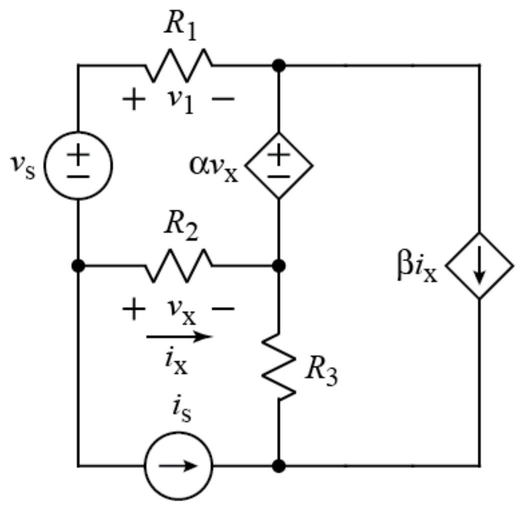
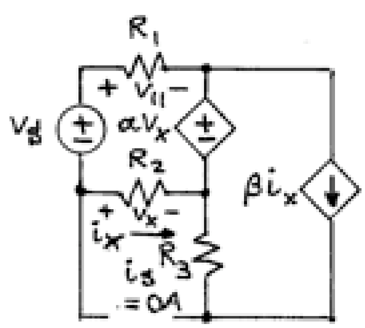
Ex: Using superposition, find an expression for v_1 in the circuit shown below.



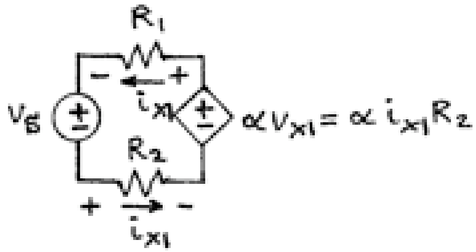
sol'n: We turn on one independent source at a time. Dependent sources stay on.

case I: v_s on, i_s off = open



If we examine where i_x goes, we discover that i_x flows thru R_1. Thus, a v-loop around the upper left yields a value for i_x.

Note that $v_{x1} = i_{x1} R_2$.



v-loop: $v_S + i_{x1}R_1 - i_{x1}\alpha R_2 + i_{x1}R_2 = 0\text{ V}$

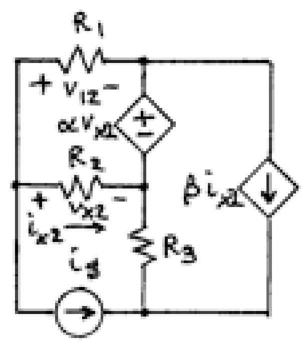
or $i_{x1}(R_1 + R_2 - \alpha R_2) = -v_S$

or $i_{x1} = \dfrac{-v_S}{R_1 + R_2 - \alpha R_2}$

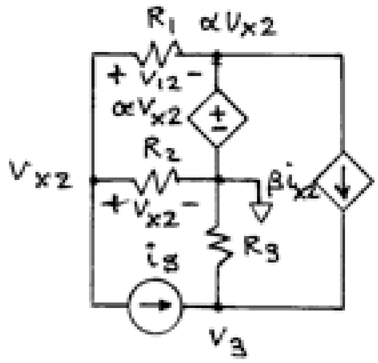
Using Ohm's law to find v_{11} we have

$$v_{11} = -i_{x1}R_1 = v_S \frac{R_1}{R_1 + R_2 - \alpha R_2}$$

case II: v_S off = wire, i_S on



We can always use node-voltage. Putting a reference in the center is convenient.



Consider the v_{x2} node:

$$\frac{v_{x2} - \alpha v_{x2}}{R_1} + \frac{v_{x2}}{R_2} + i_s = 0\,\text{A}$$

or $$v_{x2}\left(\frac{1}{R_1} - \frac{\alpha}{R_1} + \frac{1}{R_2}\right) = -i_s$$

or $$v_{x2}\left(R_2 - \alpha R_2 + R_1\right) = -i_s R_1 R_2$$

or $$v_{x2} = \frac{-i_s R_1 R_2}{R_1 + R_2(1-\alpha)}$$

$$v_{12} = v_{x2} - \alpha v_{x2} = (1-\alpha)\, v_{x2}$$

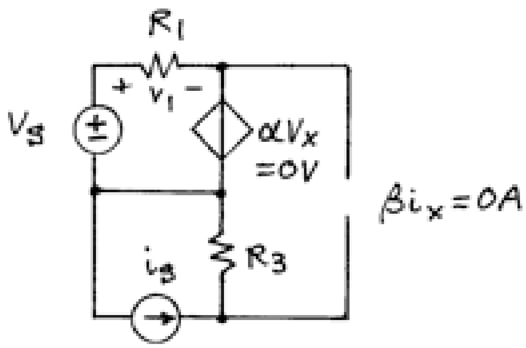
or $$v_{12} = -\frac{i_s R_1 R_2 (1-\alpha)}{R_1 + R_2(1-\alpha)}$$

Now we sum the results from the two cases.

$$\boxed{v_1 = v_{11} + v_{12} = \frac{v_s R_1 - i_s R_1 R_2 (1-\alpha)}{R_1 + R_2(1-\alpha)}}$$

We perform some consistency checks. We set some component values to zero to create a circuit with an obvious solution. Then we see if our above expression for v_1 gives the correct answer.

Check 1: Set $R_2 = 0$ so $v_x = 0\,V$.
Set $\beta = 0$ so $\beta i_x = 0A$.



From the v-loop in the upper left we have $v_1 = v_S$.

Our answer above gives

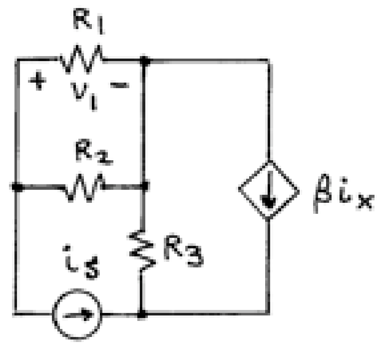
$$v_1 = \frac{v_S \cancel{R_1} - i_S \cancel{R_1} \underbrace{(0)(1-\alpha)}_{0}}{\cancel{R_1} + \underbrace{0(1-\alpha)}_{0}} = v_S \quad ✓$$

Check 2: Set $R_1 = 0$, then $v_1 = 0\,V$.

Our answer for v_1 gives

$$v_1 = \frac{v_S(0) - i_S(0)R_2(1-\alpha)}{0 + R_2(1-\alpha)} = 0 \, ✓$$

Check 3: Set $v_s = 0\,V$ and $\alpha = 0$.



Careful inspection reveals that R_1 and R_2 are in parallel and i_s flows thru R_1 and R_2.

So we have a current divider, and voltage v_1 is given by

$$v_1 = -i_s \frac{R_2}{R_1 + R_2} \cdot R_1$$

Our answer above gives

$$v_1 = \frac{\cancelto{0}{0 \cdot R_1} - i_s R_1 R_2 (1-0)}{R_1 + R_2(1-0)}$$

or $v_1 = \dfrac{-i_s R_1 R_2}{R_1 + R_2}$ ✓

All checks thus far are satisfied.