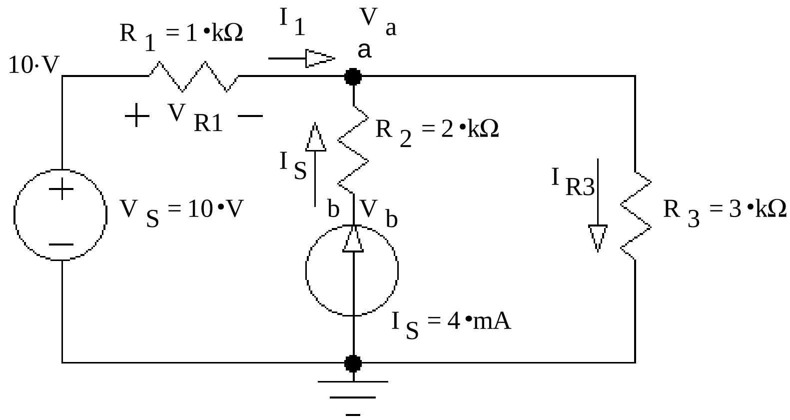
Nodal Analysis Examples

Ex 1 Use nodal analysis to find the voltage across R_1 (V_{R1}).



1) See next page
   Label one node as ground (zero voltage). By choosing the negative side of a voltage source as ground, the upper-left node is known (10V). Label the remaining nodes, either with known voltages or with letters, a, b, ...

2) Label unknown node voltages as V_a, V_b, ...
and label the current in each resistor as I_1, I_2, ...



3) Write Kirchoff's current equations for node a.

$$I_1 + I_S = I_{R3}$$

4) Replace the currents in the **KCL** equations with Ohm's law relationships.

$$\frac{V_S - V_a}{R_1} + I_S = \frac{V_a - 0}{R_3}$$

$$\frac{V_S}{R_1} - \frac{V_a}{R_1} + I_S = \frac{V_a}{R_3}$$

5) Solve:

$$\frac{V_S}{R_1} + I_S = \frac{V_a}{R_3} + \frac{V_a}{R_1}$$

$$\frac{V_S}{R_1} + I_S = V_a \cdot \left(\frac{1}{R_1} + \frac{1}{R_3}\right)$$

$$V_a := \frac{\frac{V_S}{R_1} + I_S}{\left(\frac{1}{R_1} + \frac{1}{R_3}\right)} \qquad V_a = 10.5 \cdot V$$

Usually it's easier to put in the numbers at this point

$$\frac{10 \cdot V}{1 \cdot k\Omega} + 4 \cdot mA = \frac{V_a}{3 \cdot k\Omega} + \frac{V_a}{1 \cdot k\Omega}$$

Multiply both sides by a value that will clear the denominators.

$$3 \cdot k\Omega \cdot \left(\frac{10 \cdot V}{1 \cdot k\Omega} + 4 \cdot mA\right) = \left(\frac{V_a}{3 \cdot k\Omega} + \frac{V_a}{1 \cdot k\Omega}\right) \cdot 3 \cdot k\Omega$$

$$30 \cdot V + 3 \cdot k\Omega \cdot 4 \cdot mA = V_a + 3 \cdot V_a$$

$$30 \cdot V + 12 \cdot V = 4 \cdot V_a$$

$$V_a = \frac{42 \cdot V}{4} = 10.5 \cdot V$$

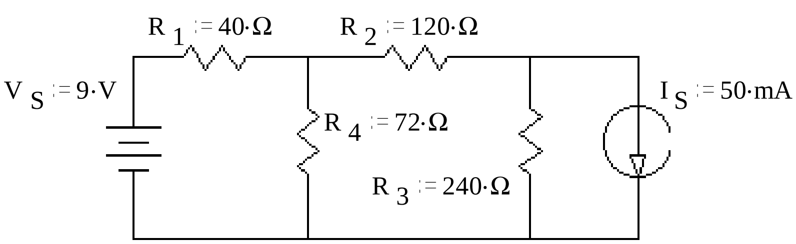
Either way, you still have to find V_{R1} from V_a.

$$V_{R1} := V_S - V_a \qquad V_{R1} = -0.5 \cdot V$$

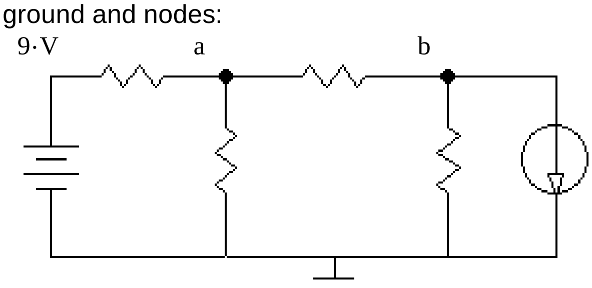
V_b doesn't matter in this case

b) Find the current through R_3 (I_{R3}). $\quad I_{R3} = \frac{V_a}{R_3} = 3.5 \cdot mA$

Ex 2 Same circuit used in Thévenin example, where R_4 was R_L.

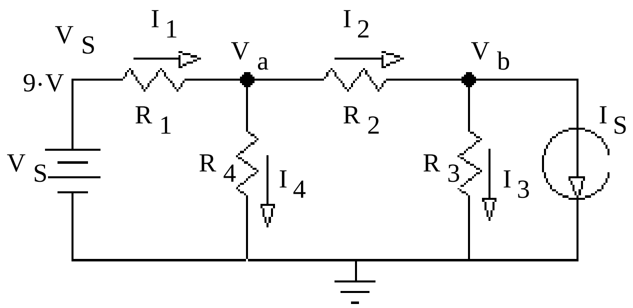


1) Define ground and nodes:



2 unknown nodes ---> will need 2 equations

2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...
It doesn't matter if these currents are in the correct directions.



3) Write Kirchoff's current equations for each unknown node.

node a $\quad I_1 = I_2 + I_4$

node b $\quad I_2 = I_3 + I_S$

4) Replace the currents in your **KCL** equations with expressions like this. $\dfrac{V_a - V_b}{R_1}$

node a $\quad I_1 = I_2 + I_4$

$$\frac{V_S - V_a}{R_1} = \frac{V_a - V_b}{R_2} + \frac{V_a - 0\cdot V}{R_4}$$

node b $\quad I_2 = I_3 + I_S$

$$\frac{V_a - V_b}{R_2} = \frac{V_b - 0\cdot V}{R_3} + I_S$$

Now you have just as many equations as unknowns.

5) Solve the multiple equations for the multiple unknown voltages. Solve by any method you like:

$$\frac{V_S}{R_1} - \frac{V_a}{R_1} = \frac{V_a}{R_2} - \frac{V_b}{R_2} + \frac{V_a}{R_4}$$

$$\frac{V_a}{R_2} - \frac{V_b}{R_2} = \frac{V_b}{R_3} + I_S \qquad V_b = \frac{\dfrac{V_a}{R_2} - I_S}{\dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

$$\frac{V_S}{R_1} - \frac{V_a}{R_1} = \frac{V_a}{R_2} - \frac{\dfrac{V_a}{R_2} - I_S}{R_2\cdot\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)} + \frac{V_a}{R_4}$$

$$V_a := \frac{\left[\dfrac{V_S}{R_1} - \dfrac{1}{\left[R_2\cdot\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)\right]}\cdot I_S\right]}{\left[\dfrac{1}{R_1} + \dfrac{1}{R_2} - \dfrac{1}{R_2^{\,2}\cdot\left(\dfrac{1}{R_2} + \dfrac{1}{R_3}\right)} + \dfrac{1}{R_4}\right]} \qquad V_a = 4.6\,\bullet V$$

$$V_b := \frac{\dfrac{V_a}{R_2} - I_S}{\dfrac{1}{R_2} + \dfrac{1}{R_3}} \qquad V_b = -0.933\,\bullet V$$

Or, with numbers

node a

$$360\cdot\Omega\cdot\left(\frac{9\cdot V - V_a}{40\cdot\Omega}\right) = \left(\frac{V_a - V_b}{120\cdot\Omega} + \frac{V_a}{72\cdot\Omega}\right)\cdot 360\cdot\Omega$$

$$81\cdot V - 9\cdot V_a = 3\cdot V_a - 3\cdot V_b + 5\cdot V_a$$

node b

$$240\cdot\Omega\cdot\frac{V_a - V_b}{120\cdot\Omega} = \left(\frac{V_b - 0\cdot V}{240\cdot\Omega} + 50\cdot mA\right)\cdot 240\cdot\Omega$$

$$2\cdot V_a - 2\cdot V_b = V_b + 48\cdot mA\cdot 240\cdot\Omega$$

$$V_a = \frac{2\cdot V_b + V_b + 12\cdot V}{2} = 1.5\cdot V_b + 6\cdot V$$

<-- substitute for V_a

$$81\cdot V - 9\cdot\left(1.5\cdot V_b + 6\cdot V\right) = 3\cdot\left(1.5\cdot V_b + 6\cdot V\right) - 3\cdot V_b + 5\cdot\left(1.5\cdot V_b + 6\cdot V\right)$$

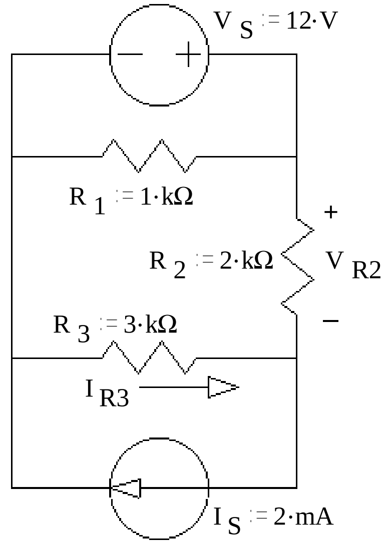
$$81\cdot V - 13.5\cdot V_b - 54\cdot V = 4.5\cdot V_b + 18\cdot V - 3\cdot V_b + 7.5\cdot V_b + 30\cdot V$$

$$81\cdot V - 54\cdot V - 18\cdot V - 30\cdot V = -21\,\bullet V = 4.5\cdot V_b - 3\cdot V_b + 7.5\cdot V_b + 13.5\cdot V_b = 22.5\cdot V_b$$

$$V_b = \frac{-21\cdot V}{22.5} = -0.933\,\bullet V \qquad V_a = 1.5\cdot V_b + 6\cdot V = 4.6\,\bullet V$$

Same as V_L of Ex 4 of Thévenin examples:

Ex 3 Like Superposition Ex.2

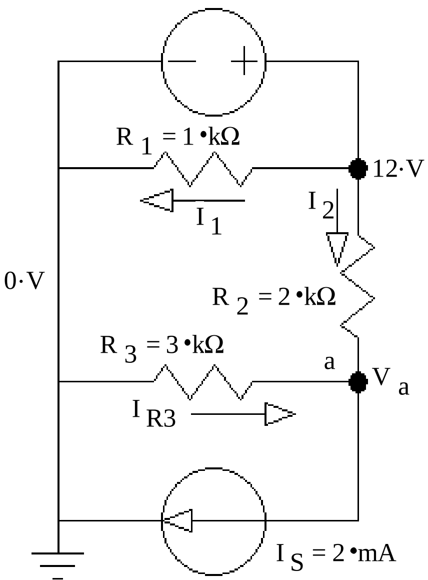


a) Use nodal analysis to find the voltage across R_2 (V_{R2}).

You **MUST** show all the steps of nodal analysis work to get credit, including drawing appropriate symbols and labels on the circuit shown.

1) Define ground and nodes:

2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...



3) Write Kirchoff's current equations for each unknown node.

node a: $I_2 + I_{R3} = I_S$

4) Replace the currents in the **KCL** equations with Ohm's law relationships.

$$\frac{V_S - V_a}{R_2} + \frac{0 - V_a}{R_3} = I_S$$

5) Solve the equation for the unknown voltage.

$$\frac{V_S}{R_2} - \frac{V_a}{R_2} - \frac{V_a}{R_3} = I_S$$

$$\frac{V_S}{R_2} = \frac{V_a}{R_2} + \frac{V_a}{R_3} + I_S$$

$$\frac{V_S}{R_2} - I_S = V_a \cdot \left(\frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$V_a := \frac{\frac{V_S}{R_2} - I_S}{\left(\frac{1}{R_2} + \frac{1}{R_3}\right)} \qquad V_a = 4.8 \cdot V$$

Usually it's easier to put in the numbers at this point

$$\frac{12 \cdot V - V_a}{2 \cdot k\Omega} + \frac{0 - V_a}{3 \cdot k\Omega} = 2 \cdot mA$$

Multiply both sides by a value that will clear the denominators.

$$6 \cdot k\Omega \cdot \left(\frac{12 \cdot V - V_a}{2 \cdot k\Omega} + \frac{0 - V_a}{3 \cdot k\Omega}\right) = 2 \cdot mA \cdot 6 \cdot k\Omega$$

$$36 \cdot V - 3 \cdot V_a - 2 \cdot V_a = 12 \cdot V$$

$$-5 \cdot V_a = -24 \cdot V$$

$$V_a = \frac{-24 \cdot V}{-5} = 4.8 \cdot V$$

Remember, we needed to find the voltage across R_2 (V_{R2}).

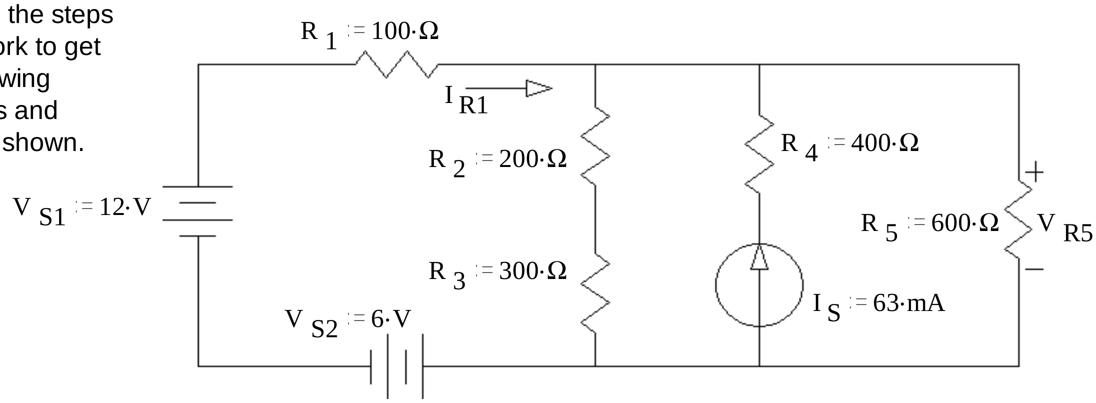
$$V_{R2} = V_S - V_a = 7.2 \cdot V$$

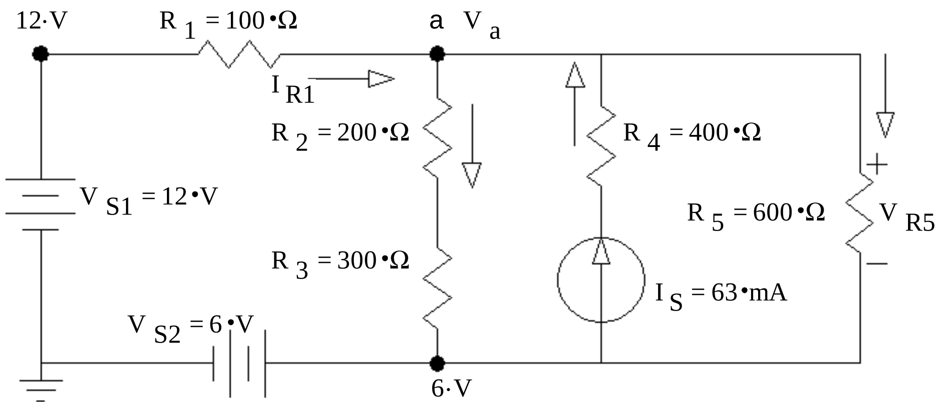
b) Find the current through R_3 (I_{R3}).

$$I_{R3} = \frac{0 - V_a}{R_3} = -1.6 \cdot mA$$ actually flows the other way

Ex 4 Use nodal analysis to find the voltage across R_5 (V_{R5}) and the current through R_1 (I_{R1}). From exam 1, F09

You **MUST** show all the steps of nodal analysis work to get credit, including drawing appropriate symbols and labels on the circuit shown.





node a:

$$I_{R1} + I_S = I_R + I_5$$

$$R_3 = 300\cdot\Omega \qquad \frac{V_{S1} - V_a}{R_1} + I_S = \frac{V_a - V_{S2}}{R_2 + R_3} + \frac{V_a - V_{S2}}{R_5}$$

$$\frac{12\cdot V}{100\cdot\Omega} - \frac{V_a}{100\cdot\Omega} + 63\cdot mA = \frac{V_a}{500\cdot\Omega} - \frac{6\cdot V}{500} + \frac{V_a}{600\cdot\Omega} - \frac{6\cdot V}{600}$$

multiply both sides by 3000Ω

$$3000\cdot\Omega\cdot\left(\frac{12\cdot V}{100\cdot\Omega} - \frac{V_a}{100\cdot\Omega} + 63\cdot mA\right) = \left(\frac{V_a}{500\cdot\Omega} - \frac{6\cdot V}{500} + \frac{V_a}{600\cdot\Omega} - \frac{6\cdot V}{600}\right)\cdot 3000\cdot\Omega$$

$$360\cdot V - 30\cdot V_a + 189\cdot V = 6\cdot V_a - 36\cdot V + 5\cdot V_a - 30\cdot V$$

$$360\cdot V + 189\cdot V + 36\cdot V + 30\cdot V = 6\cdot V_a + 5\cdot V_a + 30\cdot V_a$$

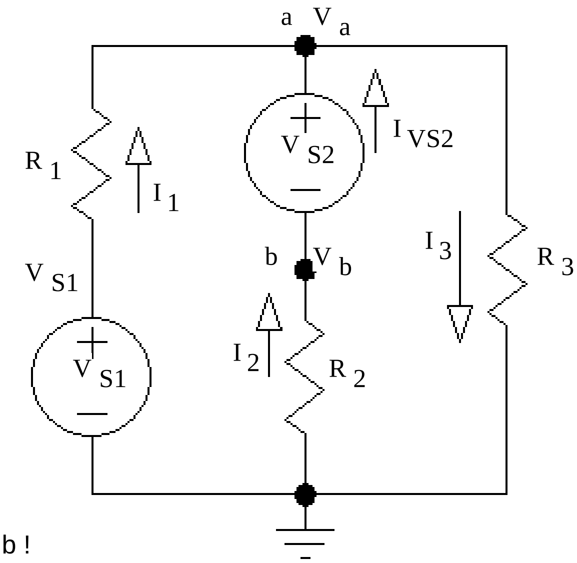
$$615\cdot V = 41\cdot V_a$$

$$V_a := \frac{615\cdot V}{41} \qquad V_a = 15\cdot V$$

$$V_{R5} = V_a - V_{S2} = 9\cdot V$$

$$I_{R1} = \frac{V_{S1} - V_a}{R_1} = -30\cdot mA$$

What if one side of a voltage source isn't ground?



$$I_1 + I_{VS2} = I_3$$

$$\frac{V_{S1} - V_a}{R_1} + \ ? \ = I_S$$

What do you put in for I_{VS2}??

Go to the other side of V_{S2}.

$$\frac{V_{S1} - V_a}{R_1} + \frac{0 - V_b}{R_2} = I_S$$

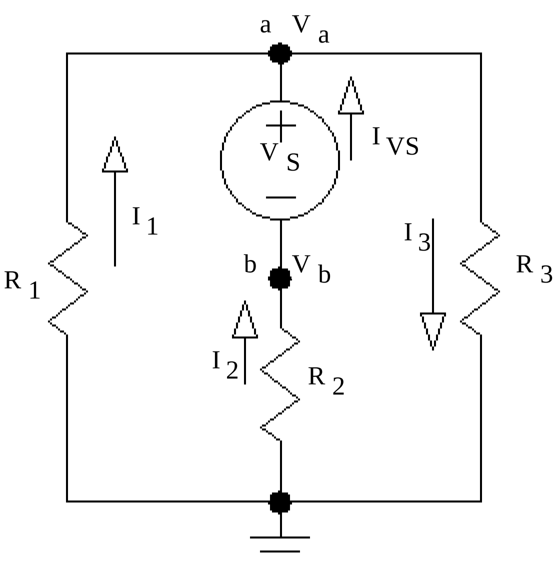
Only problem is that you get the same equation at node b !

Where does the second equation come from?

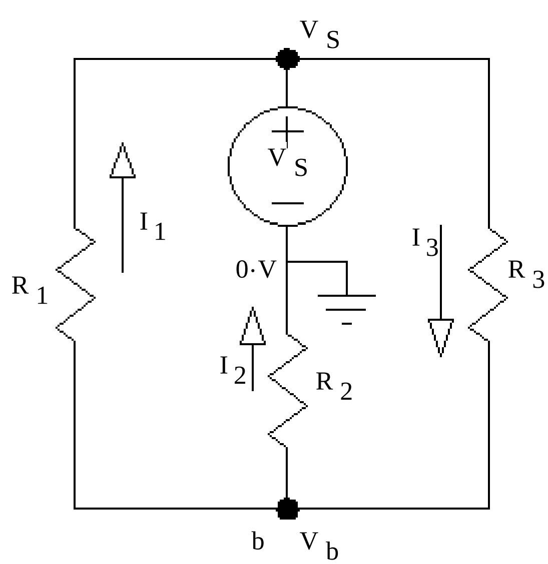
Use something like this: $V_a = V_b + V_{S2}$

Similar Circuit, but no V_{S1}.

If the ground is already at the bottom, use the same method as above.



If you can chose your ground, you can make life a little simpler.



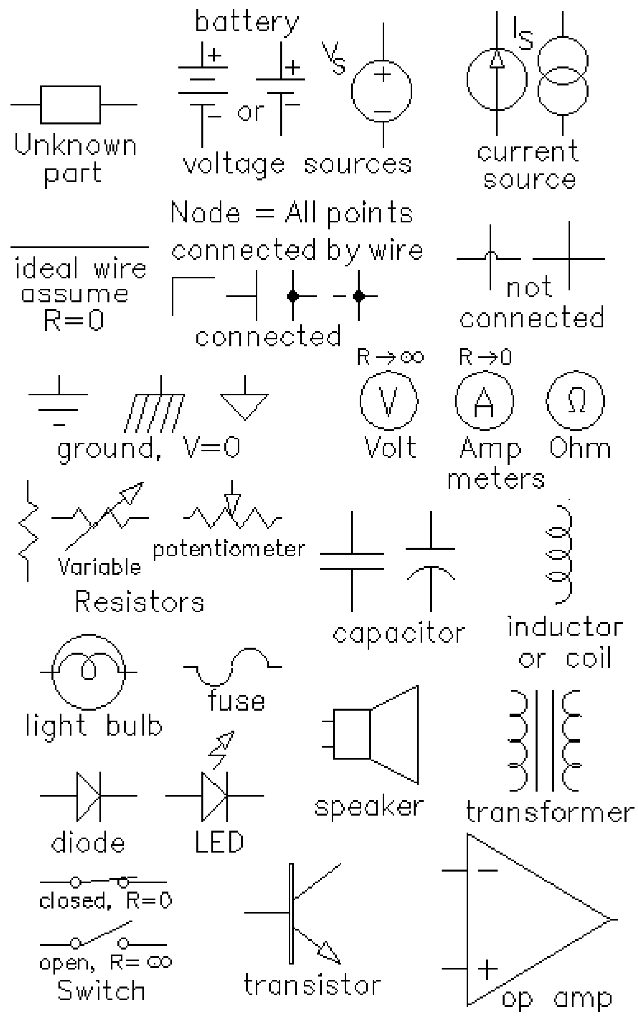
DC Notes

ECE 2210 / 00

A.Stolp
2/8/00, rev 1/28/06

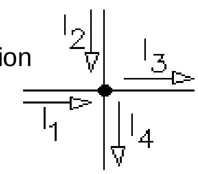
Basic electrical quantities		Unit
Charge, actually moves	Q	Coulomb (C)
Current, like fluid flow	$I = \frac{Q}{s}$	Amp (A, mA, µA,...)
Voltage, like pressure	V	volt (V, mV, kV,...)
Resistance	$R = \frac{V}{I}$	Ohm (Ω, kΩ, MΩ,...)
Conductance	$G = \frac{1}{R}$	Siemens (S, old unit mho)
Power energy/time	$P = V \cdot I$	Watt (W, mW, kW, MW,...)

Schematic symbols



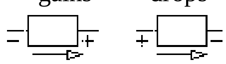
KCL, **K**irchhoff's **C**urrent **L**aw

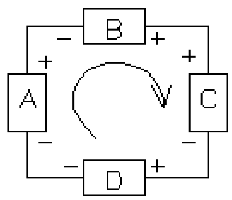
$I_{in} = I_{out}$ of any point, part, or section



KVL, **K**irchhoff's **V**oltage **L**aw

$V_{gains} = V_{drops}$ around any loop

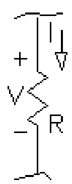




Node = all points connected by wire, all at same voltage (potential)

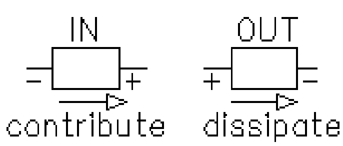
Ohm's law (resistors)

$$V = I \cdot R$$



$$I = \frac{V}{R} \qquad R = \frac{V}{I}$$

Power $P_{IN} = P_{OUT}$ for resistor circuits



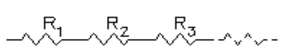
$P = V \cdot I$ for everything

$= I^2 \cdot R = \frac{V^2}{R}$ for resistors

Maximum power transfer: $R_L = R_{Th}$

Load = Thevenin's

Resistors



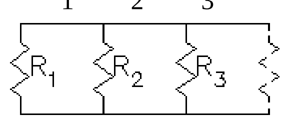
series: $R_{eq} = R_1 + R_2 + R_3 + ...$

Exactly the **same current** through each resistor

Voltage divider:

$$V_{Rn} = V_{total} \cdot \frac{R_n}{R_1 + R_2 + R_3 + ...}$$

parallel: $R_{eq} = \dfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...}$



Exactly the **same voltage** across each resistor

current divider:

$$I_{Rn} = I_{total} \cdot \frac{\frac{1}{R_n}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...}$$

Multiple unknowns:

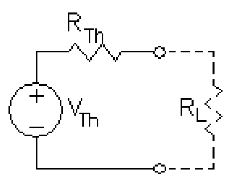
1. Combine resistors into equivalents where possible.
2. Use superposition if there are multiple sources and you know all the resistors.
3. Use KCL, KVL, & Ohm's laws to write multiple equations and solve.

DC Notes

Thévenin equivalent

To calculate a circuit's Thévenin equivalent:

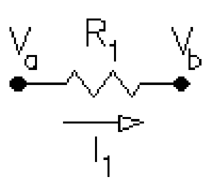
1) Remove the load and calculate the open-circuit voltage where the load used to be. This is the Thévenin voltage (V_{Th}).
2) Zero all the sources. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
3) Compute the total resistance between the load terminals. (DO NOT include the load in this resistance.) This is the Thévenin source resistance (R_{Th}).



4) Draw the Thévenin equivalent circuit and add your values.

Nodal Analysis

1) If the circuit doesn't already have a ground, label one node as ground (zero voltage). If the ground can be defined as one side of a voltage source, that will make the following steps easier.
2) Label unknown node voltages as V_a, V_b, ... and label the current in each resistor as I_1, I_2, ...
3) Write Kirchoff's current equations for each unknown node.
4) Replace the currents in your KCL equations with expressions like the one below.



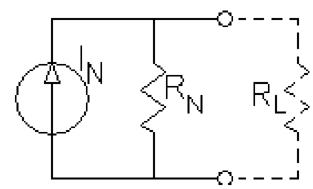
$$I_1 = \frac{V_a - V_b}{R_1}$$

5) Solve the multiple equations for the multiple unknown voltages

Norton equivalent

To calculate a circuit's Norton equivalent:

1) Replace the load with a short (a wire) and calculate the short-circuit current in this wire. This is the Norton current (I_N). Remove the short.
2) Zero all the sources. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
3) Compute the total resistance between the load terminals. (DO NOT include the load in this resistance.) This is the Norton source resistance (R_N). (Exactly the same as theThévenin source resistance (R_{Th})).



4) Draw the Norton equivalent circuit and add your values.

OR (the more common way)...

1) Find the Thévenin equivalent circuit.
2) Convert to Norton circuit, $R_N = R_{Th}$ and $I_N = V_{Th}/R_{Th}$.

Superposition

For circuits with **more than 1 source**.

1) Zero all but one source. (To zero a voltage source, replace it with a short. To zero a current source, replace it with an open.)
2) Compute your wanted voltage or current due to the remaining source. Careful, some may be negative.
3) Repeat the first two steps for all the sources.
4) Sum all the contributions from all the sources to find the actual voltage or current. **Watch your signs!**

ECE 2210 / 00 Lecture 8 Notes Basic AC

A. Stolp
1/27/03,
2/27/07

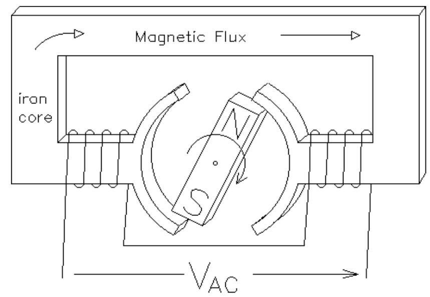
AC stands for **A**lternating **C**urrent as opposed to DC, **D**irect **C**urrent. AC refers to voltages and currents that change with time, usually the voltage is + sometimes and - at other times. This results in currents with go one direction when the voltage is + and the reverse direction when the voltage is -.

AC is important for two reasons. Power is created and distributed as AC. Signals are AC.

AC Power

Power is generated by rotating magnetic fields.
This naturally produces sinusoidal AC waveforms.

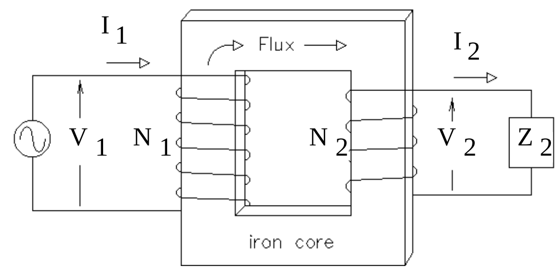
It is easier to make AC motors than DC motors.



AC Power allows use of transformers to reduce line losses

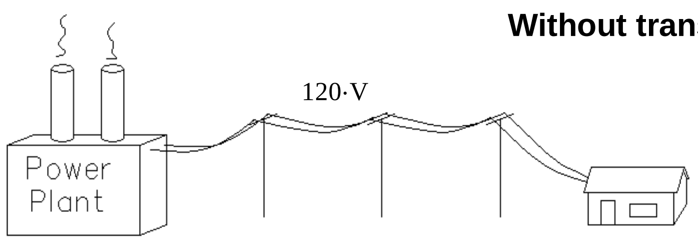
Transformers work with AC, but not DC. Transformers can be used to raise or lower AC voltages (with an opposite change of current). This can be very useful in power distribution systems. Power is voltage times current. You can distribute the same amount of power with high voltage and low current as you can with low voltage and high current. However, the lower the current, the lower the I^2R loses in the wires (all real wires have some resistance). So you'd like to distribute power at the highest possible voltage. Transformers allow you to do this with AC, but won't work with DC.

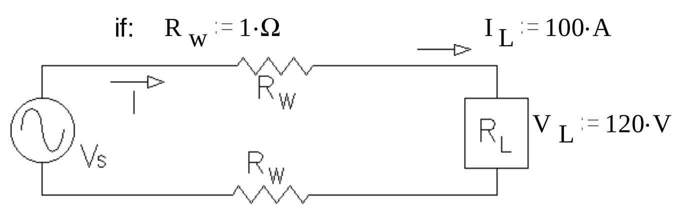
<u>Iron-core transformer</u>
primary secondary



Example:

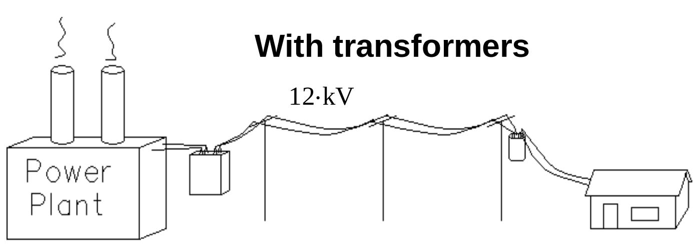
Without transformers

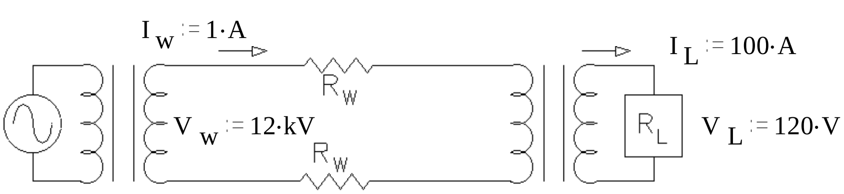




Wire loss: $P_W = I_L^{\,2}\cdot 2\cdot R_w = 20\cdot kW$

With transformers





Wire loss: $P_W = I_w^{\,2}\cdot 2\cdot R_w = 2\cdot W$

In this example, the power lost in the transmission lines is only 1/10,000th what it is without transformers.

That's why they raise the voltage in transmission lines to the point where they crackle and buzz. That crackle is the sound of the losses into the surrounding air and can become significant if the voltage is *too* high.

Signals

A time-varying voltage or current that carriers information. If it varies in time, then it has an AC component.

| |
| Audio, video, position, temperature, digital data, etc...

In some unpredictable fashion

DC is not a signal, Neither is a pure sine wave. If you can predict it, what information can it provide?
Neither DC nor pure sine wave have any "bandwidth". In fact, no periodic waveform is a signal & no periodic waveform has bandwidth. You need bandwidth to transmit information.

Signal sources

Microphone	Audio
Camera	Video
Thermistor or other thermal sensor	Temperature
Potentiometer	Position
LVDT (Linear Variable Differential Transformer)	Position
Light sensor	
Computer	
switch	
etc...	

A transducer is a device which transforms one form of energy to another. Some sensors are transducers, many are not

Most often a signal comes from some other system.

Periodic waveforms: Waveshape repeats

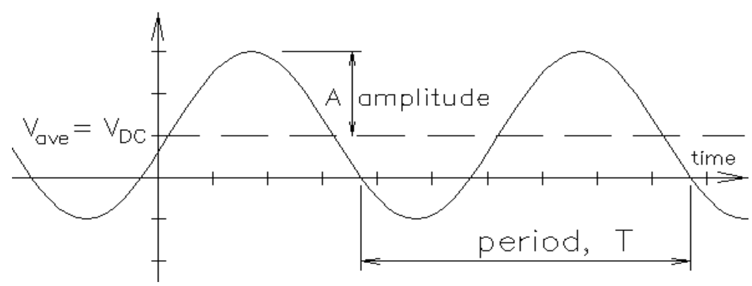
T = Period = repeat time

f = frequency, cycles / second $\quad f = \frac{1}{T} = \frac{\omega}{2\cdot\pi}$

ω = radian frequency, radians/sec $\quad \omega = 2\cdot\pi\cdot f$

A = amplitude

DC = average



Sinusoidal AC

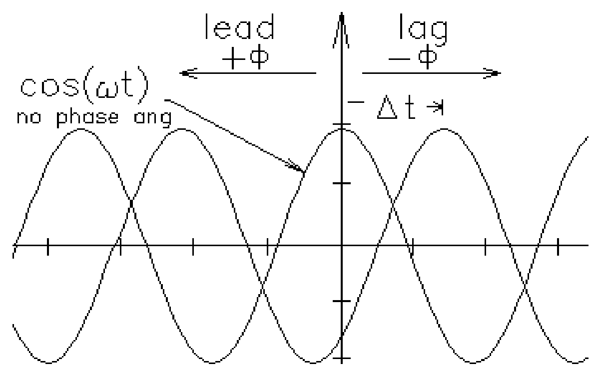
$y(t) = A\cdot\cos(\omega\cdot t + \phi)$

voltage: $v(t) = V_p\cdot\cos(\omega\cdot t + \phi)$

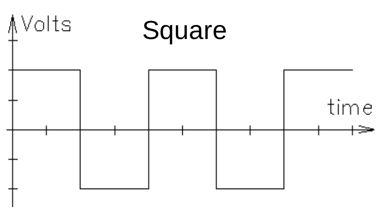
current: $i(t) = I_p\cdot\cos(\omega\cdot t + \phi)$

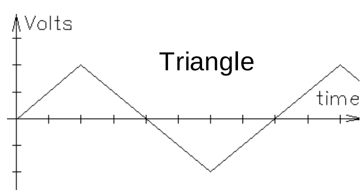
Phase: $\phi = -\frac{\Delta t}{T}\cdot 360\cdot\text{deg}$ or: $\phi = -\frac{\Delta t}{T}\cdot 2\cdot\pi\cdot\text{rad}$

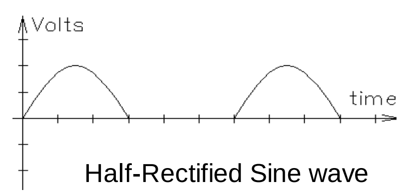
Phase:



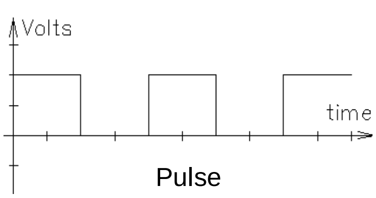
Other common periodic waveforms



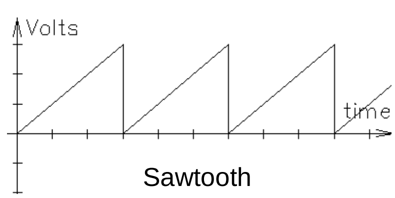




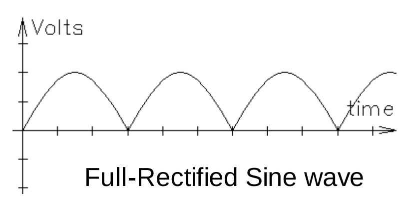
Half-Rectified Sine wave



Pulse



Sawtooth



Full-Rectified Sine wave

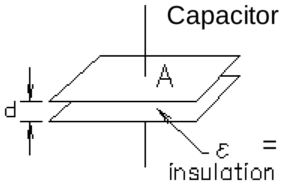
All but the square and triangle waves have a DC component as well as AC.

ECE 2210 / 00 Capacitor Lecture Notes

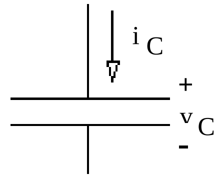
A. Stolp
2/17/03
rev 9/16/09
12/15 & 9/19

Now that we have voltages and currents which can be functions of time, it's time to introduce the capacitor and the inductor.

Electrical equivalent:

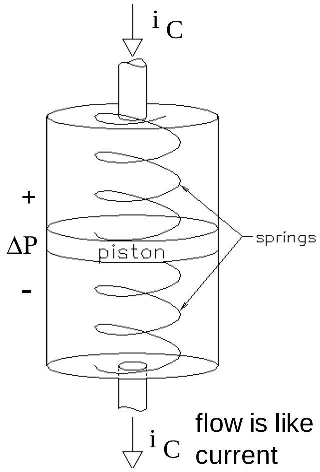


ε = permittivity



like pressure ΔP

Fluid Model:



flow is like current

$$C = \varepsilon \cdot \frac{A}{d} = \frac{Q}{V} = \frac{dq}{dv}$$

Units: $\text{farad} = \frac{\text{coul}}{\text{volt}} = \frac{\text{amp} \cdot \text{sec}}{\text{volt}}$ $\mu F = 1 \cdot 10^{-6} \cdot \text{farad}$ $pF = 1 \cdot 10^{-12} \cdot \text{farad}$

For drawings of capacitors and info about tolerances, see Ch.3 of textbook.

Basic equations you should know: $C = \frac{Q}{V}$

$$v_C = \frac{1}{C} \cdot \int_{-\infty}^{t} i_C \, dt$$

$$i_C = C \cdot \frac{d}{dt} v_C$$

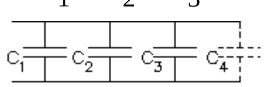
Or... $$v_C = \frac{1}{C} \cdot \int_{0}^{t} i_C \, dt + v_C(0)$$ initial voltage

Or... $$\Delta v_C = \frac{1}{C} \cdot \int_{t_1}^{t_2} i_C \, dt$$

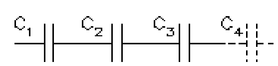
Energy stored in electric field: $W_C = \frac{1}{2} \cdot C \cdot V_C^{\,2}$

Capacitor voltage **cannot** change instantaneously

parallel: $C_{eq} = C_1 + C_2 + C_3 + \ldots$



series: $$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots}$$



Capacitors are the only "backwards" components.

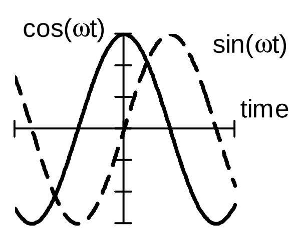
Sinusoids

$i_C(t) = I_p \cdot \cos(\omega \cdot t)$

$$v_C(t) = \frac{1}{C} \cdot \int i_C \, dt = \underbrace{\frac{1}{C} \cdot \frac{1}{\omega} \cdot I_p}_{V_p} \cdot \sin(\omega \cdot t) = \underbrace{\frac{1}{C} \cdot \frac{1}{\omega} \cdot I_p}_{V_p} \cdot \cos(\omega \cdot t - 90 \cdot \text{deg})$$

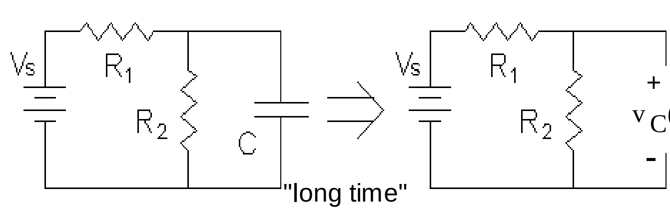
indefinite integral

Voltage "lags" current, makes sense, current has to flow in first to charge capacitor.



Steady-state or Final conditions

If a circuit has been connected for "a long time", then it has reached a steady state condition. that means the currents and voltages are no longer changing.



$$v_C(\infty) = V_S \cdot \frac{R_2}{R_1 + R_2}$$

$\frac{d}{dt} v_C = 0$ $i_C = C \cdot \frac{d}{dt} v_C = 0$

no current means it looks like an open

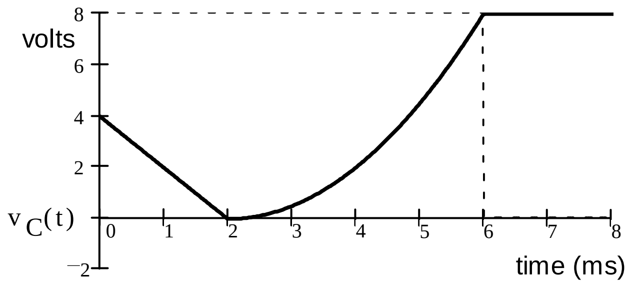
Example

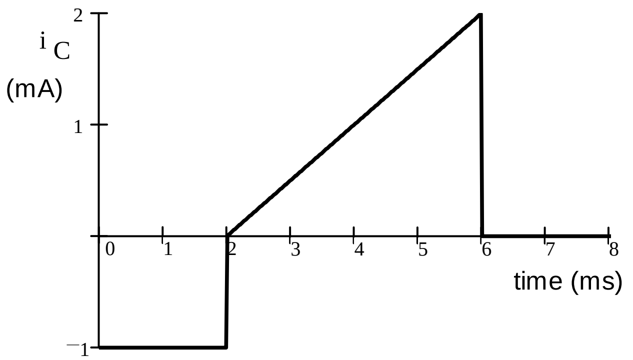
The voltage across a $0.5\ \mu F$ capacitor is shown below. Make an accurate drawing of the capacitor current. Label the y-axis of your graph (I've already done the time-axis).

The accuracy of your plot at 0, 2, 6, and 8 ms is important, so calculate those values and plot or label them carefully. Between those points your plot must simply be the correct shape.

$C := 0.5\cdot\mu F$

The curve is 2nd order





1 - 2ms: $i_C = C\cdot\frac{\Delta V}{\Delta t} = 0.5\cdot\mu F\cdot\frac{-4\cdot V}{2\cdot ms} = -1\cdot mA$

2ms - 6ms: Initial slope is zero and the final slope is positive, so the current must be a triangle that starts at zero and ends at some height.

$$\Delta v_C(t) = \frac{1}{C}\cdot\int_0^t i_C(t)\,dt$$

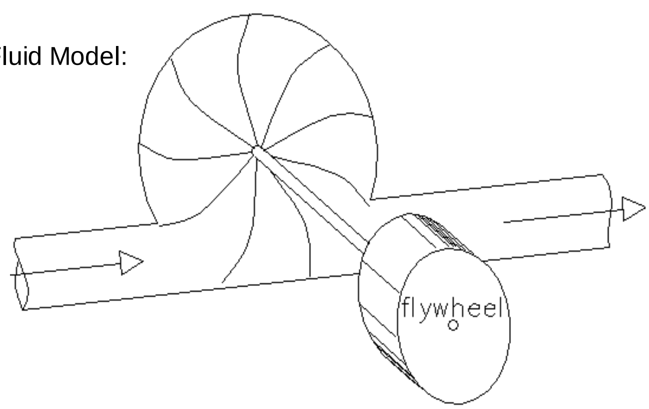
$$8\cdot V = \frac{1}{C}\cdot\left(\frac{4\cdot ms\cdot height}{2}\right)$$

$$height = 8\cdot V\cdot\frac{C\cdot 2}{4\cdot ms} = 2\cdot mA$$

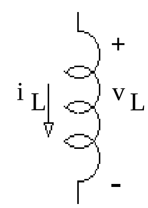
6ms - 8ms: Slope is zero, so the current must be zero.

ECE 2210 / 00 Inductor Lecture Notes

Fluid Model:



Electrical equivalent:



$L = \mu_o\cdot N^2\cdot K$

μ is the permeability of the inductor core

K is a constant which depends on the inductor geometry

N is the number of turns of wire

Basic equations you should know: $v_L = L\cdot\frac{d}{dt}i_L$

$$i_L = \frac{1}{L}\cdot\int_{-\infty}^{t} v_L\,dt$$

initial current

Or... $i_L = \frac{1}{L}\cdot\int_0^t v_L\,dt + i_L(0)$

Or... $\Delta i_L = \frac{1}{L}\cdot\int_{t_1}^{t_2} v_L\,dt$

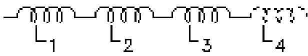
Energy stored in electric field: $W_L = \frac{1}{2}\cdot L\cdot I_L^{\,2}$

Inductor current **cannot** change instantaneously

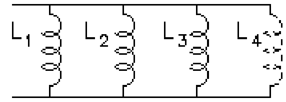
Units: $henry = \frac{volt\cdot sec}{amp}$

$mH = 10^{-3}\cdot H \qquad \mu H = 10^{-6}\cdot H$

series: $L_{eq} = L_1 + L_2 + L_3 + \ldots$



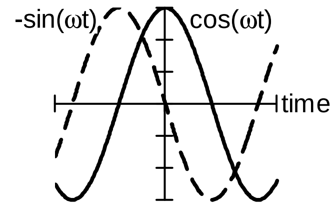
parallel: $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + \dfrac{1}{L_3} + \ldots}$



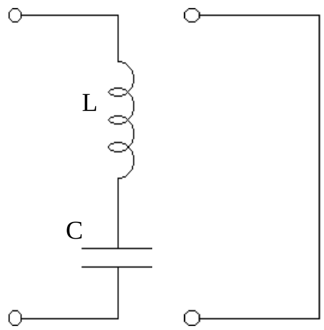
Sinusoids $i_L(t) = I_p \cdot \cos(\omega \cdot t)$

$$v_L(t) = L \cdot \frac{d}{dt} i_L = \underbrace{L \cdot \omega}_{V_p} \cdot \left(-I_p \cdot \sin(\omega \cdot t)\right) = \underbrace{L \cdot \omega \cdot I_p}_{V_p} \cdot \cos(\omega \cdot t + 90 \cdot \deg)$$

Voltage "leads" current, makes sense, voltage has to present to make current change, so voltage comes first.

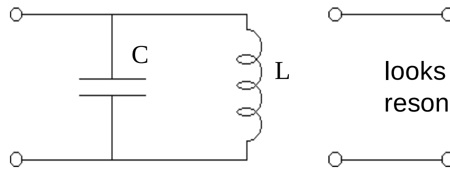


Resonance Series resonance



looks like a short at resonance frequency

Parallel resonance



looks like an open at resonance frequency

The resonance frequency is calculated the same way for either case:

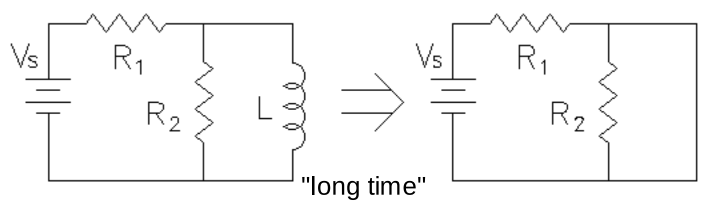
$\omega_o = \dfrac{1}{\sqrt{L \cdot C}} \left(\dfrac{rad}{sec}\right)$ OR.. $\omega_o = \dfrac{1}{\sqrt{L_{eq} \cdot C_{eq}}}$ If you have multiple capacitors or inductors which can be combined. $f_o = \dfrac{\omega_o}{2 \cdot \pi}$ (Hz)

Steady-state of Final conditions

If a circuit has been connected for "a long time", then it has reached a steady state condition. that means the currents and voltages are no longer changing.

$\dfrac{d}{dt} i_L = 0$ $v_L = L \cdot \dfrac{d}{dt} i_L = 0$

no voltage means it looks like a short



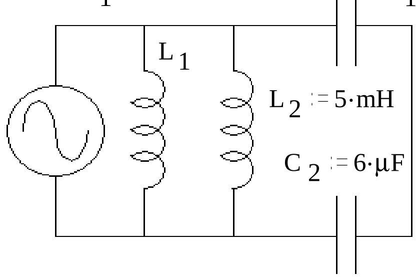
$i_L(\infty) = \dfrac{V_S}{R_1}$

Examples

Ex 1

Find the resonant frequency (or frequencies) of the circuit shown (in cycles/sec or Hz).

$L_1 := 5 \cdot mH$ $C_1 := 6 \cdot \mu F$



$L_2 := 5 \cdot mH$ $C_2 := 6 \cdot \mu F$

$L_{eq} := \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}$ $L_{eq} = 2.5 \cdot mH$

$C_{eq} := \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$ $C_{eq} = 3 \cdot \mu F$

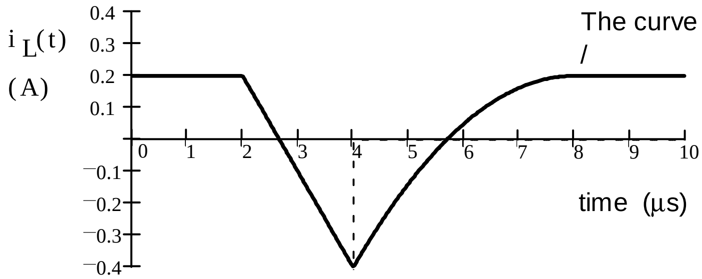
$\omega_o := \dfrac{1}{\sqrt{L_{eq} \cdot C_{eq}}}$ $\omega_o = 11547 \cdot \dfrac{rad}{sec}$ $f_o = \dfrac{\omega_o}{2 \cdot \pi} = 1838 \cdot Hz$

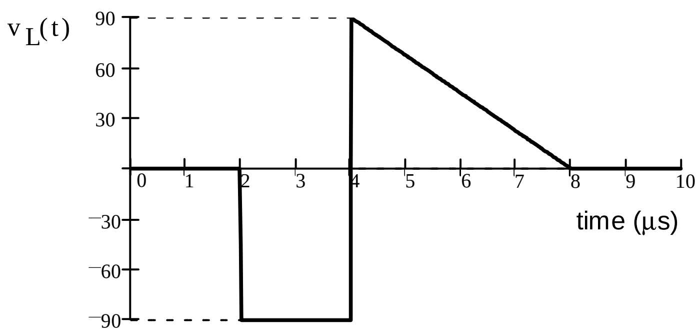
Ex 2

The current through a 0.3mH inductor is shown below. Make an accurate drawing of the inductor voltage. Make reasonable assumptions where necessary. Label your graph.

$L := 0.3\cdot mH$

The curve is 2nd order and ends at 8μs





0 - 2μs: No change in current, so: $v_L = 0$

2μs - 4μs: $v_L = L\cdot\frac{\Delta I}{\Delta t} = 0.3\cdot mH\cdot\frac{-0.6\cdot A}{2\cdot\mu s} = -90\cdot V$

4μs - 8μs: Initial slope is positive and the final slope is zero, so the voltage must be a triangle that starts at some height and ends at zero.

$$\Delta i_L(t) = \frac{1}{L}\cdot\int_0^t v_L(t)\,dt$$

$$0.6\cdot A = \frac{1}{0.3\cdot mH}\cdot\left(\frac{4\cdot\mu s\cdot height}{2}\right)$$

$$height = 0.6\cdot A\cdot\frac{0.3\cdot mH\cdot 2}{4\cdot\mu s} = 90\cdot V$$

8μs - 10μs: No change in current, so: $v_L = 0$

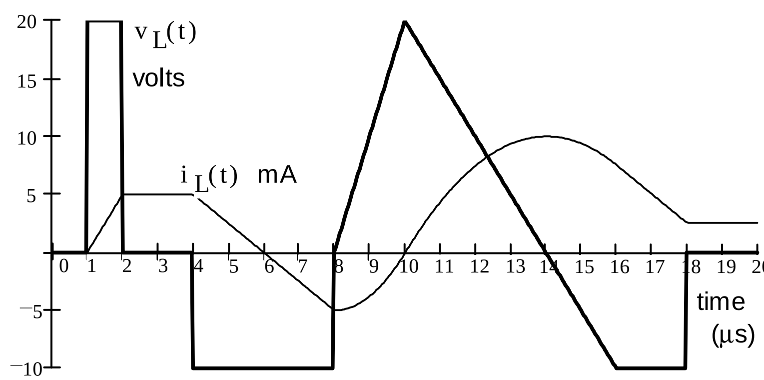
Ex 3 Given a voltage, find the current, $L := 4\cdot mH$

$$\Delta i_L(t) = \frac{1}{L}\cdot\int_{1\cdot\mu s}^{2\cdot\mu s} 20\cdot V\,dt = 5\cdot mA$$

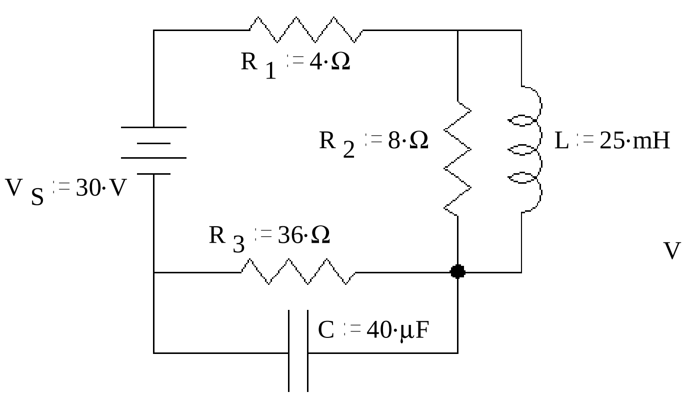
$$\frac{1}{L}\cdot\int_{4\cdot\mu s}^{8\cdot\mu s} -10\cdot V\,dt + 5\cdot mA = -5\cdot mA$$

$$\frac{1}{L}\cdot\int_{8\cdot\mu s}^{10\cdot\mu s} V(t)\,dt + -5\cdot mA$$

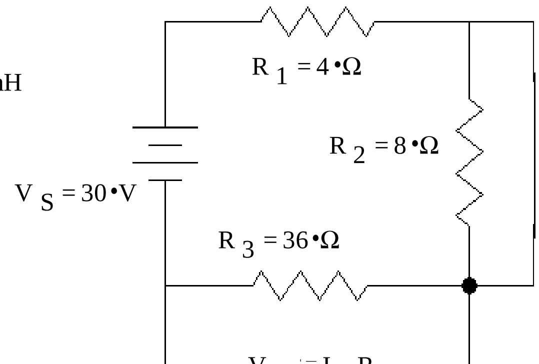
$$= \frac{1}{L}\cdot\frac{20\cdot V\cdot 2\cdot\mu s}{2} - 5\cdot mA = 0\cdot mA \quad \text{etc...}$$



Ex 4 The following circuit has been connected as shown for a long time. Find the energy stored in the capacitor and the inductor.



Redraw:



$V_C := I_L\cdot R_3$

$V_C = 27\cdot V$

$W_C := \frac{1}{2}\cdot C\cdot V_C^2$

$I_L := \frac{V_S}{R_1 + R_3}$

$I_L = 0.75\cdot A$

$W_L := \frac{1}{2}\cdot L\cdot I_L^2$

$W_L = 7.031\cdot mJ$

$W_C = 14.58\cdot mJ$

Capacitor, Inductor Notes

ECE 2210 / 00

A.Stolp
2/27/00,
9/13/05

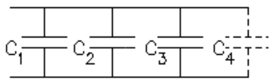
Capacitors

$C = \frac{Q}{V}$ $\quad$ $\text{farad} = \frac{\text{coul}}{\text{volt}} = \frac{\text{amp}\cdot\text{sec}}{\text{volt}}$ $\quad$ $v_C = \frac{1}{C}\cdot\int_{-\infty}^{t} i_C\,dt = \frac{1}{C}\cdot\int_{0}^{t} i_C\,dt + \underset{\text{initial voltage}}{v_C(0)}$ $\quad$ $i_C = C\cdot\frac{d}{dt}v_C$

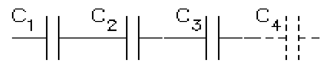
Energy stored in electric field: $W_C = \frac{1}{2}\cdot C\cdot V_C^{\,2}$

Capacitor voltage **cannot** change instantaneously

parallel: $C_{eq} = C_1 + C_2 + C_3 + \dots$



series: $C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} + \dots}$



Steady-state sinusoids:
Impedance: $Z_C = \frac{1}{j\cdot\omega\cdot C} = \frac{-j}{\omega\cdot C}$ Current leads voltage by 90 deg

Inductors

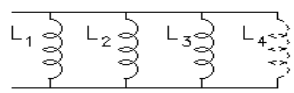
$\text{henry} = \frac{\text{volt}\cdot\text{sec}}{\text{amp}}$ $\quad$ $i_L = \frac{1}{L}\cdot\int_{-\infty}^{t} v_L\,dt = \frac{1}{L}\cdot\int_{0}^{t} v_L\,dt + \underset{\text{initial current}}{i_L(0)}$ $\quad$ $v_L = L\cdot\frac{d}{dt}i_L$

Energy stored in magnetic field: $W_L = \frac{1}{2}\cdot L\cdot I_L^{\,2}$

Inductor current **cannot** change instantaneously

series: $L_{eq} = L_1 + L_2 + L_3 + \dots$

parallel: $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + \dfrac{1}{L_3} + \dots}$



Steady-state sinusoids:
Impedance: $Z_L = j\cdot\omega\cdot L$ Current lags voltage by 90 deg

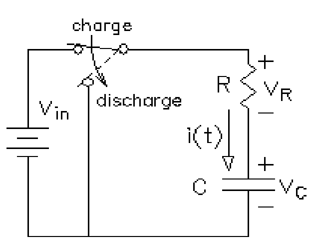
RC and RL first-order transient circuits

For all first order transients: $v_X(t) = v_X(\infty) + \left(v_X(0) - v_X(\infty)\right)\cdot e^{-\frac{t}{\tau}}$ $\quad$ $i_X(t) = i_X(\infty) + \left(i_X(0) - i_X(\infty)\right)\cdot e^{-\frac{t}{\tau}}$

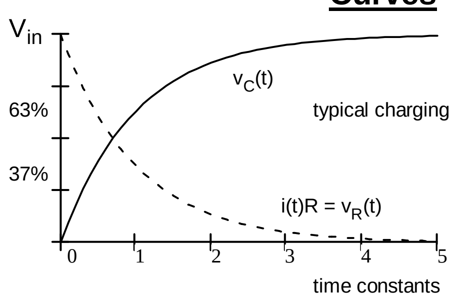
Find initial Conditions (v_C and/or i_L)
Find conditions just before time t = 0, $v_C(0-)$ and $i_L(0-)$. These will be the same just after time t = 0, $v_C(0+)$ and $i_L(0+)$ and will be your initial conditions. (If initial conditions are zero: Capacitors are shorts, Inductors are opens.)
Use normal circuit analysis to find your desired variable: $v_X(0)$ or $i_X(0)$

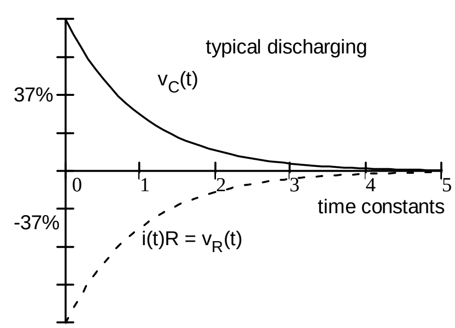
Find final conditions ("steady-state" or "forced" solution)
Inductors are shorts Capacitors are opens Solve by DC analysis $v_X(\infty)$ or $i_X(\infty)$

RC Time constant $= \tau = RC$

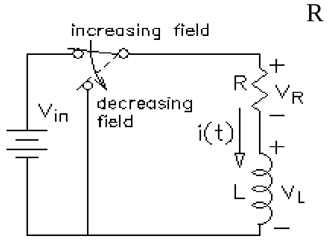


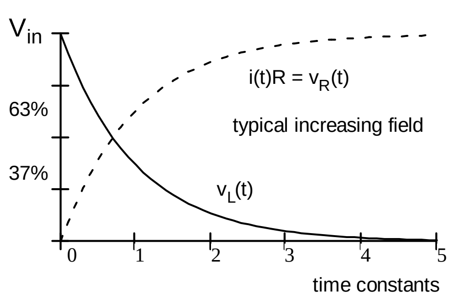
Curves

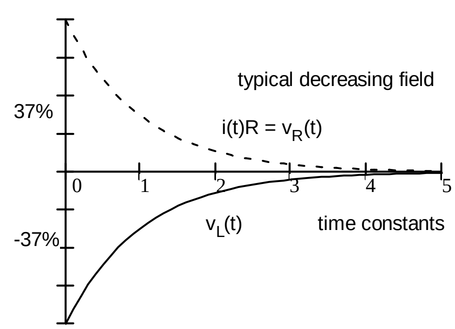




RL Time constant $= \tau = \frac{L}{R}$







$e^{-1} = 0.368$ $\quad$ $1 - e^{-1} = 0.632$

Circuit Transients

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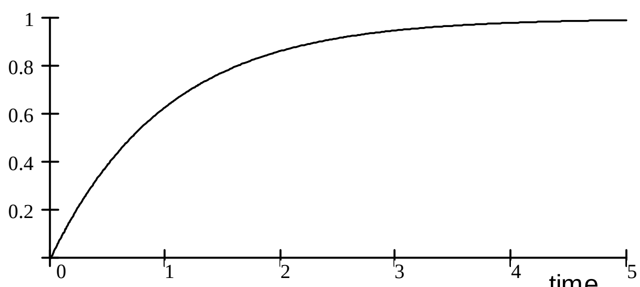
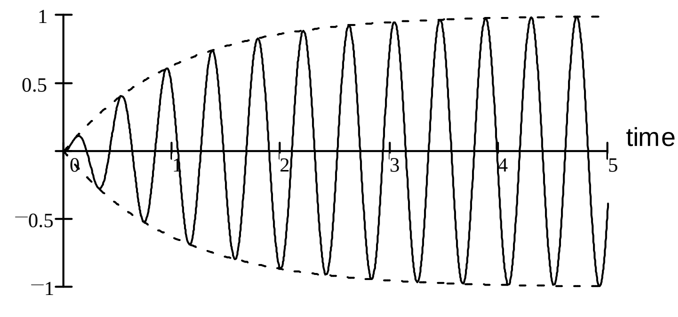
3/29/01,
9/29/04

1.1 Introduction

Transient: A transient is a transition from one state to another. If the voltages and currents in a circuit do not change with time, we call that a "steady state". In fact, as long as the voltages and currents are steady AC sinusoidal values, we can call that a steady state as well. Up until now we've only discussed circuits in a single steady state But what happens when the state of a circuit changes, say from "off" to "on"? Can the state of the circuit change instantaneously? No, nothing ever changes instantaneously, the circuit state will go through some transition from the initial state, "off" to the final state, "on" and that change will take some amount of time. The same is true in mechanical systems. If you want to change the velocity of a mass or the level of fluid in a tank or the temperature of your coffee, that transition from one state to another will take some time.

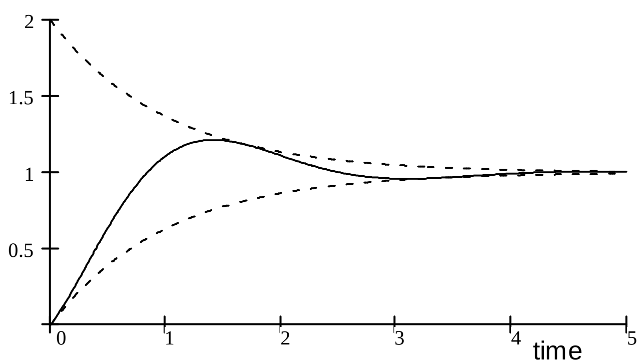


Simple exponential curve

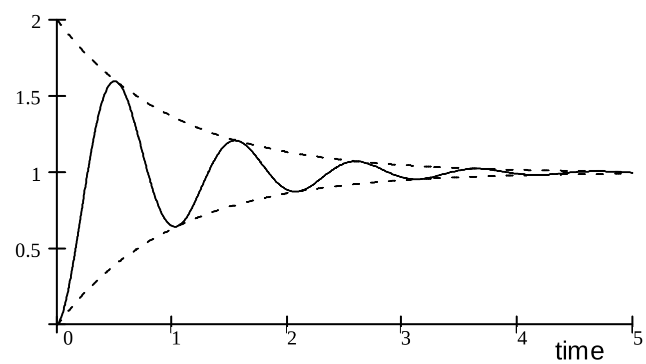


An exponential buildup of a sine wave

The drawings on this page show some typical transients that can occur when a circuit is first turned on. The initial state of all the waveforms is 0. The final state is either 1 or a sine wave with an amplitude of 1. Notice that in all four cases the transient effects decay exponentially and that all four waveforms have pretty nearly reached their steady-state values by the end of the graph.



Overshoot



Ringing

Transient analysis: Needless to say, the analysis of these transients is a bit more involved than the steady state. In fact, it usually involves two steady state analyses just to find the initial and final states of the circuit, and then you still need to figure out what happens in between.

Transients are not instant because capacitors and inductors in the circuit store energy, and moving the energy into or out of these parts takes some time. The voltage-current relationships of capacitors and inductors are differential equations, so transient analysis will involve solving differential equations. But don't panic, you'll learn some nice tricks and techniques for dealing with these equations— tricks and techniques that you can use in any engineering field, not just EE. Actually, all that phasor stuff you used with AC circuits was also a trick to simplify the differential equations, unfortunately, that trick only works for sinusoids in steady state.

DC circuits with only resistors also experience transients, but these are due to non-ideal capacitance and inductance of the parts and wires that we haven't considered before. These transients happen so fast that we won't worry about them.

Importance: So why are transients important? Two reasons really. DC and steady-state AC are fine for moving and using electrical power, but sometimes you need to turn them on and off and you may need to know what happens at those times. That need turns out to be relatively rare and probably couldn't justify the time we'll spend studying transients. It's signals processing and control systems really drive our study of transients.

Signals are electrical voltages and currents that carry information. The information could be audio or video or the information might be about the position or speed of mechanical parts, or about the temperature or level of fluids or chemicals or practically anything you can imagine. To carry information signals have to change in some way that we can't predict and we'll need to have some idea how a circuit will respond to those changes. Changes are transients. However, since these changes can't be known beforehand we usually find a circuit's response to specific types of inputs and then draw conclusions about the effectiveness or stability in the general case. Often the electrical circuit is just one part of a larger system that may include mechanical, hydraulic, or thermal systems. See box.

Printer Design

Lets think about some of the transients and signals involved with moving a print head and putting ink on a page of paper.

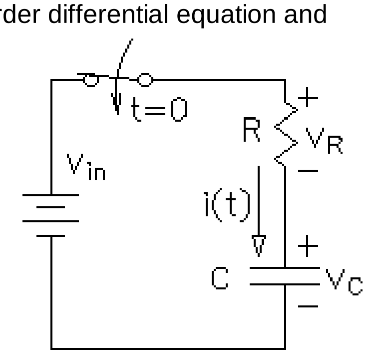
First, there's the mechanical system to move the print head. How quickly does the movement respond to an electrical signal sent to the motor? How powerful do those signals have to be? Does it have a natural frequency where it might vibrate of oscillate? These are all questions for the transient analysis of the mechanical system.

The electrical circuit would take a signal from some sensor that indicates the position of the print head and, using other information about where the next character should be printed, send the right signals to the motor. You'd use transient analysis to make sure that it could handle any combination of inputs without overshooting the position or oscillating or going too slowly. Besides this, the electrical system may have to compensate for properties of the mechanical system.

Finally, there's the system that actually puts the ink on the paper, let's say it's an ink jet. Transient considerations here would include the time it takes for the print head to heat the ink to the point where it spits a bubble and how that should all be timed with the head movement to place that bubble on the paper at just the right place.

1.2 First-order transients

Analysis of a circuit with only one capacitor or one inductor results in a first-order differential equation and the transients are called first-order transients.



Series RC circuit, traditional way: Look at the circuit at right. It shows a capacitor and a resistor connected to a voltage source by way of a switch that is closed at time t=0. Before the switch is closed the current i(t) and the voltage v_R are both 0, but the voltage v_C is unknown. Remember a capacitor is capable of storing a charge, so we don't know what its charge might be unless we or can measure it or its is given. I'll call it the initial voltage ($v_C(0)$). Because the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor just after the switch closes must be the same as it was just before the switch closes.

Now we just have to apply the basic circuit laws

$$V_{in} = v_R + v_C \qquad i \cdot R + v_C = i \cdot R + \frac{1}{C} \cdot \int_{-\infty}^{t} i_C \, dt$$ Making the obvious substitution.

The next step here would be to differential both sides of the equation, but if you're a little more clever, there's an easier way, check this out:

Make this substitution instead $i = i_C = C \cdot \frac{d}{dt} v_C$, to get $V_{in} = R \cdot C \cdot \frac{d}{dt} v_C + v_C$

Waa-laa, no integration. Always try to write your differential equations without integrals, it will eliminate one more source of mistakes. We now have a differential equation in terms of v_C. If v_C isn't the variable we want to find in our analysis then we can always go back to the circuit later and find the current or the voltage v_R by simple circuit analysis *after* we've found v_C.

So now we have to solve the differential equation. Recall from your differential equations class the that first order differential equations are always solved by equations of the following form.

Standard first order differential equation answer: $v_C(t) = A + B\cdot e^{s\cdot t}$

And, by differentiation: $\frac{d}{dt}v_C = B\cdot s\cdot e^{s\cdot t}$

Substitute these back into the original equation:

$$V_{in} = R\cdot C\cdot\frac{d}{dt}v_C + v_C = R\cdot C\cdot B\cdot s\cdot e^{s\cdot t} + \left(A + B\cdot e^{s\cdot t}\right) = R\cdot C\cdot B\cdot s\cdot e^{s\cdot t} + B\cdot e^{s\cdot t} + A$$

We can separate this equation into two parts, one which is time dependent and one which is not. Each part must still be an equation.

Time independent (forced) part: $V_{in} = A$, $A = V_{in} =$ final condition $= v_C(\infty)$

Time dependent (transient) part: $0 = R\cdot C\cdot B\cdot s\cdot e^{s\cdot t} + B\cdot e^{s\cdot t}$,

Divide both sides by $B\cdot e^{s\cdot t}$ to get $0 = R\cdot C\cdot s + 1$, $s = -\frac{1}{R\cdot C} = -\frac{1}{\tau}$, where $\tau = R\cdot C$

This τ is called the "time constant" and will become a rather important little character.

Put the parts we know back into the expression for $v_C(t) = V_{in} + B\cdot e^{-\frac{t}{R\cdot C}} = v_C(\infty) + B\cdot e^{-\frac{t}{R\cdot C}}$

at time t = 0: $v_C(0) = V_{in} + B$, $B = v_C(0) - V_{in} = v_C(0) - v_C(\infty)$ B is the difference between v_C at the start and v_C at the end.

And finally: $v_C(t) = V_{in} + B\cdot e^{-\frac{t}{R\cdot C}} = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{R\cdot C}}$

It turns out that all first-order transient solutions will have the same form, just different variables and time constants.

Once you have $v_C(t)$, you can also find $v_R(t)$ and/or i(t) from $v_C(t)$ if you want.

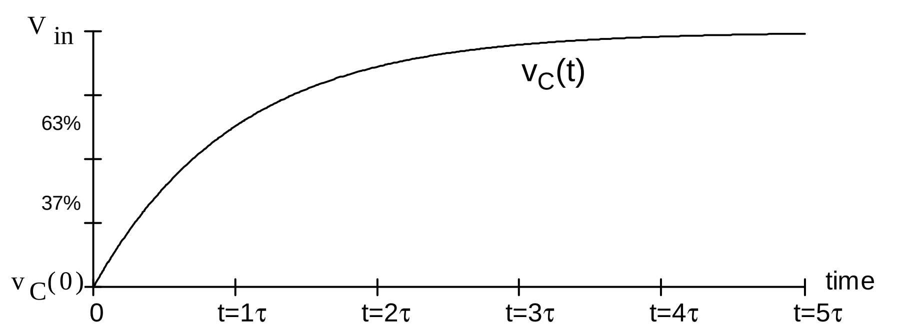
$$v_R(t) = V_{in} + v_C(t) = V_{in} - \left(B\cdot e^{-\frac{t}{R\cdot C}} + V_{in}\right) = -B\cdot e^{-\frac{t}{R\cdot C}} = -B\cdot e^{-\frac{t}{\tau}} = \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{R\cdot C}}$$

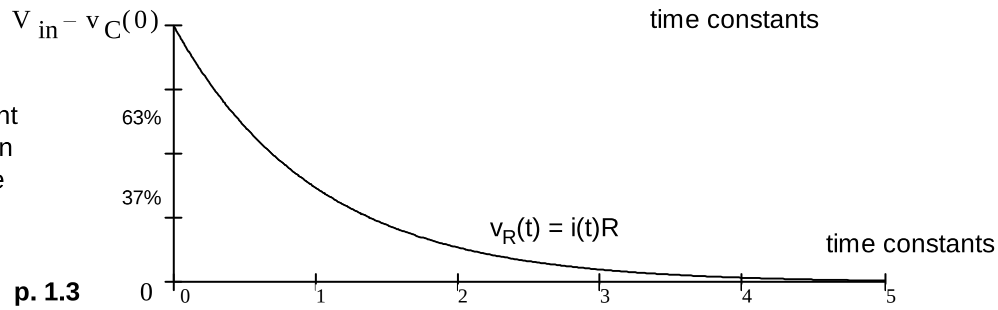
$$i(t) = C\cdot\frac{d}{dt}v_C = -C\cdot B\cdot\frac{1}{R\cdot C}\cdot e^{-\frac{t}{R\cdot C}} = -\frac{B}{R}\cdot e^{-\frac{t}{\tau}} = \frac{-\left(v_C(0) - v_C(\infty)\right)}{R}\cdot e^{-\frac{t}{R\cdot C}}$$

Let's plot these and see what they actually look like. These graphs show the capacitor charging from it's initial value to Vin and v_R falling to 0 (same for i_R)

The curves are generalized based on the concept of the time constant, which is why we introduced th time constant. Later we'll look at these kind of curves in greater detail.

Ok, that was fun, but you might ask at this point if there isn't an easier way. Yes, in fact, there is. We'll look at next.





First-Order Transients the Easy Way

Notice in the preceeding analysis that I made a very standard guess at the solution of the differential equation.

Standard first order differential equation answer: $v_C(t) = A + B \cdot e^{s \cdot t}$

Further notice that A turned out to be the final condition and that B turned out to be the difference between the initial and final conditions. Finally, remember that I renamed s to -1/τ. All of this can be generalized to any first order system. The answer will always be in this form:

final condition

For all first order transients: $x(t) = x(\infty) + (x(0) - x(\infty)) \cdot e^{-\frac{t}{\tau}}$ --- time constant

initial condition

x(t) could be any variable in any first-order system. It could be a temperature, or a fluid level, or a velocity, but for us it usually means voltages and currents, so we'll have solutions like these.

$$v_X(t) = v_X(\infty) + \left(v_X(0) - v_X(\infty)\right) \cdot e^{-\frac{t}{\tau}} \quad \text{or} \quad i_X(t) = i_X(\infty) + \left(i_X(0) - i_X(\infty)\right) \cdot e^{-\frac{t}{\tau}}$$

You find Initial and final conditions from steady-state analysis. That leaves only one thing that you have to find from the differential equation-- the time constant. If we could only figure out what the time constant of a circuit (or system) is, then we could almost jump straight to the solution.

The first way to find the time constant is to simply remember it's form for a few cases, like the for RC circuit. Even if the circuit doesn't look exactly like the standard RC series circuit, Thevenin can help us make it look that way. Since nearly all of our first order circuits will involve a single capacitor or a single inductor this is not an impractical method at all.

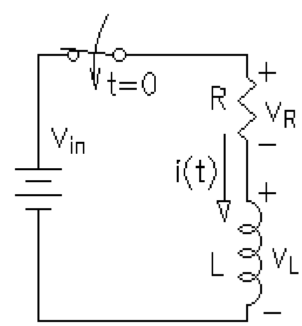
Another way to find the time constant is to manipulate the differential equation into this particular form

constant = $X + \tau \cdot \frac{dX}{dt}$ with no factor in front of the "X" term. Whatever the factor in front of $\frac{dX}{dt}$ turns out to be, that will be τ. For the RC circuit the differential equation could be written as

$V_{in} = R \cdot C \cdot \frac{d}{dt} v_C + v_C$ notice that the factor in front of $\frac{d}{dt} v_C$ is indeed τ.

Finally, there is an even easier way based on the LaPlace "s" and s-impedances that we can use in circuits and equations in place of differentials and integrals. You'll see this last method later, after second-order transients. (Incidentally, this is the reason that I chose to use an s as the unknown in the exponential.)

Series RL circuit: OK, if it's so easy, let's try it with a series RL circuit.



$$v_{in} = v_R + v_L \qquad V_{in} = i \cdot R + L \cdot \frac{d}{dt} i \qquad \frac{V_{in}}{R} = i + \frac{L}{R} \cdot \frac{d}{dt} i$$

So, the time constant must be $\tau = \frac{L}{R}$ That wasn't too bad.

Initial condition: $i_L(0) = 0$ If the switch was initially open the the current just before the switch was closed was 0, and inductor current can't change instantly.

Final condition: $i_L(\infty) = \frac{V_{in}}{R}$ The inductor looks like an short for steady-state DC.

So: $$i_L(t) = i_L(\infty) + \left(i_L(0) - i_L(\infty)\right) \cdot e^{-\frac{t}{\tau}} = \frac{V_{in}}{R} + \left(0 - \frac{V_{in}}{R}\right) \cdot e^{-\frac{R}{L} \cdot t} = \frac{V_{in}}{R} \cdot \left(1 - e^{-\frac{R}{L} \cdot t}\right)$$

Well, that's wasn't too painfull, was it?

1.3 Initial and Final Conditions

More than once I've said that the initial and final conditions are found from steady-state analysis of the circuit. It's about time I said how.

Initial Conditions: There are two very important concepts that you use to find the initial conditions.

1) Capacitor voltage cannot change instantaneously, $v_C(0+) = v_C(0-)$.

If you can find the capacitor voltage just before time t = 0 (or whatever starts the transient), then you know what it is just after time t = 0, $v_C(0+) = v_C(0-)$. It cannot change instantaneously. Often you'll use the methods outlined below to find the final condition of the previous circuit, especially if the circuit's been in that condition for "a long time". Sometimes you'll have to solve the previous transient to find the initial condition for the next transient.

If you cannot find the capacitor voltage just before time t = 0 from the circuit, then you'll have to be told what the initial voltage or charge is. Capacitors can hold a charge for a long time, and can be moved from one circuit to another without losing the charge. High school electronics students like to charge capacitors and leave them where they'll shock some poor unsuspecting soul. Of course you'd never do something as childish as that. Occasionally you may be told what the initial charge is in terms of coulombs. In that case remember the definition of capacitance.

$$C = \frac{Q}{V} \quad \text{which can be rearranged to} \quad V = \frac{Q}{C}$$

If you have nothing else to go on, assume the initial voltage is 0.

2) Inductor current cannot change instantaneously, $i_L(0+) = i_L(0-)$.

If you can find the inductor current just before time t = 0 (or whatever starts the transient), then you know what it is just after time t = 0, $i_L(0+) = i_L(0-)$. It cannot change instantaneously.

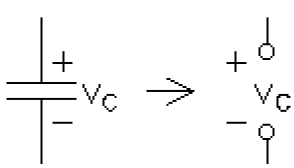
If you cannot find the inductor current just before time t = 0 from the circuit, then assume it's 0. Real circuits and real inductors always have some resistance so inductor currents just don't last very long (unless you're dealing with superconductors). Inductors would be very difficult to move from one circuit to another without losing the current. If you're given an initial current for a problem, realize that this is probably just to make the problem more interesting, or the initial current comes from previous analysis.

Do not mix these two concepts up. Capacitor current and inductor voltage can both change instantly with no problem at all.

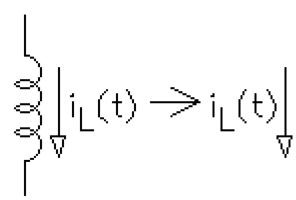
Final Conditions: This <u>is</u> steady-state analysis. The steady-state is the final condition.

DC sources

If all the voltage and current sources are DC, then at the final condition the capacitors are all done done charging so $i_C = 0$, and you can treat them as open circuits. When you find the voltage across the open, that will be the final capacitor voltage. You've done this sort of thing before to find the energy stored in a capacitor.



Replace capacitors with opens

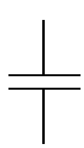


Replace inductors with wires

At the final condition the inductor currents are also no longer changing, so the voltage across an inductor is 0. Treat inductors as wires (short circuits). When you find the current through the wire, that will be the final inductor current.

AC sources

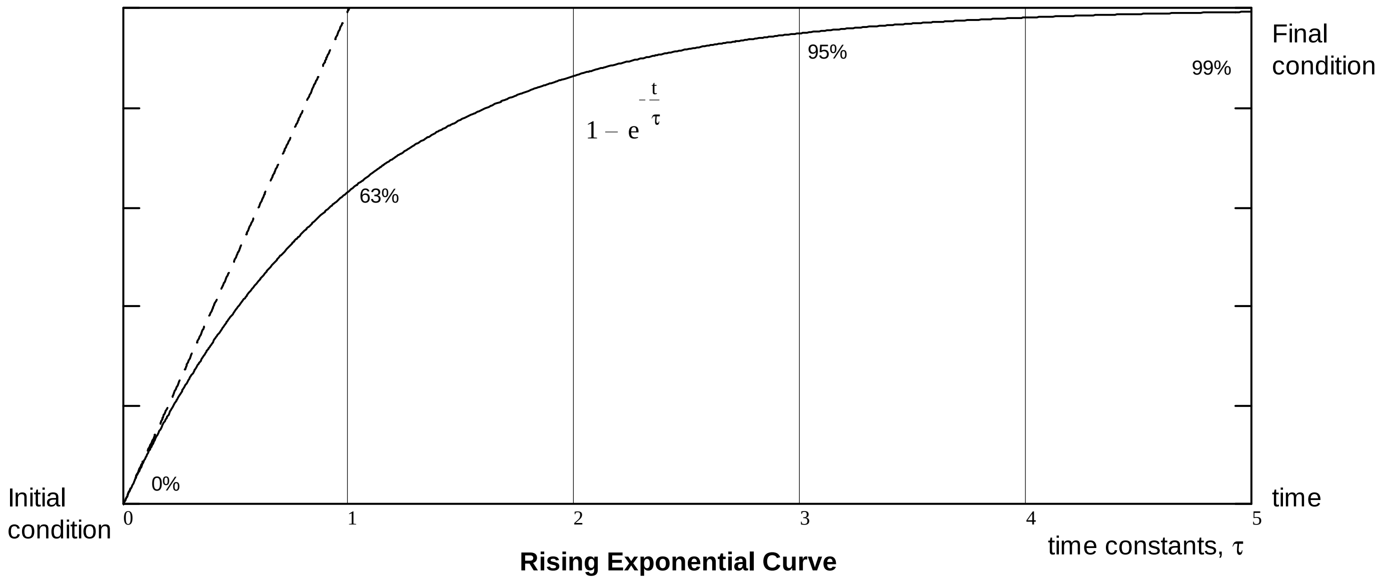
Use phasor analysis (jω). Remember that phasor analysis was also called "steady-state AC". One of the primary assumptions was that the transients had all died out.

 $\frac{1}{j\cdot\omega\cdot C}$

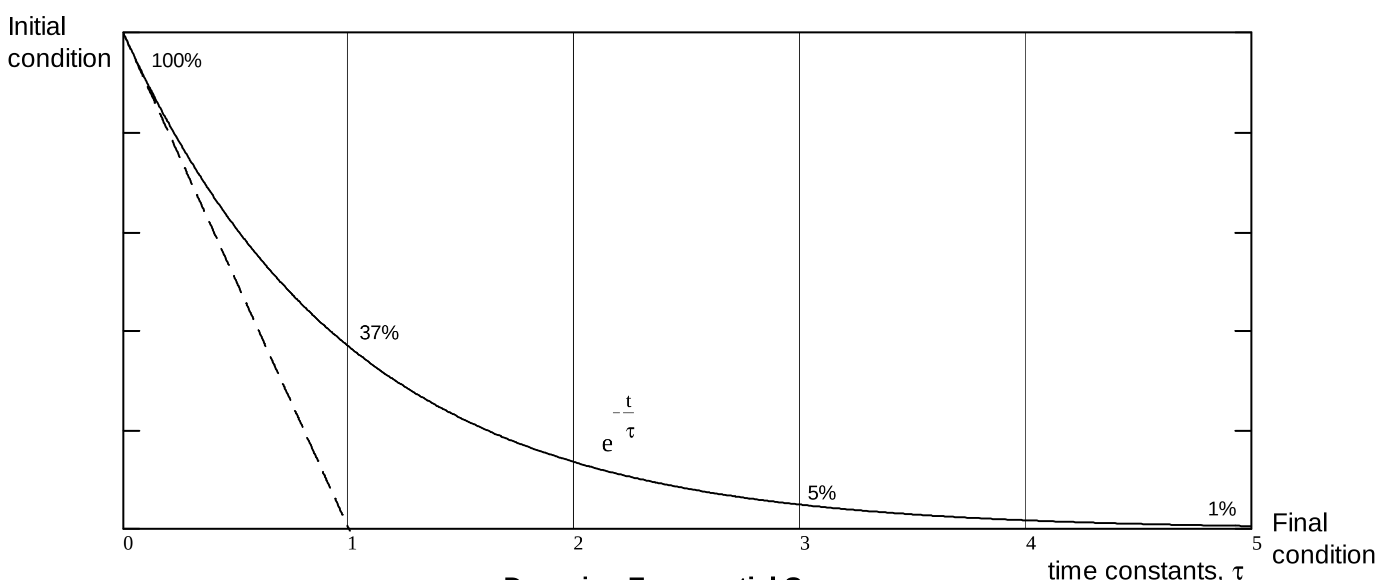
 $j\cdot\omega\cdot L$

1.4 Exponential Curves

Before we go on to second-order transients we should take a closer look at some of the characteristics of exponential curves. The curves that show up as answers to our transient problems are shown below. The transient effects always die out after some time, so the exponents are always negative. Just think about what a positive exponent would mean. That wouldn't be a transient-- that would be exponential growth, like the population.



Rising Exponential Curve



Decaying Exponential Curve

Some important features:

1) These curves proceed from an initial condition to a final condition. If the final condition is greater than the initial, then the curve is said to be a "rising" exponential. If the final condition is less than the initial, then the curve is called a "decaying" exponential.

2) The curves' initial slope is $\pm$ $1/\tau$. If they continued at this initial slope they'd be done in one time constant.

3) In the first time constant the curve goes 63% from initial to the final condition.

4) After three time constants the curve is 95% of the way to the final condition.

5) By five time constants the curve is within 1% of the final condition and is usually considered finished. Mathematically, the curve approaches the final condition asymptotically and never reaches it. In reality, of course, this is nonsense. Whatever difference there may be between the mathematical solution and the final condition will soon be overshadowed by random fluctuations (called noise) in the real circuit.

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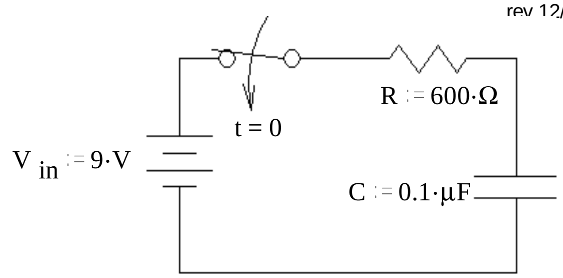
rev 12/19/15

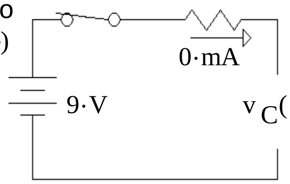
Ex1 a) Find the expression for $v_c(t)$ if the switch is closed at time $t = 0$ and $v_c(0) = 0$.



$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{\tau}}$$

$\tau := R\cdot C \qquad \tau = 60\cdot\mu s$

redraw to find $v_c(\infty)$

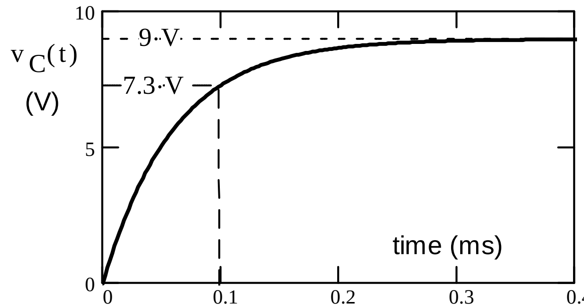


$v_C(\infty) = 9\cdot V$

$$v_C(t) = 9\cdot V + (0\cdot V - 9\cdot V)\cdot e^{-\frac{t}{60\cdot\mu s}}$$

b) What is the voltage across the capacitor, C, at $t = 0.1ms$?

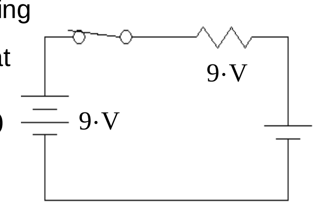
$$v_C(25\cdot\mu s) = 9\cdot V - 9\cdot V\cdot e^{-\frac{100\cdot\mu s}{60\cdot\mu s}} = 7.3\cdot V$$



c) When will the current through the resistor be $i_R := 5\cdot mA$?

$i_R(\infty) = 0\cdot mA \qquad i_R(0) = \frac{9\cdot V}{R} = 15\cdot mA$ found from drawing

redraw at $t = 0^+$ to find $i_R(0)$



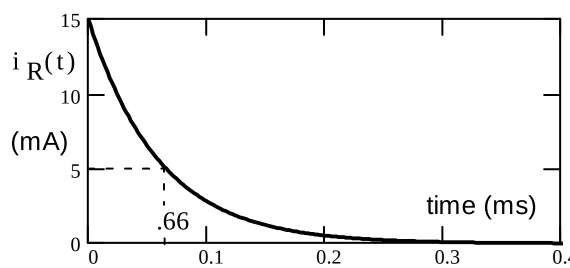
capacitor voltage can't change instantly

$$i_R(t) = i_R(\infty) + \left(i_R(0) - i_R(\infty)\right)\cdot e^{-\frac{t}{\tau}}$$

$$= 0\cdot mA + (15\cdot mA - 0\cdot mA)\cdot e^{-\frac{t}{\tau}}$$

$$= 10.976\cdot mA\cdot e^{-\frac{t}{60\cdot\mu s}} = 5\cdot mA \text{ at some time, } t$$

Solve for $t = -\tau\cdot\ln\left(\frac{5\cdot mA}{15\cdot mA}\right) = 65.92\cdot\mu s$



d) When will the current through the resistor be $i_R := 20\cdot mA$?

Since the initial condition is about $15mA$ and the final condition is $0mA$, i_R will never be $20mA$.

Ex2 A $1000\ \mu F$ capacitor has an initial charge of 12 volts. A $20\text{-}\Omega$ resistor is connected across the capacitor at time $t = 0$. Find the energy dissipated by the resistor in the first 5 time constants.

After 5 time constants nearly all of the energy initially stored in the capacitor will be dissipated by the resistor.

$C := 1000\cdot\mu F \qquad V_C := 12\cdot V \qquad W_C := \frac{1}{2}\cdot C\cdot V_C^{\ 2} \qquad W_C = 0.072\cdot joule$

You can get to this answer just by knowing a little about the exponential curve, but what if you want a more accurate answer? Then you'll have to find the remaining voltage across the capacitor at $t = 5t$ and subtract the energy left in the capacitor at that time.

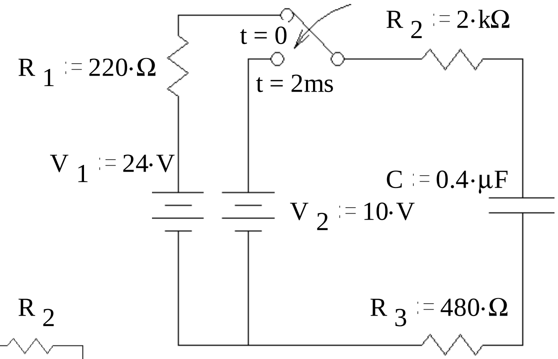
$$v_C(0) = 12\cdot V \quad v_C(\infty) = 0\cdot V \quad v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{\tau}} = 0\cdot V + (12\cdot V - 0\cdot V)\cdot e^{-\frac{t}{\tau}} = 12\cdot V\cdot e^{-\frac{t}{\tau}}$$

at $t = 5\tau$: $\quad v_C(5\cdot\tau) = 12\cdot V\cdot e^{-5} = 81\cdot mV \qquad \frac{1}{2}\cdot C\cdot(81\cdot mV)^2 = 3.281\cdot 10^{-6}\cdot joule$

Not surprisingly, this makes no significant difference: $\quad W_R = W_C - \frac{1}{2}\cdot C\cdot(81\cdot mV)^2 = 0.072\cdot joule$

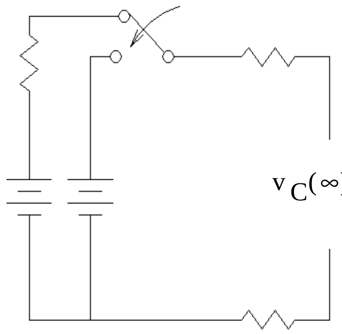
Ex3 The capacitor is initially uncharged. The switch is in the upper position from 0 to $2\mathrm{ms}$ and is switched down at time $t = 2\mathrm{ms}$.

$R_3 := 480\cdot\Omega$

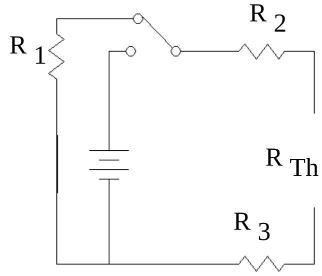


a) What is the capacitor voltage, $v_C(t)$

First interval $v_C(0) = 0\cdot V$



$v_C(\infty) = 24\cdot V$



$R_{Th} := R_1 + R_2 + R_3$

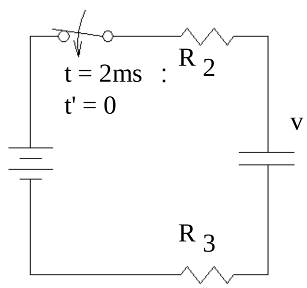
$\tau := R_{Th}\cdot C$

$\tau = 1.08\cdot\mathrm{ms}$

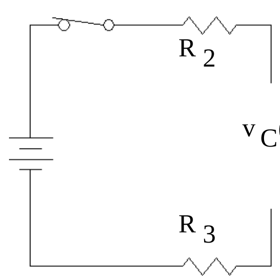
$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{\tau}} = 24\cdot V + (0\cdot V - 24\cdot V)\cdot e^{-\frac{t}{1.08\cdot ms}}$$

at 2ms: $24\cdot V - 24\cdot V\cdot e^{-\frac{2\cdot ms}{1.08\cdot ms}} = 20.23\cdot V$

Second interval, define a new time, $t' = t - 2\mathrm{ms}$



$v_C(0) = 20.23\cdot V$
t' = 0



$v_C(\infty) = 10\cdot V$
$t' = \infty$

$$\tau' := \underbrace{\left(R_2 + R_3\right)}_{R_{Th}}\cdot C$$

$\tau = 1.08\cdot\mathrm{ms}$

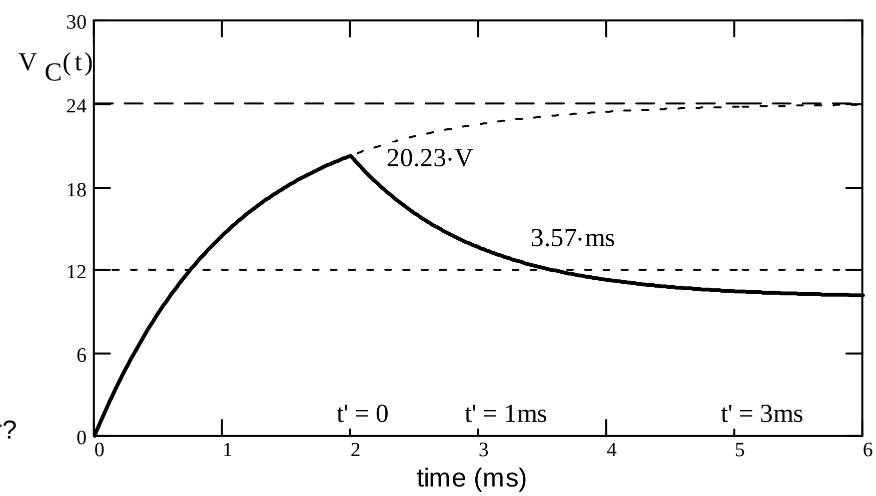
$$v_C(t') = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t'}{\tau'}} = 10\cdot V + (20.23\cdot V - 10\cdot V)\cdot e^{-\frac{t'}{0.96\cdot ms}} = 10\cdot V + 10.23\cdot V\cdot e^{-\frac{t - 2\cdot ms}{0.96\cdot ms}}$$

$0 < t < 2\mathrm{ms}$

$$V_C(t) = 24\cdot V - 24\cdot V\cdot e^{-\frac{2\cdot ms}{1.08\cdot ms}}$$

$t > 2\mathrm{ms}$

$$V_C(t) = 10\cdot V + 10.23\cdot V\cdot e^{-\frac{t - 2\cdot ms}{0.96\cdot ms}}$$



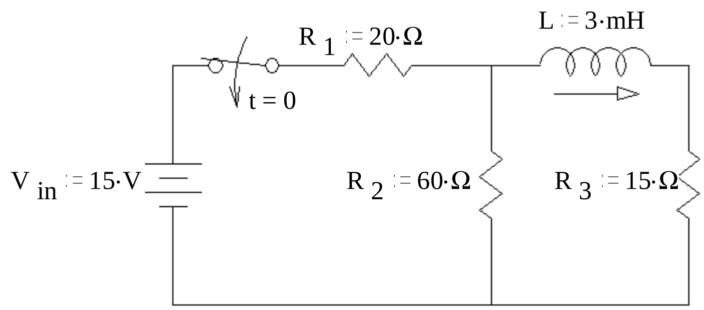
b) When is voltage across the capacitor $12\mathrm{V}$ AND getting smaller?

$$12\cdot V = 10\cdot V + 10.233\cdot V\cdot e^{-\frac{t_{12}}{0.96\cdot ms}}$$

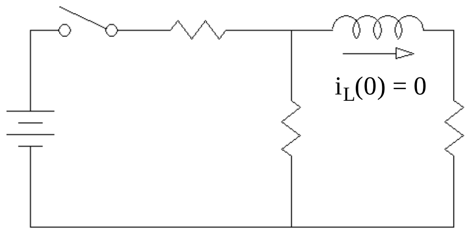
$$\frac{12\cdot V - 10\cdot V}{10.23\cdot V} = e^{-\frac{t_{12}}{0.96\cdot ms}} \qquad \ln\left(\frac{12\cdot V - 10\cdot V}{10.23\cdot V}\right) = -\left(\frac{t_{12}}{0.96\cdot ms}\right) \qquad t_{12} = -0.96\cdot ms\cdot\ln\left(\frac{12\cdot V - 10\cdot V}{10.23\cdot V}\right) = 1.57\cdot\mathrm{ms}$$

$2\cdot ms + 1.57\cdot ms = 3.57\cdot\mathrm{ms}$

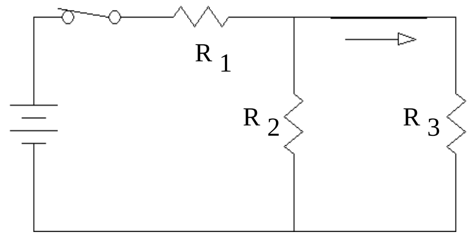
Ex4 a) Find the complete expression for $i_L(t)$.



Before the switch closes, $t = 0-$



Final time, $t = \infty$



$$R_{23} := \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}}$$

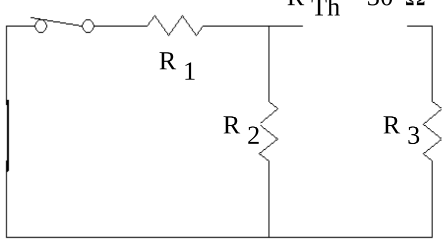
$$R_{23} = 12 \cdot \Omega$$

$$v_{R3}(\infty) = \frac{R_{23}}{R_1 + R_{23}} \cdot V_{in} = 5.625 \cdot V$$

$$i_L(\infty) = \frac{v_{R3}(\infty)}{R_3} = \frac{5.625 \cdot V}{15 \cdot \Omega} = 375 \cdot mA$$

$$R_{Th} := \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} + R_3$$

$$R_{Th} = 30 \cdot \Omega$$



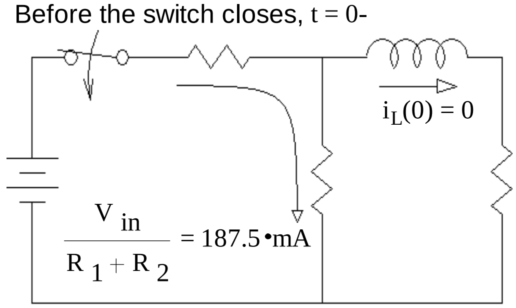
$$\tau := \frac{L}{R_{Th}}$$

$$\tau = 100 \cdot \mu s$$

$$i_L(t) = i_L(\infty) + \left(i_L(0) - i_L(\infty)\right) \cdot e^{-\frac{t}{\tau}} = 375 \cdot mA + (0 \cdot mA - 375 \cdot mA) \cdot e^{\frac{-t}{100 \cdot \mu s}} = 375 \cdot mA - 375 \cdot mA \cdot e^{\frac{-t}{100 \cdot \mu s}}$$

b) When is the voltage across $R_2 = 10V$?

Before the switch closes, $t = 0-$



$$\frac{V_{in}}{R_1 + R_2} = 187.5 \cdot mA$$

$$v_{R2}(0) = \frac{R_2}{R_1 + R_2} \cdot V_{in} = 11.25 \cdot V$$

From drawing above at $t = \infty$

$$v_{R2}(\infty) = v_{R3}(\infty) = \frac{R_{23}}{R_1 + R_{23}} \cdot V_{in} = 5.625 \cdot V$$

$$v_{R2}(t) = v_{R2}(\infty) + \left(v_{R2}(0) - v_{R2}(\infty)\right) \cdot e^{-\frac{t}{\tau}}$$

$$= 5.625 \cdot V + (11.25 \cdot V - 5.625 \cdot V) \cdot e^{-\frac{t}{100 \cdot \mu s}}$$

$= 10 \cdot V$ at some time, solving for that time...

$$t = -\tau \cdot \ln\left(\frac{10 \cdot V - 5.625 \cdot V}{11.25 \cdot V - 5.625 \cdot V}\right) = 25 \cdot \mu s$$

Alternatively, when $v_{R2}(t) = 10V$, then $v_{R1}(t) = 5V$ and $i_L(t) = \frac{5 \cdot V}{R_1} - \frac{10 \cdot V}{R_2} = 83.333 \cdot mA$

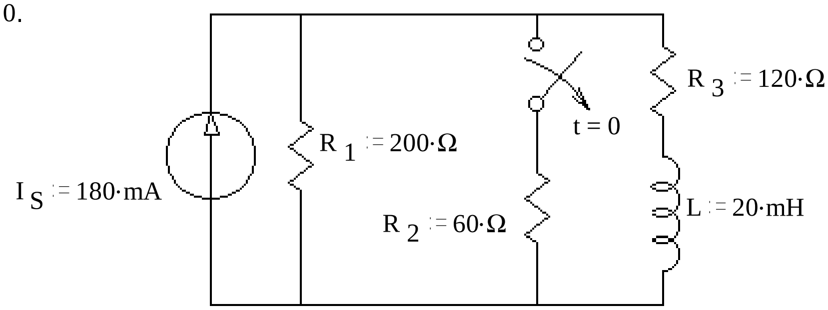
$$t = -\tau \cdot \ln\left(\frac{83.333 \cdot mA - 375 \cdot mA}{-375 \cdot mA}\right) = 25 \cdot \mu s$$

c) What is the $v_L(t)$ expression?

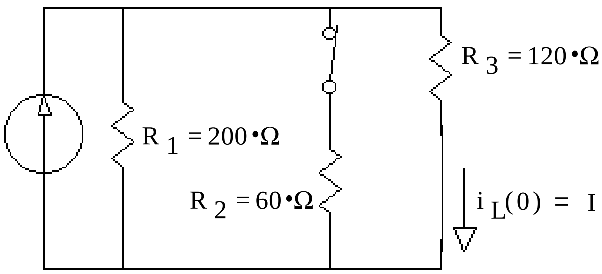
$$v_L(t) = v_L(\infty) + \left(v_L(0) - v_L(\infty)\right) \cdot e^{-\frac{t}{\tau}} = 0 \cdot V + (11.25 \cdot V - 0 \cdot V) \cdot e^{-\frac{t}{100 \cdot \mu s}}$$

Ex5 The switch has been closed for a long time and is opened (as shown) at time $t = 0$.

a) Find the complete expression for $i_L(t)$.

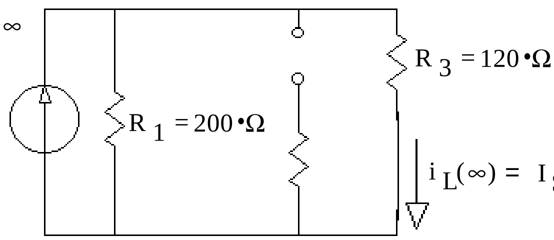


Before the switch opens, $t = 0-$

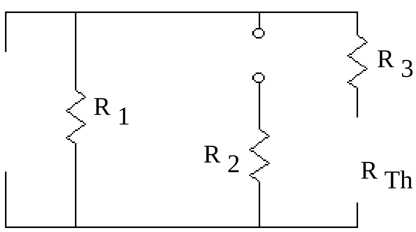


$$i_L(0) = I_S \cdot \frac{\dfrac{1}{R_3}}{\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}} = 50 \cdot \text{mA}$$

Final time, $t = \infty$



$$i_L(\infty) = I_S \cdot \frac{\dfrac{1}{R_3}}{\dfrac{1}{R_1}+\dfrac{1}{R_3}} = 112.5 \cdot \text{mA}$$

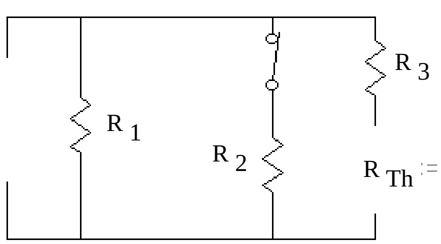


$R_{Th} := R_1 + R_3$ $\qquad R_{Th} = 320 \cdot \Omega$ $\qquad \tau := \dfrac{L}{R_{Th}}$ $\qquad \tau = 62.5 \cdot \mu s$

$$i_L(t) = i_L(\infty) + \left(i_L(0) - i_L(\infty)\right) \cdot e^{-\frac{t}{\tau}} = 112.5 \cdot \text{mA} + (50 \cdot \text{mA} - 112.5 \cdot \text{mA}) \cdot e^{\frac{-t}{62.5 \cdot \mu s}} = 112.5 \cdot \text{mA} - 62.5 \cdot \text{mA} \cdot e^{\frac{-t}{62.5 \cdot \mu s}}$$

b) Find i_L at time $t = 1.4\tau$. $\quad i_L(1.4 \cdot \tau) = 112.5 \cdot \text{mA} - 62.5 \cdot \text{mA} \cdot e^{-\frac{1.4 \cdot \tau}{\tau}} = 112.5 \cdot \text{mA} - 62.5 \cdot \text{mA} \cdot e^{-1.4} = 97.088 \cdot \text{mA}$

c) At time $t = 1.4\tau$ the switch is closed again. Find the complete expression for $i_L(t')$, where t' starts at $t = 1.4\tau$. Be sure to clearly show the time constant.



$R_{Th} := \dfrac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}} + R_3$ $\qquad R_{Th} = 166.2 \cdot \Omega$

$\tau := \dfrac{L}{R_{Th}}$ $\qquad \tau = 120.4 \cdot \mu s$

$i_L(0) = 97.1 \cdot \text{mA}$ from part b)

$i_L(\infty) = 50 \cdot \text{mA}$ initial value from part a)

$$i_L(t) = i_L(\infty) + \left(i_L(0) - i_L(\infty)\right) \cdot e^{-\frac{t}{\tau}} = 50 \cdot \text{mA} + (97.1 \cdot \text{mA} - 50 \cdot \text{mA}) \cdot e^{\frac{-t'}{120.4 \cdot \mu s}} = 50 \cdot \text{mA} + 47.1 \cdot \text{mA} \cdot e^{\frac{-t'}{120.4 \cdot \mu s}}$$

Complex Numbers

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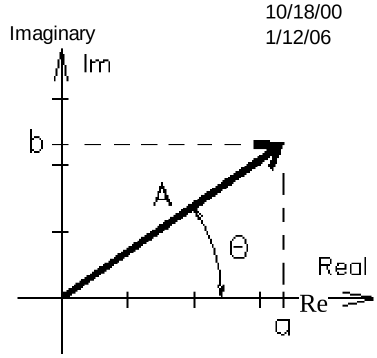


$j = \sqrt{-1}$ the imaginary number

Rectangular Form $\mathbf{A} = a + b \cdot j$

$Re(\mathbf{A}) = a \qquad Im(\mathbf{A}) = b$

Polar Form $\mathbf{A} = A \cdot e^{j \cdot \theta}$

$Re(\mathbf{A}) = A \cdot \cos(\theta) \qquad Im(\mathbf{A}) = A \cdot \sin(\theta)$

Conversions $A = |\mathbf{A}| = \sqrt{a^2 + b^2} \qquad \theta = \arg(\mathbf{A}) = \operatorname{atan}\left(\frac{b}{a}\right)$

$a = A \cdot \cos(\theta) \qquad b = A \cdot \sin(\theta)$

$\mathbf{A} = A \cdot e^{j \cdot \theta} = A \cdot \cos(\theta) + A \cdot \sin(\theta) \cdot j \qquad \mathbf{A} = a + b \cdot j = \left(\sqrt{a^2 + b^2}\right) \cdot e^{j \cdot \operatorname{atan}\left(\frac{b}{a}\right)}$

Special Cases $j := \sqrt{-1} = e^{j \cdot 90 \cdot deg} \qquad \frac{1}{j} = -j = e^{-j \cdot 90 \cdot deg} \qquad e^{j \cdot 0 \cdot deg} = 1 \qquad e^{-j \cdot 180 \cdot deg} = e^{-j \cdot 180 \cdot deg} = -1$

$j \cdot e^{j \cdot \theta} = e^{j \cdot (\theta + 90 \cdot deg)}$

Define a 2nd number: rect: $\mathbf{D} = c + d \cdot j$ polar: $\mathbf{D} = D \cdot e^{j \cdot \phi}$

Equality $\mathbf{A} = \mathbf{D}$ if and only if $a = c$ and $b = d$ OR $A = D$ and $\theta = \phi$

Addition and Subtraction $\mathbf{A} + \mathbf{D} = (a + b \cdot j) + (c + d \cdot j) = (a + c) + (b + d) \cdot j$

$\mathbf{A} - \mathbf{D} = (a + b \cdot j) - (c + d \cdot j) = (a - c) + (b - d) \cdot j$

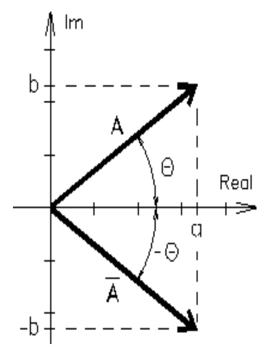
Convert polars to rectangular form first

Multiplication and Division $\mathbf{A} \cdot \mathbf{D} = (a + b \cdot j) \cdot (c + d \cdot j) = (a \cdot c - b \cdot d) + (b \cdot c + a \cdot d) \cdot j$

Rectangular: $\frac{\mathbf{A}}{\mathbf{D}} = \frac{a + b \cdot j}{c + d \cdot j} = \frac{a + b \cdot j}{c + d \cdot j} \cdot \frac{c - d \cdot j}{c - d \cdot j} = \frac{a \cdot c + b \cdot d}{c^2 + d^2} + \frac{b \cdot c - a \cdot d}{c^2 + d^2} \cdot j$

Polar: $\mathbf{A} \cdot \mathbf{D} = A \cdot e^{j \cdot \theta} \cdot D \cdot e^{j \cdot \phi} = A \cdot D \cdot e^{j \cdot (\theta + \phi)}$

$\frac{\mathbf{A}}{\mathbf{D}} = \frac{A \cdot e^{j \cdot \theta}}{D \cdot e^{j \cdot \phi}} = \frac{A}{D} \cdot e^{j \cdot (\theta - \phi)}$



Powers $\mathbf{A}^n = A^n \cdot e^{j \cdot n \cdot \theta} = A^n \cdot \cos(n \cdot \theta) + A^n \cdot \sin(n \cdot \theta) \cdot j$ Convert rectangulars first, usually

Conjugates

complex number	Conjugate	
$\mathbf{A} = a + b \cdot j$	$\overline{\mathbf{A}} = a - b \cdot j$	$\overline{\overline{\mathbf{A}}} = \mathbf{A}$
$\mathbf{A} = A \cdot e^{j \cdot \theta}$	$\overline{\mathbf{A}} = A \cdot e^{-j \cdot \theta}$	
$\mathbf{F} = \frac{3 + 5 \cdot j}{(2 - 6 \cdot j) \cdot e^{j \cdot 40 \cdot deg}}$	$\overline{\mathbf{F}} = \frac{3 - 5 \cdot j}{(2 + 6 \cdot j) \cdot e^{-j \cdot 40 \cdot deg}}$	

Euler's equation $e^{j \cdot \alpha} = \cos(\alpha) + j \cdot \sin(\alpha)$ OR: $\cos(\alpha) = \frac{e^{j \cdot \alpha} + e^{-j \cdot \alpha}}{2} \qquad \sin(\alpha) = \frac{e^{j \cdot \alpha} - e^{-j \cdot \alpha}}{2 \cdot j}$

$e^{j \cdot (\omega \cdot t + \theta)} = \cos(\omega \cdot t + \theta) + j \cdot \sin(\omega \cdot t + \theta)$

$Re\left[e^{j \cdot (\omega \cdot t + \theta)}\right] = \cos(\omega \cdot t + \theta)$

If we freeze this at time t=0, then we can represent $\cos(\omega \cdot t + \theta)$ by $e^{j \cdot \theta}$

Calculus Remember, when we write $e^{j \cdot \theta}$, we really mean $e^{j \cdot (\omega \cdot t + \theta)}$

$\frac{d}{dt}\mathbf{A} = \frac{d}{dt}\left(A \cdot e^{j \cdot \theta}\right) = j \cdot \omega \cdot A \cdot e^{j \cdot \theta} = \omega \cdot A \cdot e^{j \cdot (\theta + 90 \cdot deg)}$

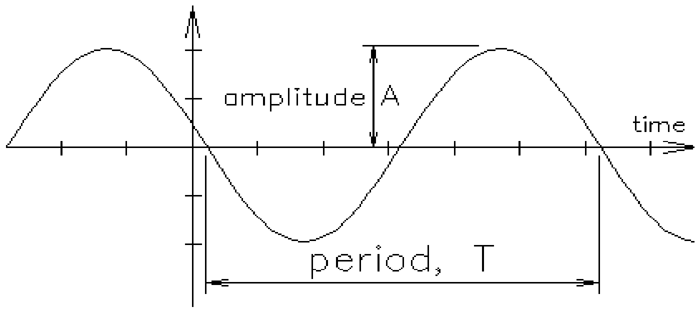
$\int \mathbf{A} \, dt = \int A \cdot e^{j \cdot \theta} \, dt = \frac{1}{j \cdot \omega} \cdot A \cdot e^{j \cdot \theta} = \frac{1}{\omega} \cdot A \cdot e^{j \cdot (\theta - 90 \cdot deg)}$

Lecture 12 Introduction to AC Phasors

A.Stolp 10/7/08
rev,

Phasor analysis with impedances, For steady-state sinusoidal response ONLY

Sinusoidal AC



T = Period = repeat time

f = frequency, cycles / second $\quad f = \frac{1}{T} = \frac{\omega}{2\cdot\pi}$

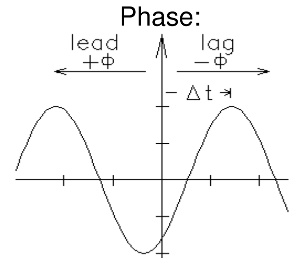
ω = radian frequency, radians/sec $\quad \omega = 2\cdot\pi\cdot f$

A = amplitude

Phase: $\quad \phi = -\frac{\Delta t}{T}\cdot 360\cdot deg$

or: $\quad \phi = -\frac{\Delta t}{T}\cdot 2\cdot\pi\cdot rad$

$y(t) = A\cdot\cos(\omega\cdot t+\theta)$

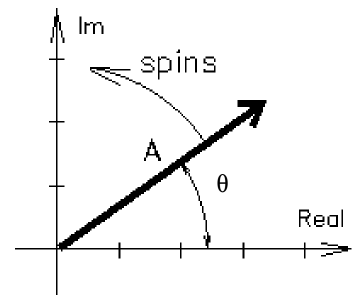


Phasor analysis

The math is all based on the Euler's equation

Euler's equation $\quad e^{j\cdot\alpha} = \cos(\alpha)+j\cdot\sin(\alpha)$

OR: $\quad \cos(\alpha) = \frac{e^{j\cdot\alpha}+e^{-j\cdot\alpha}}{2}$ $\quad \sin(\theta) = \frac{e^{j\cdot\alpha}-e^{-j\cdot\alpha}}{2\cdot j}$



$$e^{j\cdot(\omega\cdot t+\theta)} = \cos(\omega\cdot t+\theta)+j\cdot\sin(\omega\cdot t+\theta)$$

$$\mathrm{Re}\left[e^{j\cdot(\omega\cdot t+\theta)}\right] = \cos(\omega\cdot t+\theta)$$

If we freeze this at time t=0, then we can represent $\cos(\omega\cdot t+\theta)$ by $e^{j\cdot\theta}$ That's the phasor

<u>Phasor</u>

voltage: $\quad v(t) = V_p\cdot\cos(\omega\cdot t+\phi) \qquad \mathbf{V}(\omega) = V_p\cdot e^{j\cdot\phi}$

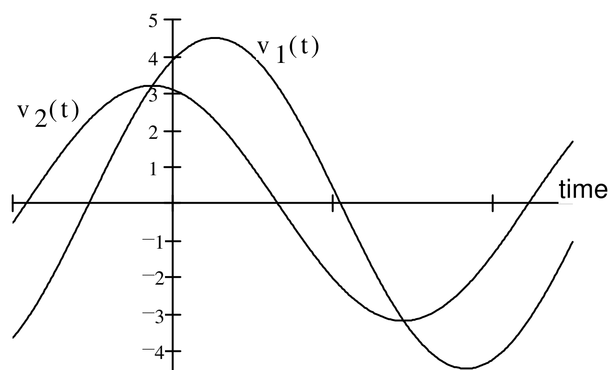
current: $\quad i(t) = I_p\cdot\cos(\omega\cdot t+\phi) \qquad \mathbf{I}(\omega) = I_p\cdot e^{j\cdot\phi}$

Phasors are drawn on a complex plane.

Phasors are used for adding and subtracting sinusoidal waveforms.

Ex1. Add the sinusoidal voltages $\quad v_1(t) = 4.5\cdot V\cdot\cos(\omega\cdot t-30\cdot deg)$

and $\quad v_2(t) = 3.2\cdot V\cdot\cos(\omega\cdot t+15\cdot deg)$



using phasor notation, draw a phasor diagram of the three phasors, then convert back to time domain form.

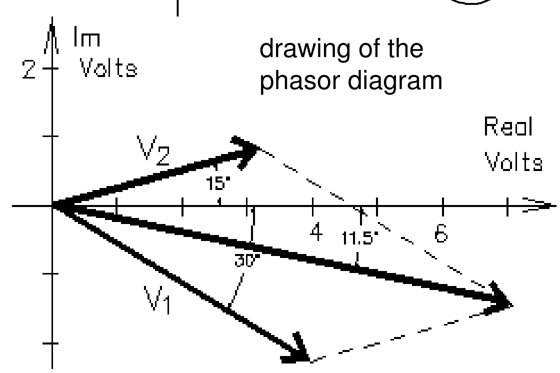
$v_1(t) = 4.5\cdot V\cdot\cos(\omega\cdot t-30\cdot deg)$

$\mathbf{V_1}(\omega) = 4.5V\ \underline{/-30^o}$ $\quad$ or: $\quad \mathbf{V_1}(\omega) = 4.5\cdot V\cdot e^{-j\cdot 30\cdot deg}$

and

$v_2(t) = 3.2\cdot V\cdot\cos(\omega\cdot t+15\cdot deg)$

$\mathbf{V_2}(\omega) = 3.2V\ \underline{/15^o}$ $\quad$ or: $\quad \mathbf{V_2}(\omega) = 3.2\cdot V\cdot e^{j\cdot 15\cdot deg}$



drawing of the phasor diagram

I'm going to drop the (ω) notation from the phasor notation, it gets cumbersome, but remember that phasors are in the frequency domain..

$\mathbf{V_1} = 4.5V\ \underline{/-30^o}$ $\quad$ or: $\quad \mathbf{V_1} := 4.5\cdot V\cdot e^{-j\cdot 30\cdot deg}$

$\mathbf{V_2} = 3.2V\ \underline{/15^o}$ $\quad$ or: $\quad \mathbf{V_2} := 3.2\cdot V\cdot e^{j\cdot 15\cdot deg}$

Add like vectors, first change to the rectangular form

$\mathbf{V_1} = 4.5V\ \angle{-30^o}$ $\quad 4.5\cdot V\cdot\cos(-30\cdot deg) = 3.897\cdot V$ $\quad 4.5\cdot V\cdot\sin(-30\cdot deg) = -2.25\cdot V$ $\quad \mathbf{V_1} = 3.897 - 2.25j\ \cdot V$ \

} add

$\mathbf{V_2} = 3.2V\ \angle{15^o}$ $\quad 3.2\cdot V\cdot\cos(15\cdot deg) = 3.091\cdot V$ $\quad 3.2\cdot V\cdot\sin(15\cdot deg) = 0.828\cdot V$ $\quad \mathbf{V_2} = 3.091 + 0.828j\ \cdot V$ /

Add real parts: $3.897 + 3.091 = 6.988$ $\quad \mathbf{V_3} := \mathbf{V_1} + \mathbf{V_2}$

Add imaginary parts: $-2.25 + 0.828 = -1.422$ $\quad \mathbf{V_3} = 6.988 - 1.422j\ \cdot V$ sum

Change $\mathbf{V_3}$ back to polar coordinates:

$$\sqrt{6.988^2 + 1.422^2} = 7.131 \qquad \operatorname{atan}\left(\frac{-1.422}{6.988}\right) = -11.502\cdot deg$$

OR, in Mathcad notation (you' ll see these in future solutions):

$$\left|\mathbf{V_3}\right| = 7.131\cdot V \qquad \arg\left(\mathbf{V_3}\right) = -11.5\cdot deg$$

Change $\mathbf{V_3}$ back to the time domain:

$$v_3(t) = v_1(t) + v_2(t) = 7.13\cdot\cos(\omega\cdot t - 11.5\cdot deg)\cdot V$$

Ex 2. Two sinusoidal voltages: $v_1(t) = 5\cdot V\cdot\cos(\omega\cdot t + 36.87\cdot deg)$ and $v_2(t) = 3.162\cdot V\cdot\cos(\omega\cdot t - 18.44\cdot deg)$

a) using phasor notation, find $v_3 = v_1 - v_2$

$\mathbf{V_1} := 5\cdot V\cdot e^{j\cdot(36.87\cdot deg)}$ $\quad 5\cdot V\cdot\cos(36.87\cdot deg) = 4\cdot V$

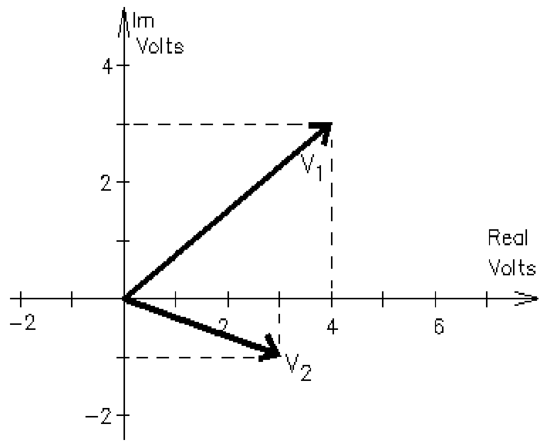
$5\cdot V\cdot\sin(36.87\cdot deg) = 3\cdot V$

$\mathbf{V_1} = 4 + 3j\ \cdot V$

$\mathbf{V_2} := 3.162\cdot V\cdot e^{j\cdot(-18.44\cdot deg)}$ $\quad 3.162\cdot V\cdot\cos(-18.44\cdot deg) = 3\cdot V$

$3.162\cdot V\cdot\sin(-18.44\cdot deg) = -1\cdot V$

$\mathbf{V_2} = 3 - j\ \cdot V$



Subtract real parts: $4\cdot V - 3\cdot V = 1\cdot V$

Subtract imaginary parts: $3\cdot V - -1\cdot V = 4\cdot V$

$\mathbf{V_3} := \mathbf{V_1} - \mathbf{V_2}$ $\qquad \mathbf{V_3} = 1 + 4j\ \cdot V$

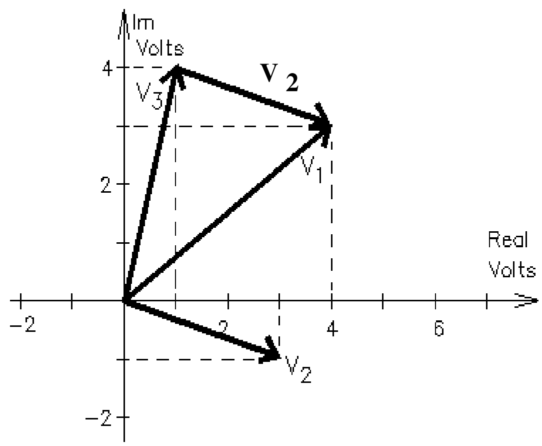
Magnitude: $\sqrt{(1\cdot V)^2 + (4\cdot V)^2} = 4.123\cdot V$

Angle: $\operatorname{atan}\left(\frac{4\cdot V}{1\cdot V}\right) = 75.96\cdot deg$

OR:

$\left|\mathbf{V_3}\right| = 4.123\cdot V$

$\arg\left(\mathbf{V_3}\right) = 75.96\cdot deg$



So: $v_3(t) = v_1(t) - v_2(t) = 4.123\cdot V\cdot\cos(\omega\cdot t + 75.96\cdot deg)\cdot V$

What about Capacitors and Inductors?

Capacitors and Inductors in AC circuits cause 90º phase shifts between voltages and currents because they integrate and differentiate. But... integration and differentiation is a piece-of-cake in phasors.

Calculus

$$\frac{d}{dt}\left[A\cdot e^{j\cdot(\omega\cdot t+\theta)}\right] = j\cdot\omega\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta)} = \omega\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta+90\cdot deg)} = \omega\cdot A\cdot e^{j\cdot(\theta+90\cdot deg)}$$

Drop the ωt (t=0) to get:

$$\int A\cdot e^{j\cdot(\omega\cdot t+\theta)}\,dt = \frac{1}{j\cdot\omega}\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta)} = \frac{1}{\omega}\cdot A\cdot e^{j\cdot(\omega\cdot t+\theta-90\cdot deg)} = \frac{1}{\omega}\cdot A\cdot e^{j\cdot(\theta-90\cdot deg)}$$

Impedance (like resistance)

Inductor

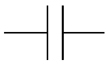
$$v_L = L\cdot\frac{d}{dt}i_L = L\cdot\frac{d}{dt}I_p\cdot e^{j\cdot(\omega\cdot t+\theta)} = j\cdot\omega\cdot L\cdot\left[I_p\cdot e^{j\cdot(\omega\cdot t+\theta)}\right]$$

\ in phasor notation ----> $\mathbf{V}_L(\omega) = j\cdot\omega\cdot L\cdot\mathbf{I}(\omega)$

AC impedance

$$\mathbf{Z}_L = j\cdot\omega\cdot L$$

Capacitor



$$i_C = C\cdot\frac{d}{dt}v_C = C\cdot\frac{d}{dt}V_p\cdot e^{j\cdot(\omega\cdot t+\theta)} = j\cdot\omega\cdot C\cdot\left[V_p\cdot e^{j\cdot(\omega\cdot t+\theta)}\right]$$

\ in phasor notation ----> $\mathbf{I}_C(\omega) = j\cdot\omega\cdot C\cdot\mathbf{V}(\omega)$

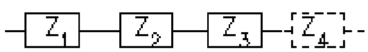
$$\mathbf{V}_C(\omega) = \frac{1}{j\cdot\omega\cdot C}\cdot\mathbf{I}(\omega) \qquad \mathbf{Z}_C = \frac{1}{j\cdot\omega\cdot C} = \frac{-j}{\omega\cdot C}$$

Resistor

$$v_R = i_R\cdot R \qquad \mathbf{V}_R(\omega) = R\cdot\mathbf{I}(\omega) \qquad \mathbf{Z}_R = R$$

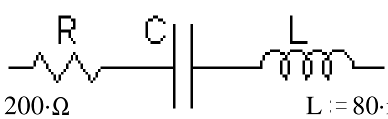
You can use impedances just like resistances as long as you deal with the complex arithmetic.
ALL the DC circuit analysis techniques will work with AC.

series:



$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \ldots$$

Example:



$$f := 500\cdot Hz \qquad R := 200\cdot\Omega \qquad C := 0.6\cdot\mu F \qquad L := 80\cdot mH$$

$$\omega := 2\cdot\pi\cdot f = \omega = 3141.6\cdot\frac{rad}{sec} \qquad j\cdot\omega\cdot L = 251.327j\cdot\Omega$$

$$\frac{1}{j\cdot\omega\cdot C} = -530.516j\cdot\Omega$$

$$\mathbf{Z}_{eq} := R + \frac{1}{j\cdot\omega\cdot C} + j\cdot\omega\cdot L = 200\cdot\Omega - 530.5\cdot j\cdot\Omega + 251.3\cdot j\cdot\Omega = 200 - 279.2j\cdot\Omega$$ rectangular form

$$\sqrt{(200\cdot\Omega)^2 + (279.2\cdot\Omega)^2} = 343.4\cdot\Omega \qquad \text{atan}\left(\frac{-279.2\cdot\Omega}{200\cdot\Omega}\right) = -54.38\cdot deg$$

$\mathbf{Z}_{eq} = 343.4\Omega\ \underline{/-54.4^o}$ polar form

If: $\mathbf{V} := 12\cdot V\cdot e^{j\cdot 0\cdot deg}$ $\quad \mathbf{I} := \frac{\mathbf{V}}{\mathbf{Z}_{eq}} = \frac{12\cdot V}{343.4\cdot\Omega} = 34.945\cdot mA \quad \underline{/}\ 0 - -54.4 = 54.4$ deg

$\mathbf{I} = 34.95mA\ \underline{/54.4^o} = \mathbf{I} = 20.348 + 28.405j\cdot mA$

Voltage divider:

$$V_{Zn} = V_{total}\cdot\frac{\mathbf{Z}_n}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} + \ldots$$

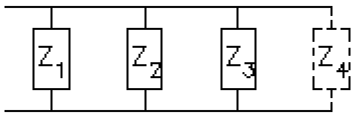
Note: $\frac{1}{j} = -j = 1\ \underline{/-90^o}$

Eg: $\mathbf{V}_C := \mathbf{V}\cdot\frac{\frac{1}{j\cdot\omega\cdot C}}{\mathbf{Z}_{eq}} = 12\cdot V\cdot e^{j\cdot 0\cdot deg}\cdot\frac{530.516\cdot e^{-j\cdot 90\cdot deg}\cdot\Omega}{343.4\cdot e^{-j\cdot 54.38\cdot deg}\cdot\Omega}$

$12\cdot V\cdot\frac{530.516\cdot\Omega}{343.4\cdot\Omega} = 18.539\cdot V \quad \underline{/}\ 0 + -90 - -54.4 = -35.6$ deg

$\mathbf{V}_C = 18.54V\ \underline{/-35.6^o} = \mathbf{V}_C = 15.069 - 10.795j\cdot V$

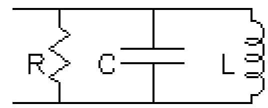
parallel:



$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots}$$

Example:

$f := 500 \cdot Hz \qquad \omega := 2 \cdot \pi \cdot f \qquad \omega = 3141.6 \cdot \frac{rad}{sec}$



$L := 80 \cdot mH$

$j \cdot \omega \cdot L = 251.327j \cdot \Omega$

$R := 200 \cdot \Omega \qquad C := 0.6 \cdot \mu F \qquad \frac{1}{\omega \cdot L} = 3.979 \cdot 10^{-3} \cdot \frac{1}{\Omega}$

$$\frac{1}{j \cdot \omega \cdot C} = -530.516j \cdot \Omega$$

$$\omega \cdot C = 1.885 \cdot 10^{-3} \cdot \frac{1}{\Omega}$$

$$\mathbf{Z}_{eq} := \frac{1}{\frac{1}{R} + \frac{1}{\left(\frac{1}{j \cdot \omega \cdot C}\right)} + \frac{1}{j \cdot \omega \cdot L}} = \frac{1}{\frac{1}{R} + j \cdot \omega \cdot C - \frac{j}{\omega \cdot L}} = \frac{1}{\frac{1}{200 \cdot \Omega} + 1.885 \cdot 10^{-3} \cdot j \cdot \frac{1}{\Omega} - 3.979 \cdot 10^{-3} \cdot j \cdot \frac{1}{\Omega}}$$

$$= \frac{1}{\left(5 \cdot 10^{-3} - 2.094 \cdot 10^{-3} \cdot j\right) \cdot \frac{1}{\Omega}} \cdot \frac{5 \cdot 10^{-3} + 2.094 \cdot 10^{-3} \cdot j}{\left(5 \cdot 10^{-3} + 2.094 \cdot 10^{-3} \cdot j\right)} = 170.156 + 71.261j \cdot \Omega$$

$2.93848 \cdot 10^{-5}$

If you want the answer in polar form, it's easier to convert the denominator first.

$$\sqrt{\left(5 \cdot 10^{-3} \cdot \frac{1}{\Omega}\right)^2 + \left(2.094 \cdot 10^{-3} \cdot \frac{1}{\Omega}\right)^2} = 5.4 \cdot 10^{-3} \cdot \frac{1}{\Omega} \qquad \operatorname{atan}\left(\frac{-2.094 \cdot 10^{-3} \cdot \Omega}{5 \cdot 10^{-3} \cdot \Omega}\right) = -22.72 \cdot deg$$

$$\frac{1}{5.4 \cdot 10^{-3} \cdot \frac{1}{\Omega}} = 185.185 \cdot \Omega$$

$\mathbf{Z}_{eq} = 185.2 \underline{/22.7^o}$

If: $\quad \mathbf{V} := 12 \cdot V \cdot e^{j \cdot 0 \cdot deg} \qquad \mathbf{I} := \frac{\mathbf{V}}{\mathbf{Z}_{eq}} = \frac{12 \cdot V}{185.2 \cdot \Omega} = 64.795 \cdot mA \quad \underline{/} \; 0 - 22.7 = -22.7 \; deg$

$\mathbf{I} = 60 - 25.127j \cdot mA$

Current divider:

$$\mathbf{I}_{Zn} = \mathbf{I}_{total} \cdot \frac{\frac{1}{\mathbf{Z}_n}}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots}$$

Eg: $\quad \mathbf{I}_L := \mathbf{I} \cdot \frac{\frac{1}{j \cdot \omega \cdot L}}{\frac{1}{R} + j \cdot \omega \cdot C + \frac{1}{j \cdot \omega \cdot L}} = \mathbf{I} \cdot \frac{\mathbf{Z}_{eq}}{j \cdot \omega \cdot L}$

$$= 64.795 \cdot mA \cdot e^{j \cdot 22.7 \cdot deg} \cdot \frac{185.2 \cdot e^{-j \cdot 22.7 \cdot deg} \cdot \Omega}{251.327 \cdot e^{j \cdot 90 \cdot deg} \cdot \Omega}$$

$$= 64.795 \cdot mA \cdot \frac{185.2 \cdot \Omega}{251.327 \cdot \Omega} = 47.747 \cdot mA \quad \underline{/} \; 22.7 + -22.7 - 90 = -90 \; deg$$

$\mathbf{I}_L = -47.746j \cdot mA$

Duh... $\quad \frac{\mathbf{V}}{j \cdot \omega \cdot L} = -47.746j \cdot mA$

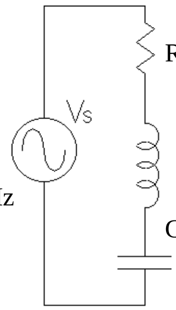
ECE 2210 / 00 Phasor Examples

Ex 1. Find $\mathbf{V_R}$, $\mathbf{V_L}$, and $\mathbf{V_C}$ in polar phasor form. $f := 2 \cdot kHz$

$\mathbf{V}(j\omega)$

$6 \cdot V \cdot e^{j0}$

$f := 2 \cdot kHz$

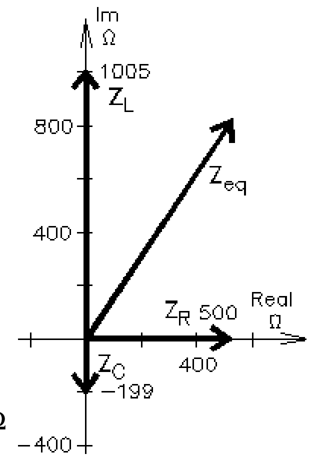


$R := 500 \cdot \Omega$ $\qquad \omega := 2 \cdot \pi \cdot f$ $\qquad \omega = 1.257 \cdot 10^4 \cdot \frac{rad}{sec}$

$L := 80 \cdot mH$ $\qquad \mathbf{Z_L} := j \cdot \omega \cdot L$ $\qquad \mathbf{Z_L} = 1.005j \cdot k\Omega$

$C := 0.4 \cdot \mu F$ $\qquad \mathbf{Z_C} := \frac{1}{j \cdot \omega \cdot C}$ $\qquad \mathbf{Z_C} = -0.199j \cdot k\Omega$

$\mathbf{Z_{eq}} := R + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}$ $\qquad \mathbf{Z_{eq}} = 500 + 806.366j \cdot \Omega$



$\sqrt{500^2 + 806^2} = 948.491$ $\qquad \operatorname{atan}\left(\frac{806}{500}\right) = 58.187 \cdot deg$ $\qquad \mathbf{Z_{eq}} = 948.5\Omega\ \underline{/\ 58.}2^o$

find the current: $\mathbf{I} := \frac{6 \cdot V \cdot e^{j \cdot 0}}{\mathbf{Z_{eq}}}$ $\qquad$ magnitude: $\frac{6 \cdot V}{948.5 \cdot \Omega} = 6.326 \cdot mA$ $\qquad$ angle: $0 \cdot deg - 58.2 \cdot deg = -58.2 \cdot deg$

$\mathbf{I} = 6.326mA\ \underline{/-58.}2^o$

	<u>find the magnitude</u>	<u>find the angle</u>	
$\mathbf{V_R} := \mathbf{I} \cdot R$	$6.326 \cdot mA \cdot 500 \cdot \Omega = 3.163 \cdot V$	$-58.2 \cdot deg + 0 \cdot deg = -58.2 \cdot deg$	$\mathbf{V_R} = 3.163V\ \underline{/-58.}2^o$
$\mathbf{V_L} := \mathbf{I} \cdot \mathbf{Z_L}$	$6.326 \cdot mA \cdot 1005 \cdot \Omega = 6.358 \cdot V$	$-58.2 \cdot deg + 90 \cdot deg = 31.8 \cdot deg$	$\mathbf{V_L} = 6.358V\ \underline{/\ 31.8}^o$
$\mathbf{V_C} := \mathbf{I} \cdot \mathbf{Z_C}$	$6.326 \cdot mA \cdot (-199) \cdot \Omega = -1.259 \cdot V$	$-58.2 \cdot deg + (90) \cdot deg = 31.8 \cdot deg$	$\mathbf{V_C} = -1.259V\ \underline{/\ 31.8}^o$
OR:	$6.326 \cdot mA \cdot (199) \cdot \Omega = 1.259 \cdot V$	$-58.2 \cdot deg + (-90) \cdot deg = -148.2 \cdot deg$	$\mathbf{V_C} = 1.259V\ \underline{/-148.2}^o$

OR, you can also find these voltages directly, using a voltage divider. I.E. to find $\mathbf{V_C}$ directly:

$$\mathbf{V_C} := \frac{\frac{1}{j \cdot \omega \cdot C}}{R + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}} \cdot 6 \cdot V = \frac{1}{R \cdot (j \cdot \omega \cdot C) + j \cdot \omega \cdot L \cdot (j \cdot \omega \cdot C) + 1} \cdot 6 \cdot V = \frac{1}{R \cdot (j \cdot \omega \cdot C) - \omega^2 \cdot L \cdot C + 1} \cdot 6 \cdot V$$

$$= \frac{1}{\left(1 - \omega^2 \cdot L \cdot C\right) + j \cdot \omega \cdot R \cdot C} \cdot 6 \cdot V \qquad \left(1 - \omega^2 \cdot L \cdot C\right) = -4.053 \qquad j \cdot \omega \cdot R \cdot C = 2.513j$$

$$= \frac{6 \cdot V}{-4.053 + 2.513 \cdot j} \cdot \frac{(-4.053 - 2.513 \cdot j)}{(-4.053 - 2.513 \cdot j)} = \frac{6 \cdot V \cdot (-4.053 - 2.513 \cdot j)}{(-4.053)^2 + 2.513^2}$$

$6 \cdot V \cdot (-4.053 - 2.513 \cdot j) = -24.318 - 15.078j \cdot V$

$(-4.053)^2 + 2.513^2 = 22.742$

$$= \left(\frac{-24.318}{22.742} - \frac{15.078 \cdot j}{22.742}\right) \cdot V = -1.069 - 0.663j \cdot V$$

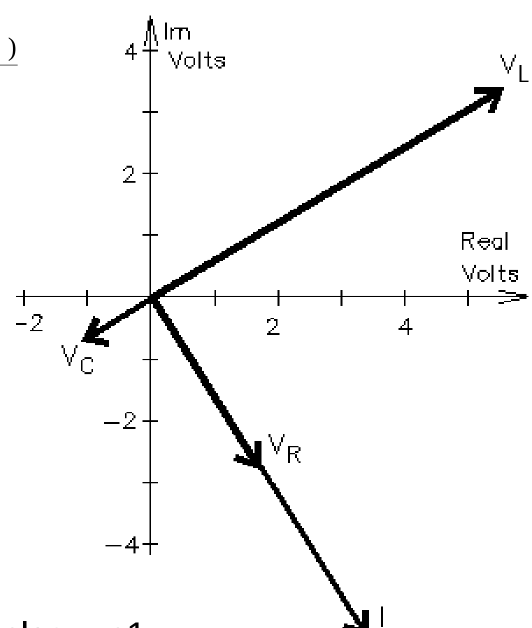
magnitude: $\sqrt{1.069^2 + 0.663^2} = 1.258$

angle: $\operatorname{atan}\left(\frac{-0.663}{-1.069}\right) = 31.81 \cdot deg$

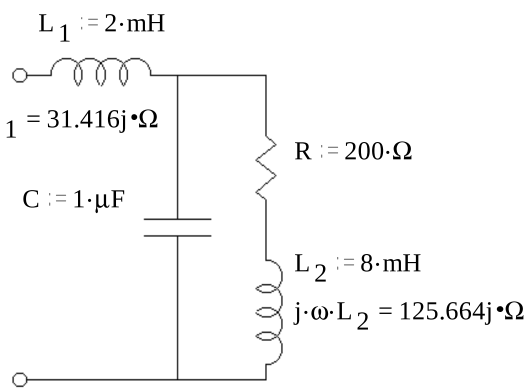
but this is actually in the third quadrant,
so modify your calculator's results:

$31.81 \cdot deg - 180 \cdot deg = -148.19 \cdot deg$

$= 1.258V\ \underline{/-148.2}^o$



Ex 2. a) Find $\mathbf{Z_{eq}}$. $f := 2.5 \cdot kHz$ $\quad \omega := 2 \cdot \pi \cdot f$ $\quad \omega = 1.571 \cdot 10^{4} \cdot \frac{rad}{sec}$



$$\mathbf{Z_{eq}} = j \cdot \omega \cdot L_1 + \cfrac{1}{\cfrac{1}{\left(\cfrac{1}{j \cdot \omega \cdot C}\right)} + \cfrac{1}{R + j \cdot \omega \cdot L_2}}$$

But it's easier to split the problem up

Left branch

$$\mathbf{Z_l} := \frac{1}{j \cdot \omega \cdot C} \qquad \mathbf{Z_l} = -63.662j \cdot \Omega$$

$$\frac{1}{\left(\frac{1}{j \cdot \omega \cdot C}\right)} = j \cdot \omega \cdot C = 0.01571i \cdot \frac{1}{\Omega}$$

Right branch

$$\mathbf{Z_r} := j \cdot \omega \cdot L_2 + R \qquad \mathbf{Z_r} = 200 + 125.664j \cdot \Omega$$

$$\frac{1}{200 + 125.664 \cdot j} = 3.585 \cdot 10^{-3} - 2.252 \cdot 10^{-3} j$$

denominator: $j \cdot \omega \cdot C + \frac{1}{R + j \cdot \omega \cdot L_2} = 0.01571 \cdot j + \left(3.585 \cdot 10^{-3} - 2.252 \cdot 10^{-3} \cdot j\right) = 3.585 \cdot 10^{-3} + 1.346 \cdot 10^{-2} i \quad \frac{1}{\Omega}$

rectangular division:

$$\frac{1}{\left(3.585 \cdot 10^{-3} + 1.346 \cdot 10^{-2} \cdot j\right)} \cdot \frac{\left(3.585 \cdot 10^{-3} - 1.346 \cdot 10^{-2} \cdot j\right)}{\left(3.585 \cdot 10^{-3} - 1.346 \cdot 10^{-2} \cdot j\right)} = \frac{3.585 \cdot 10^{-3} - 1.346 \cdot 10^{-2} \cdot j}{1.94 \cdot 10^{-4}} = 18.479 - 69.381j \quad \Omega$$

$$\left(3.585 \cdot 10^{-3}\right)^2 + \left(1.346 \cdot 10^{-2}\right)^2 = 1.94 \cdot 10^{-4}$$

add: $j \cdot \omega \cdot L_1 = 31.416j \cdot \Omega$ $\qquad 31.416 \cdot j + (18.479 - 69.381 \cdot j) = 18.479 - 37.965j \quad \Omega$

convert to polar (if needed): $\sqrt{18.48^2 + 37.97^2} = 42.228$ $\qquad \operatorname{atan}\left(\frac{-37.97}{18.48}\right) = -64.048 \cdot deg$ $\qquad \mathbf{Z_{eq}} = 42.23\Omega \ \underline{/-64.05^{o}}$

Another Way

Sometimes you might simplify a little before putting in numbers.

$$\mathbf{Z_{eq}} := j \cdot \omega \cdot L_1 + \cfrac{1}{\cfrac{1}{R + j \cdot \omega \cdot L_2} + \cfrac{1}{\cfrac{1}{j \cdot \omega \cdot C}}} = j \cdot \omega \cdot L_1 + \cfrac{1}{\cfrac{1}{R + j \cdot \omega \cdot L_2} + j \cdot \omega \cdot C} = j \cdot \omega \cdot L_1 + \frac{R + j \cdot \omega \cdot L_2}{1 + j \cdot \omega \cdot C \cdot \left(R + j \cdot \omega \cdot L_2\right)}$$

$$= j \cdot \omega \cdot L_1 + \frac{R + j \cdot \omega \cdot L_2}{1 - \omega^2 \cdot C \cdot L_2 + j \cdot \omega \cdot C \cdot R}$$

$$\mathbf{Z_{eq}} = 31.416 \cdot j \cdot \Omega + \frac{(200 + 125.664 \cdot j) \cdot \Omega}{-0.974 + 3.142 \cdot j} \cdot \left(\frac{-0.974 - 3.142 \cdot j}{-0.974 - 3.142 \cdot j}\right) = 31.416 \cdot j \cdot \Omega + \frac{(200 + 125.664 \cdot j) \cdot (-0.974 - 3.142 \cdot j)}{0.974^2 + 3.142^2}$$

$$= 31.416 \cdot j \cdot \Omega + \frac{((200 \cdot (-0.974) - 125.664 \cdot (-3.142)) + (125.664 \cdot (-0.974) - 200 \cdot 3.142) \cdot j) \cdot \Omega}{0.974^2 + 3.142^2}$$

$$= 31.416 \cdot j \cdot \Omega + \frac{(200.036288 - 750.796736 \cdot j) \cdot \Omega}{10.82084} = 31.416 \cdot j \cdot \Omega + 18.486 \cdot \Omega - 69.384 \cdot j \cdot \Omega = 18.486 - 37.968j \cdot \Omega$$

$\sqrt{18.49^2 + 37.97^2} = 42.233$ $\qquad \operatorname{atan}\left(\frac{-37.97}{18.49}\right) = -64.036 \cdot deg$ $\qquad \mathbf{Z_{eq}} = 42.23\Omega \ \underline{/-64.04^{o}}$

a little roundoff difference

b) $\mathbf{V}_{\mathbf{in}} := 12 \cdot V \cdot e^{j \cdot 20 \cdot deg}$ Find $\mathbf{I}_{\mathbf{L1}}$, $\mathbf{V}_{\mathbf{C}}$

$$\mathbf{I}_{\mathbf{L1}} := \frac{\mathbf{V}_{\mathbf{in}}}{\mathbf{Z}_{\mathbf{eq}}} \qquad \frac{12 \cdot V}{42.23 \cdot \Omega} = 284.16 \cdot mA \qquad 20 \cdot deg - (-64.04) \cdot deg = 84.04 \cdot deg$$

$\mathbf{I}_{\mathbf{L1}} = 284mA$ $\underline{/\ 84.04^{o}}$

$$\mathbf{V}_{\mathbf{C}} := \mathbf{I}_{\mathbf{L1}} \cdot (18.479 - 69.381 \cdot j) \cdot \Omega \qquad 284 \cdot mA \cdot \sqrt{18.479^2 + 69.381^2} \cdot \Omega = 20.391 \cdot V$$

$$84.04 \cdot deg + atan\left(\frac{-69.381}{18.479}\right) = 8.954 \cdot deg$$

$\mathbf{V}_{\mathbf{C}} = 20.4V$ $\underline{/\ 8.95^{o}}$

You could then use another voltage divider to find $\mathbf{V}_{\mathbf{R}}$ or $\mathbf{V}_{\mathbf{L2}}$.

convert to rectangular (if needed):

$$20.391 \cdot V \cdot \cos(8.954 \cdot deg) = 20.143 \cdot V$$

$$20.391 \cdot V \cdot \sin(8.954 \cdot deg) = 3.174 \cdot V$$

$\mathbf{V}_{\mathbf{C}} = 20.14 + 3.174 \cdot j \quad V$

Another Way

To find V_C directly:

$$\mathbf{V}_{\mathbf{C}} := \frac{\dfrac{1}{\dfrac{1}{R + j \cdot \omega \cdot L_2} + j \cdot \omega \cdot C}}{j \cdot \omega \cdot L_1 + \dfrac{1}{\dfrac{1}{R + j \cdot \omega \cdot L_2} + j \cdot \omega \cdot C}} \cdot \mathbf{V}_{\mathbf{in}}$$

--> math --> $\mathbf{V}_{\mathbf{C}} = 20.153 + 3.178j \cdot V$ Same but for a little roundoff difference

c) Let's find $\mathbf{I}_{\mathbf{L2}}$.

$$\mathbf{Z}_{\mathbf{r}} = 200 + 125.664j \cdot \Omega \qquad \sqrt{200^2 + 125.664^2} = 236.202 \qquad atan\left(\frac{125.664}{200}\right) = 32.142 \cdot deg$$

$$\mathbf{I}_{\mathbf{L2}} := \frac{\mathbf{V}_{\mathbf{C}}}{\mathbf{Z}_{\mathbf{r}}} = \frac{20.4 \cdot V \cdot e^{j \cdot 8.95 \cdot deg}}{236.202 \cdot \Omega \cdot e^{j \cdot 32.142 \cdot deg}} = \frac{20.4 \cdot V}{236.202 \cdot \Omega} \ \underline{/\ 8.95 - 32.142^{o}} \quad = 86.4mA \ \underline{/-23.19}$$

Another Way

Directly by Current divider:

$$\mathbf{I}_{\mathbf{L2}} := \frac{\dfrac{1}{R + j \cdot \omega \cdot L_2}}{j \cdot \omega \cdot C + \dfrac{1}{R + j \cdot \omega \cdot L_2}} \cdot \mathbf{I}_{\mathbf{L1}} = \frac{1}{j \cdot \omega \cdot C \cdot (R + j \cdot \omega \cdot L_2) + 1} \cdot \mathbf{I}_{\mathbf{L1}} = \frac{\mathbf{I}_{\mathbf{L1}}}{1 - \omega^2 \cdot C \cdot L_2 + j \cdot \omega \cdot C \cdot R}$$

real part is negative

denominator:

$$\sqrt{(1 - \omega^2 \cdot C \cdot L_2)^2 + (\omega \cdot C \cdot R)^2} = 3.289 \qquad atan\left(\frac{\omega \cdot C \cdot R}{1 - \omega^2 \cdot C \cdot L_2}\right) + 180 \cdot deg = 107.224 \cdot deg$$

$$\mathbf{I}_{\mathbf{L2}} = \frac{284 \cdot mA \cdot e^{j \cdot 84.04 \cdot deg}}{3.289 \cdot e^{j \cdot 107.224 \cdot deg}} = \frac{284 \cdot mA}{3.289} \ \underline{/\ 84.04 - 107.224^{o}} \quad = 86.4mA \ \underline{/-23.18^{o}}$$

d) How about $\mathbf{I}_{\mathbf{C}}$?

$$\mathbf{I}_{\mathbf{C}} := \frac{\mathbf{V}_{\mathbf{C}}}{\left(\dfrac{1}{j \cdot \omega \cdot C}\right)} = \mathbf{V}_{\mathbf{C}} \cdot j \cdot \omega \cdot C \quad = 20.4V \ \underline{/\ 8.95^{o}} \ 0.015708 \ \underline{/\ 90^{o}} \ \frac{1}{\Omega} \quad = 320mA \ \underline{/\ 98.95^{o}}$$

Another Way Could also be found directly by current divider:

$$\mathbf{I}_{\mathbf{C}} := \frac{j \cdot \omega \cdot C}{j \cdot \omega \cdot C + \dfrac{1}{R + j \cdot \omega \cdot L_2}} \cdot \mathbf{I}_{\mathbf{L1}} \quad = 320mA \ \underline{/\ 98.95^{o}}$$

<u>Something Weird</u>

$\mathbf{I}_{\mathbf{C}}$ is greater than the input current ($\mathbf{I}_{\mathbf{L1}}$) . What's going on?

The angle between $\mathbf{I}_{\mathbf{C}}$ & $\mathbf{I}_{\mathbf{L2}}$ is big enough that they somewhat cancel each other out (partially resonate).

Check Kirchoff's Current Law: $\mathbf{I}_{\mathbf{C}} + \mathbf{I}_{\mathbf{L2}} = 29.485 + 282.569j \cdot mA \overset{?}{\underset{yes}{=}} \mathbf{I}_{\mathbf{L1}} = 29.485 + 282.569j \cdot mA$

Ex 3. a) Find $\mathbf{Z_2}$.

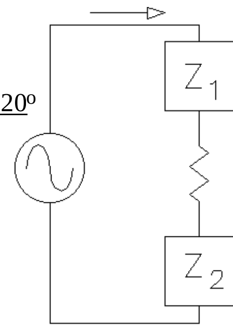
$i(t) = 25\cdot mA\cdot\cos\left(377\cdot\frac{rad}{sec}\cdot t + 10\cdot deg\right)$

$\mathbf{I} := 25\cdot mA\cdot e^{j\cdot 10\cdot deg}$

$\mathbf{V_{in}} := 10\cdot V\cdot e^{-j\cdot 20\cdot deg}$

$\mathbf{Z_T} := \frac{\mathbf{V_{in}}}{\mathbf{I}} = \frac{10\cdot V}{25\cdot mA}$ $\underline{/ -20 - 10^o}$ $= 400\Omega\ \underline{/-30^o}$

$\mathbf{Z_T} = 346.41 - 200j\ \bullet\Omega$



$\mathbf{V_{in}}$ = 10V $\underline{/-20^o}$

$\mathbf{Z_1} := (120 - 60\cdot j)\cdot\Omega$

$R := 50\cdot\Omega$

$\mathbf{Z_2}$ = ?

$\mathbf{Z_2} := \mathbf{Z_T} - R - \mathbf{Z_1} = (346.41 - 200\cdot j\ )\cdot\Omega - 50\cdot\Omega - (120 - 60\cdot j)\cdot\Omega = 176.41 - 140j\ \bullet\Omega$

b) Circle 1: i) The source current leads the source voltage <--- answer, because $10^o > -20^o$.

ii) The source voltage leads the source current

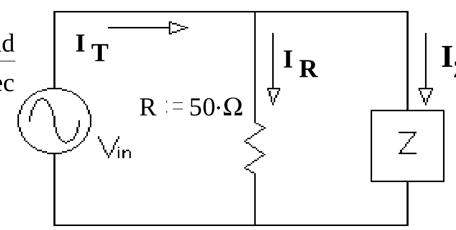
Ex 4. a) Find $\mathbf{V_{in}}$ in polar form.

$\mathbf{I_Z} := 100\cdot mA$ $\quad \mathbf{Z} := (80 - 60\cdot j)\cdot\Omega$ $\quad \omega := 1000\cdot\frac{rad}{sec}$

$\mathbf{V_{in}} := \mathbf{I_Z}\cdot\mathbf{Z}$ $\quad \mathbf{V_{in}} = 8 - 6j\ \bullet V$

$\sqrt{8^2 + 6^2} = 10$ $\quad \operatorname{atan}\left(\frac{-6}{8}\right) = -36.87\ \bullet deg$

$\mathbf{V_{in}}$ = 10V $\underline{/-36.9^o}$



$\mathbf{I_Z}$ = 100 $\underline{/\ 0^o}$ mA

$\mathbf{Z} := (80 - 60\cdot j)\cdot\Omega$

b) Find $\mathbf{I_T}$ in polar form. $\quad \mathbf{I_R} := \frac{\mathbf{V_{in}}}{R} = \frac{10\cdot V}{50\cdot\Omega}$ $\underline{/-36.9^o}$ $= \frac{10\cdot V}{50\cdot\Omega}\cdot\cos(-36.9\cdot deg) + j\cdot\frac{10\cdot V}{50\cdot\Omega}\cdot\sin(-36.9\cdot deg) = 160 - 120i\ \bullet mA$

$\mathbf{I_T} := \mathbf{I_R} + \mathbf{I_Z} = (160 - 120\cdot j\ )\cdot mA + 100\cdot mA = 260 - 120j\ \bullet mA$

$\sqrt{260^2 + 120^2} = 286.356$ $\quad \operatorname{atan}\left(\frac{-120}{260}\right) = -24.78\ \bullet deg$ $\quad \mathbf{I_T}$ = 286mA $\underline{/-24.8^o}$

c) Circle 1: i) The source current leads the source voltage answer i), $-24.8^o > -36.9^o$

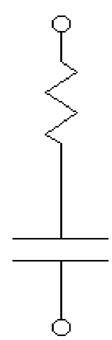
ii) The source voltage leads the source current

d) The impedance $\mathbf{Z}$ (above) is made of two components in series. What are they and what are their values?

$\mathbf{Z} = 80 - 60j\ \bullet\Omega$

Must have a resistor because there is a real part.

$R := \operatorname{Re}(\mathbf{Z})$ $\quad R = 80\ \bullet\Omega$



Must have a capacitor because the imaginary part is negative.

$\operatorname{Im}(\mathbf{Z}) = -60\ \bullet\Omega$ $\quad = \frac{-1}{\omega\cdot C}$ $\quad C := \frac{-1}{\omega\cdot\operatorname{Im}(\mathbf{Z})}$ $\quad C = 16.667\ \bullet\mu F$

Ex 5. The impedance $\mathbf{Z} = 80 - 60j \cdot \Omega$ is made of two components in <u>parallel</u>. What are they and what are their values?

Must have a resistor because there is a real part.

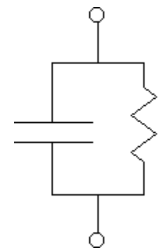
Must have an capacitor because the imaginary part is negative.

$$\mathbf{Z} = \frac{1}{\frac{1}{R} + j\cdot\omega\cdot C} \qquad \frac{1}{\mathbf{Z}} = \frac{1}{(80 - 60\cdot j)\cdot\Omega}\cdot\left(\frac{80+60\cdot j}{80+60\cdot j}\right) = \frac{80+60\cdot j}{80^2+60^2} = \frac{80+60\cdot j}{10{,}000}\cdot\frac{1}{\Omega}$$

$$\frac{1}{\mathbf{Z}} = 0.008 + 0.006i \cdot \Omega^{-1} = \frac{1}{R} + j\cdot\omega\cdot C$$

$$\frac{1}{R} = .008\cdot\frac{1}{\Omega} \qquad R := \frac{1}{.008\cdot\Omega^{-1}} \qquad R = 125\cdot\Omega$$

$$\omega\cdot C = .006\cdot\frac{1}{\Omega} \qquad C := \frac{.006\cdot\Omega^{-1}}{\omega} \qquad C = 6\cdot\mu F$$



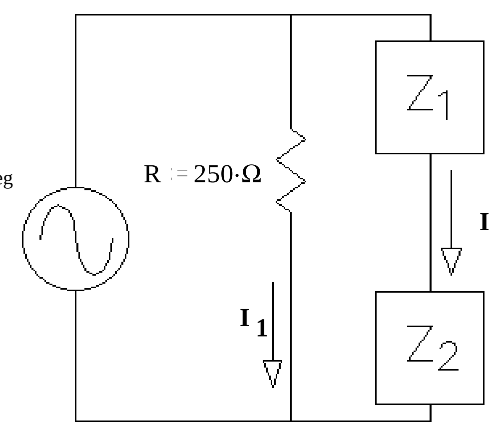
$R = 125\cdot\Omega$

Positive imaginary parts would require inductors

Ex 6. a) Find $\mathbf{I_1}$

$$\omega := 20000\cdot\frac{rad}{sec}$$

$$\mathbf{V_{in}} := 20\cdot V\cdot e^{j\cdot 30\cdot deg}$$



voltage $\mathbf{V_{Z1}} := (8 - 5\cdot j)\cdot V$

$\mathbf{I_2} := 20\cdot mA\cdot e^{j\cdot 20\cdot deg}$

$$\mathbf{I_1} := \frac{\mathbf{V_{in}}}{R} = \frac{20\cdot V}{250\cdot\Omega}\cdot e^{j\cdot 30\cdot deg} = 80\cdot mA\cdot e^{j\cdot 30\cdot deg}$$

polar division

b) Circle 1: i) $\mathbf{V_{in}}$ leads $\mathbf{I_2}$ ii) $\mathbf{V_{in}}$ lags $\mathbf{I_2}$

Why? Show numbers: __30__ > __20__ ______ < ______

c) Find $\mathbf{Z_2}$ in polar form

Convert $\mathbf{V_{in}}$ to rectangular coordinates

$20\cdot V\cdot\cos(30\cdot deg) = 17.321\cdot V$ $\qquad$ $20\cdot V\cdot\sin(30\cdot deg) = 10\cdot V$ $\qquad$ pol to rect

$\mathbf{V_{in}} = 17.321 + 10j \cdot V$

$\mathbf{V_{Z2}} := \mathbf{V_{in}} - \mathbf{V_{Z1}}$ $\qquad$ $\mathbf{V_{Z2}} = 9.321 + 15j \cdot V$ $\qquad$ subtract

rect to pol $\qquad \sqrt{9.321^2 + 15^2} = \left|\mathbf{V_{Z2}}\right| = 17.66\cdot V$

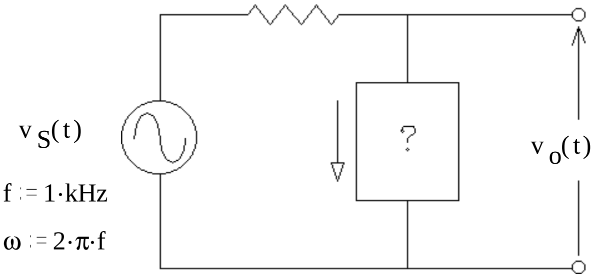
$$\text{atan}\left(\frac{15}{9.321}\right) = \arg\left(\mathbf{V_{Z2}}\right) = 58.145\cdot deg$$

div $\quad \mathbf{Z_2} := \dfrac{\mathbf{V_{Z2}}}{\mathbf{I_2}} \qquad \dfrac{17.66\cdot V}{20\cdot mA} = 883\cdot\Omega \qquad \underline{/\ } \ 58.145\cdot deg - 20\cdot deg = 38.145\cdot deg \qquad \mathbf{Z_2} = 883\ \underline{/\ 38.15^{o}}\ \Omega$

$\mathbf{Z_2} = 694.436 + 545.379j \cdot\Omega$

Ex 7. You need to design a circuit in which the "output" voltage leads the input voltage ($v_S(t)$) by 40º of phase.

$R := 400 \cdot \Omega$



$f := 1 \cdot kHz$

$\omega := 2 \cdot \pi \cdot f$

$\omega = 6.283 \cdot 10^3 \cdot \frac{rad}{sec}$

a) What should go in the box: R, L, C?

$$\mathbf{V_o} = \frac{\mathbf{Z_{box}}}{R + \mathbf{Z_{box}}} \cdot \mathbf{V_S}$$

angle of $\frac{\mathbf{Z_{box}}}{R + \mathbf{Z_{box}}}$ is 40º.

This can only happen if the angle of $\mathbf{Z_{box}}$ is positive, so $\mathbf{Z_{box}}$ is a inductor

b) Find its value. $\mathbf{V_o} =$ $\quad \mathbf{V_o} = \frac{j \cdot \omega \cdot L}{R + j \cdot \omega \cdot L} \cdot \mathbf{V_S}$ angle $\frac{j \cdot \omega \cdot L}{R + j \cdot \omega \cdot L}$ is $90 - \operatorname{atan}\left(\frac{\omega \cdot L}{R}\right)$ = 40º.

So: $\operatorname{atan}\left(\frac{\omega \cdot L}{R}\right) = 50^o$ $\qquad \frac{\omega \cdot L}{R} = \tan(50 \cdot deg) = 1.192$ $\qquad L = \frac{R \cdot 1.192}{\omega} = 75.9 \cdot mH$

c) Repeat if the "output" voltage should lag the input voltage ($v_S(t)$) by 20º of phase.

angle of $\frac{\mathbf{Z_{box}}}{R + \mathbf{Z_{box}}}$ is -20º. This can only happen if the angle of $\mathbf{Z_{box}}$ is negative, so $\mathbf{Z_{box}}$ is a capacitor

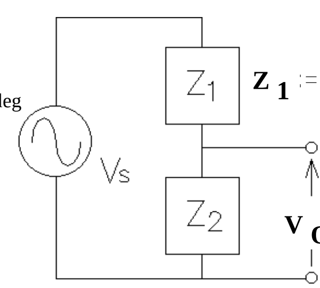
$$\mathbf{V_o} = \frac{\frac{1}{j \cdot \omega \cdot C}}{R + \frac{1}{j \cdot \omega \cdot C}} \cdot \mathbf{V_S}$$

angle $\frac{\frac{1}{j \cdot \omega \cdot C}}{R + \frac{1}{j \cdot \omega \cdot C}}$ is $-90 - \operatorname{atan}\left(\frac{-\frac{1}{\omega \cdot C}}{R}\right) = -90 - \operatorname{atan}\left(-\frac{1}{\omega \cdot C \cdot R}\right)$

$\operatorname{atan}\left(-\frac{1}{\omega \cdot C \cdot R}\right) = -70^o$. $\qquad -\frac{1}{\omega \cdot C \cdot R} = \tan(-70 \cdot deg) = -2.747$ $\qquad C = \frac{1}{\omega \cdot R \cdot 2.747} = 0.145 \cdot \mu F$

Ex 8. Find $\mathbf{V_O}$ in the circuit shown. Express it as a magnitude and phase angle (polar).

$\mathbf{V_S} := 6 \cdot V \cdot e^{j \cdot 18 \cdot deg}$



$\mathbf{Z_1} := 25 \cdot \Omega + 35j \cdot \Omega$

$\mathbf{Z_2} := 80 \cdot \Omega \cdot e^{-j \cdot 60 \cdot deg}$

$\mathbf{V_O}$ = ?

$\mathbf{V_O} := \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}} \cdot \mathbf{V_S}$ Simple voltage divider

$|\mathbf{Z_2}| \cdot \cos(-60 \cdot deg) = 40 \cdot \Omega$ $\qquad |\mathbf{Z_2}| \cdot \sin(-60 \cdot deg) = -69.282 \cdot \Omega$ $\qquad \mathbf{Z_2} = 40 - 69.282j \cdot \Omega$

$\mathbf{Z_1} + \mathbf{Z_2} = 25 \cdot \Omega + 35j \cdot \Omega + 40 \cdot \Omega - 69.282 \cdot j \cdot \Omega = 65 - 34.282j \cdot \Omega \qquad = 73.486 \cdot \Omega \cdot e^{-j \cdot 27.81 \cdot deg}$

$$\mathbf{V_O} := \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}} \cdot \mathbf{V_S} = \frac{80 \cdot \Omega \cdot e^{-j \cdot 60 \cdot deg}}{73.486 \cdot \Omega \cdot e^{-j \cdot 27.81 \cdot deg}} \cdot \left(6 \cdot V \cdot e^{j \cdot 18 \cdot deg}\right) = \frac{80 \cdot \Omega}{73.486 \cdot \Omega} \cdot 6 \cdot V \cdot e^{j \cdot (-60 - (-27.81) + 18) \cdot deg} = 6.53 \cdot V \cdot e^{-j \cdot 14.2 \cdot deg}$$