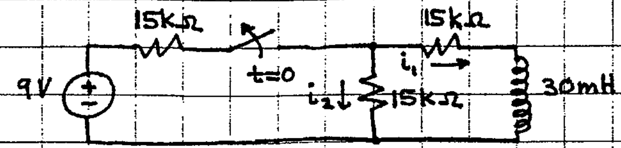
Problem Set #5

Ref: Nilsson & Riedel, ~~Rev 6, Prob 7.5~~ Rev. 8, Prob. 7.2

ex:



The switch has been closed for a long time before opening at time $t=0$.

Note: Whenever a circuit configuration has existed "for a long time" it means forever — from $t=-\infty$ to $t=0$. Thus, all currents and voltages have achieved final values. In other words, $\frac{di}{dt}=0$ for all currents, $\frac{dv}{dt}=0$ for all voltages.

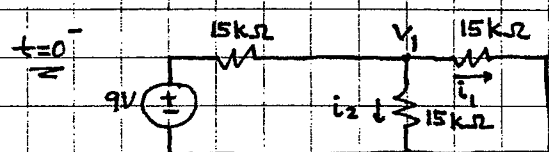
a) Find $i_1(t=0^-)$ and $i_2(t=0^-)$.

soln: At $t=0^-$ the circuit has had the same configuration "for a long time," and $\frac{di_1}{dt}=0$ $\frac{di_2}{dt}=0$.

For the L we have $v_L = L\frac{di}{dt} = L\cdot 0 = 0V$.

Thus, the 30mH L looks like a wire (as L's always do for their final values), and our circuit model is:

$\underline{\underline{t=0^-}}$



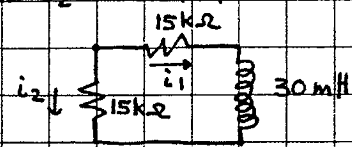
We get V_1 from V-divider: $V_1 = 9V\cdot\frac{15k\Omega \| 15k\Omega}{15k\Omega + 15k\Omega\|15k\Omega}$

$\therefore V_1 = 3V$

$i_1(t=0^-) = i_2(t=0^-) = V_1/15k\Omega = \frac{3V}{15k\Omega} = 0.2\,mA$

b) Find $i_1(t=0^+)$ and $i_2(t=0^+)$.

sol'n: With the switch open, the 9V and 15kΩ before the switch no longer affect i_1 and i_2. Our model (for i_1 and i_2) becomes:



Since the current through the L cannot change instantly, we must have $i_1(t=0^+) = i_1(t=0^-)$.

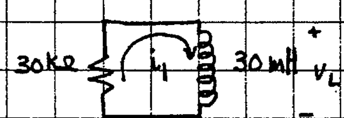
But now we also have the two 15kΩ R's in series. Thus, $i_2 = -i_1 = -i_1(t=0^-)$.

$\therefore\ i_1(t=0^+) = 0.2\text{mA} \qquad i_2(t=0^+) = -0.2\text{ mA}$

Note: The current in an R can change instantly, as happens here (with i_2 changing sign).

c) Find $i_1(t)$ for $t \geq 0$

sol'n: Our circuit model (with 15kΩ R's summed) is:



$v_L = L\dfrac{di_1}{dt}$ and $v_L = -i_1 \cdot 30\text{k}\Omega$ by Ohm's law

$\therefore\ -i_1 \cdot \underset{R}{\underset{|||}{30\text{k}\Omega}} = L\dfrac{di_1}{dt}$ or $i_1 \cdot R + L\dfrac{di_1}{dt} = 0\text{ V}$

Note: we used sum of V's around loop = 0V

From text p. 279, $i_1(t) = i(0^+)e^{-tR/L} = 0.2\text{mA}\,e^{-t\,30\text{k}\Omega/30\text{mH}}$

$\therefore\ i_1(t) = 0.2\text{ mA}\cdot e^{-t\cdot 10^6/s}$

d) Find $i_2(t)$ for $t \geq 0^+$

sol'n: From part (b) we have $i_2(t) = -i_1(t)$ for $t \geq 0^+$

$$\therefore \; i_2(t \geq 0^+) = -0.2\,mA \cdot e^{-t \cdot 10^6/s}$$

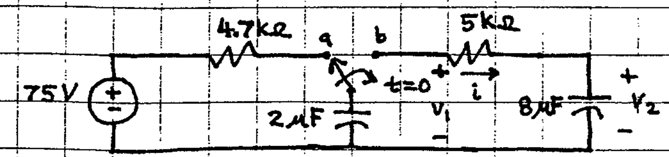
e) Explain why $i_2(0^-) \neq i_2(0^+)$.

answer: As noted in the solution to (b), the current in an R can change instantly. It does so here to keep $i_1(t)$ the same when the switch opens.

Note: The R current can change instantly because an R stores no energy. It takes time to change the energy stored in an L, and energy $= \frac{1}{2} L i^2$.

Ref: Nilsson & Riedel, ~~Rev 6, Prob 7.22~~ Rev. 8, Prob. 7.21

ex:



$v_2(t \le 0) = 0\text{V}$

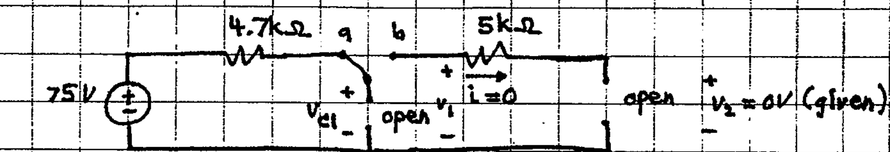
The switch in the circuit has been in position 'a' for a long time before it changes to position 'b' at $t=0$.

a) Find i, v_1, and v_2 for $t \ge 0^+$.

At $t=0^-$ we have $\frac{dv_c}{dt} = 0$ both C's.

$\therefore i_c = 0$ for both C's. $\therefore$ C's look like open circuits

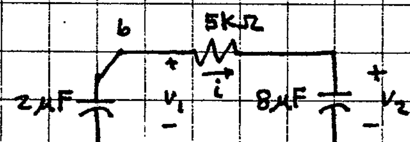
$t = 0^-$



Since no current flows on the left side, there is no V-drop across the 4.7kΩ R. Thus, $v_{c1} = 75\text{V}$.

When we throw the switch at $t=0$, the left side 75V and 4.7kΩ no longer affect i, v_1, and v_2:

$t \ge 0^+$

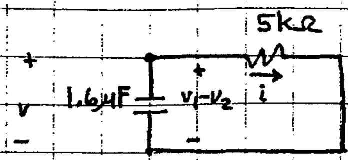


We combine the two C's in series into one $C_{eq} = 2\mu F \| 8\mu F$;

$$C_{eq} = \frac{2\mu F \cdot 8\mu F}{2\mu F + 8\mu F} = \frac{16}{10}\mu F = 1.6\,\mu F.$$

The initial voltage on C_{eq} is $v_1(0^+) - v_2(0^+) = \underset{75V}{v_1(0^-)} - \underset{0V}{v_2(0^-)}$, since V across C's cannot change instantly.

Now our equivalent circuit is an RC with initial voltage $v_{c1} - v_{c2} = 75\text{V} - 0 = 75\text{V}$:



We sum currents flowing out of node between R & C, (instead of summing V's around loop as we did for RL problems).

In other words, we equate the i in the R & C.

$$-i = C\frac{d\,(v_1-v_2)}{dt} = C\frac{dv}{dt} \quad \text{and} \quad i = \frac{v_2-v_1}{R} = \frac{v}{R}$$

$$\therefore \quad \frac{v}{R} = -C\frac{dv}{dt} \quad \text{or} \quad \frac{v}{R} + C\frac{dv}{dt} = 0\,A$$

As shown on p.287 of text, the solution is

$$v(t) = v(t=0^+)\,e^{-t/RC_{eq}} = 75V\cdot e^{-t/5k\Omega\cdot 1.6\mu F}$$
$$= 75V\cdot e^{-t/8\,ms}$$

$$i(t) = \frac{v(t)}{R} = \frac{75V\,e^{-t/8ms}}{5k\Omega} = 15\,mA\,e^{-t/8ms}$$

Now that we have $i(t)$, we can integrate $i(t)$ to find $v_1(t)$ and $v_2(t)$.

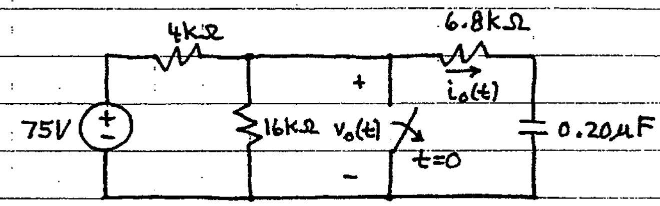
$$-i(t) = \overset{2\mu F}{C_1}\frac{dv_1}{dt} \qquad\qquad i(t) = \overset{8\mu F}{C_2}\frac{dv_2}{dt}$$

$$-\int_{t'=0^+}^{t'=t} i(t')\,dt' = C_1\int_{v_1(t=0^+)}^{v_1(t)} dv_1 \qquad \int_{t'=0^+}^{t'=t} i(t')\,dt' = C_2\int_{v_2(t=0^+)}^{v_2(t)} dv_2$$

$$\underbrace{\frac{--8ms\cdot 15mA}{2\mu F}}_{60V}\,e^{-t'/8ms}\Big|_{t'=0}^{t'=t} = v_1(t)-v_1(0^+) \qquad \underbrace{\frac{-8ms\cdot 15mA}{8\mu F}}_{-15}\,e^{-t'/8ms}\Big|_{t'=0}^{t'=t} = v_2(t)-v_2(0^+)$$

$$v_1(t) = 60V\left(e^{-t/8ms}-1\right) + 75V \qquad v_2(t) = -\ \ V\left(e^{-t/8ms}-1\right)$$

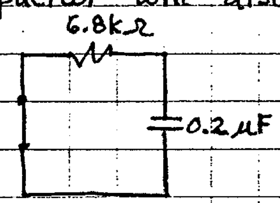
ex:



Switch has been closed for a long time before opening at time $t=0$.

a) Find initial value of $i_o(t)$.

sol'n: If switch was closed for a long time, then the capacitor will discharge through 6.8kΩ:

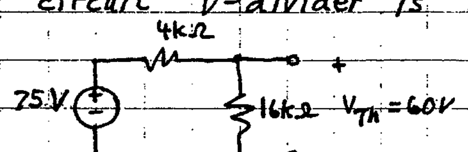


(Right side of circuit, see 'Note' below.)

Note: The closed switch creates a short circuit. The left and right sides of the circuit may then be treated as though they are totally independent. Why? Because the currents flowing through the short create no V drop. Thus, the mesh currents on the two sides of the short do not interact.

Thus, at $t=0^-$ the C has no charge, and $v_C = 0$.
∴ C acts like short at $t=0^-$. Since v_C cannot change instantly, $v_C(t=0^+) = 0\,V$, too.

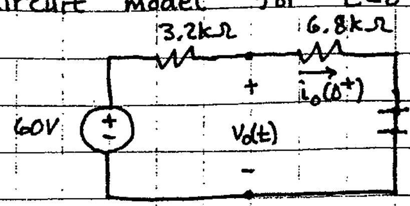
Also, we replace the 75V and 4kΩ and 16kΩ with a Thevenin equivalent. V_{Th} from open-circuit V-divider is $75V \cdot \dfrac{16k\Omega}{4k\Omega + 16k\Omega} = 60V = V_{Th}$



Turn 75V down to 0V and connect 1V source to output to get $R_{Th} = 1V / [1V/(4k\Omega \| 16k\Omega)] = 4k\Omega \| 16k\Omega$

$$R_{Th} = 4k\Omega \cdot 1 \| 4 = 4k\Omega \cdot \frac{4}{5} = 3.2k\Omega$$

$\therefore$ Circuit model for $t = 0^+$ is:



0V (C like short circuit)

$$i_o(t=0^+) = \frac{60V}{3.2k\Omega + 6.8k\Omega} = \frac{60V}{10k\Omega} = \frac{6V}{1k\Omega} = 6\,mA$$

b) Find $i_o(t \to \infty)$.

sol'n: When the C is charged, it looks like open circuit.

Note: $i_C(t \to \infty) = 0\,A$ for all C's in all switching probs.

$v_C(t=0^-) = v_C(t=0^+)$ " " " " " " "

But $v_C(t=0^-) = 0V$ only if circuit discharges C completely for $t < 0$. Otherwise, $v_C(t=0^-)$ is some nonzero value.

If C is open circuit, then $i_o(t \to \infty) = 0\,A$.

c) Find time constant for $t \geq 0$.

sol'n: $\tau = R_{eq} \cdot C$ $\quad R_{eq} = 10k\Omega = 3.2k\Omega + 6.8k\Omega$

$= 10k\Omega \cdot 0.2\mu F$ $\quad C = 0.2\mu F$

$= 2\,ms$

d) Find expression for $i_o(t)$ when $t \geq 0^+$

sol'n: General sol'n for $i_o(t)$ is:

$$i_o(t) = i_o(t=0^+) + [i_o(t \to \infty) - i_o(t=0^+)][1 - e^{-t/RC}]$$

$$= 6\,mA + [0 - 6mA][1 - e^{-t/2ms}]$$

$$= 6\,mA\, e^{-t/2ms} \qquad t \geq 0^+$$

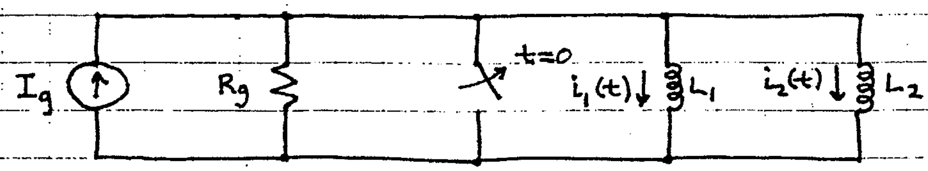
e) Find expression for $v_o(t)$ when $t \geq 0^+$.

sol'n: $v_o(t) = 60\text{V} - i_o(t) \cdot 3.2\text{k}\Omega = 60\text{V} - 6\text{mA} \cdot e^{-t/2\text{ms}} \cdot 3.2\text{k}\Omega$

$= 60\text{V} - 19.2\text{V}\, e^{-t/2\text{ms}} \qquad t \geq 0^+$

Ref: Nilsson & Riedel, ~~Rev 6, Prob 7.50~~ Rev.8, Prob. 7.43

ex:



No energy stored in L_1 and L_2 when switch opens

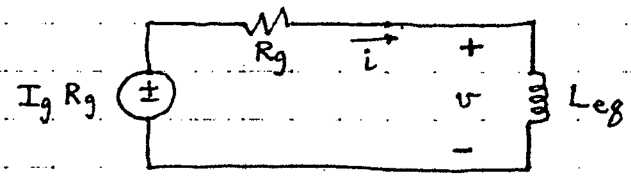
a) Find $i_1(t \geq 0)$ and $i_2(t \geq 0)$. b) Find $i_1(t \to \infty)$ and $i_2(t \to \infty)$

ans: a) $i_1(t \geq 0) = I_g \dfrac{L_2}{L_1 + L_2}\left(1 - e^{-t/\tau}\right)$ where $\tau \equiv L_1 \| L_2$

$i_2(t \geq 0) = I_g \dfrac{L_1}{L_1 + L_2}\left(1 - e^{-t/\tau}\right)$ " "

b) $i_1(t \to \infty) = I_g \dfrac{L_2}{L_1 + L_2}$ $i_2(t \to \infty) = I_g \dfrac{L_1}{L_1 + L_2}$

sol'n: a) Take Thevenin equiv of I_g and R_g on left. Solve for v across L's by replacing L's with equivalent L:



circuit for $t \geq 0$

L's in parallel give $L_{eq} = L_1 \| L_2 = \dfrac{L_1 L_2}{L_1 + L_2}$.

(L's in parallel are like R's in parallel in terms of the formula we use.)

Now we use the general solution for $v(t \geq 0)$:

$$v(t \geq 0) = v(t \to \infty) + [v(0^+) - v(t \to \infty)]\, e^{-t/(L_{eq}/R_g)}$$

time constant

To find $v(0^+)$, we use $i_1(0^+) = i_1(0^-)$ and $i_2(0^+) = i_2(0^-)$. But $i_1(0^-) = i_2(0^-) = 0$ since no energy is stored in L_1 and L_2 at $t=0$.

a) cont.

Since $i_1(0^+)$ and $i_2(0^+) = 0$, we must have no current through R_g at $t = 0^+$.

$\therefore$ At $t=0^+$, we have no v drop across R_g.

$\therefore$ $v(t=0^+) = V_{Th} = I_g R_g$

For $v(t\to\infty)$ we observe that the L's act like wires, and $v(t\to\infty) = 0$.

Plugging into the general soln gives

$$v(t\geq 0) = I_g R_g \, e^{-t/(L_{eq}/R_g)}$$

Note: The time constant for circuit with L and R is L_{eq}/R_{Thev}. Taking the Thevenin equivalent always gives the needed R.

Now we can also write down a formula for $i(t) = i_1(t) + i_2(t)$ for $t \geq 0$:

$$i(t\geq 0) = i(t\to\infty) + [i(0^+) - i(t\to\infty)]\, e^{-t/(L_{eq}/R_g)}$$

Note: All i's and v's have same time constant.

We know $i(0^+) = i_1(0^+) + i_2(0^+) = i_1(0^-) + i_2(0^-) = 0$

At $t\to\infty$, the L's act like wires, giving $i = I_g$.

$$\therefore \quad i(t\geq 0) = I_g\left[1 - e^{-t/(L_{eq}/R_g)}\right]$$

Now we determine how $i(t\geq 0)$ is divided between the two L's to give $i_1(t\geq 0)$ and $i_2(t\geq 0)$.

a) cont.

Since both L's have same V across them, we have

$$v = L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt} \quad \therefore \frac{\frac{di_1}{dt}}{\frac{di_2}{dt}} = \frac{L_2}{L_1}$$

Now we calculate currents:

$$i_1(t) = \int \frac{di_1}{dt}\,dt = \int \frac{L_2}{L_1}\frac{di_2}{dt}\,dt = \frac{L_2}{L_1}\int di_2$$

or $i_1(t) = \frac{L_2}{L_1} i_2(t)$

Also, $i_1(t) + i_2(t) = i(t)$.

Solving these two eq'ns gives $i_1(t) = \frac{L_2}{L_1+L_2} i(t)$

$$i_2(t) = \frac{L_1}{L_1+L_2} i(t)$$

Thus, $i_1(t \geq 0) = I_g \frac{L_2}{L_1+L_2}\left(1 - e^{-t/(L_{eq}/R_g)}\right)$

$$i_2(t \geq 0) = I_g \frac{L_1}{L_1+L_2}\left(1 - e^{-t/(L_{eq}/R_g)}\right)$$

b) At $t \to \infty$ we have $e^{-t/(L_{eq}/R_g)} \to 0$.

$$\therefore i_1(t \to \infty) = I_g \frac{L_2}{L_1+L_2}$$

$$i_2(t \to \infty) = I_g \frac{L_1}{L_1+L_2}$$