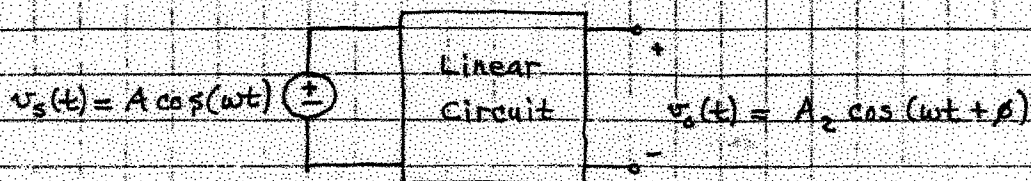


Sinusoidal Signals

19 Mar 01

Neil E. Lott



- input sinusoid, output also sinusoid!
freq unchanged!

amplitude & phase shift change
calc these

- Can write arbitrary input (periodic) as sum of sinusoids of different frequencies

Imaginary #'s

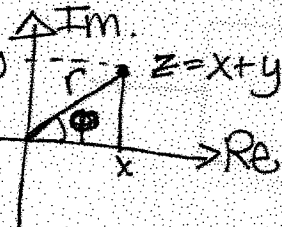
What is j ?0) j allows to find roots of all polynomials1) $j = \sqrt{-1}$ 2) j is a unit vector $\perp$ to real axis3) j defines rule for multiplying vecs4) $j = 90^\circ$ rotation5) j allows us to use phasors for $A \cos(\omega t + \phi)$ 6) j is a token placed in front of $\sin()$ to make $\cos()$ 7) j allows us to write $\cos(\omega t + \phi)$ as $a + jb = \cos + j \sin$ 8) j allows us to use complex exp for polar form $A e^{j\phi}$ W/ no phase shift draw real polar plot9) j results from taking $\frac{d}{dt} e^{j\phi(t)} = j e^{j\phi(t)} \frac{d\phi(t)}{dt}$ L's and C's10) j allows us to write $\cos = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$

Complex #'s:

$a+bj$ ← used so that current (i symbol) not get confused (131)

a = real part, b = imaginary part

complex plane
or (Argand)
diagram



$z = x + jy$ ← **Rectangular** or Cartesian form

polar form exponential form

$$z = r e^{i\phi} \quad \text{where } \phi = \begin{cases} -\tan^{-1}\left(\frac{y}{x}\right) & x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & x < 0, y \geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & x < 0, y < 0 \\ +\pi/2 & x = 0, y > 0 \\ -\pi/2 & x = 0, y < 0 \end{cases}$$

$$r = \sqrt{x^2 + y^2}$$

trigonometric form:

$$z = r(\cos \phi + j \sin \phi)$$

Suggestions $\Rightarrow$

add/sub. using rect. form $\rightarrow a+bj + (c+dj) = (a+c) + (b+d)j$
mult./div. " exponential form

$$(r_1 e^{i\phi_1}) * (r_2 e^{i\phi_2}) = r_1 \cdot r_2 e^{i(\phi_1 + \phi_2)}$$

EX: Express $3e^{j32^\circ} + 4e^{-j40^\circ}$ in rectangular and polar form.

ANS: $5.608 - j0.981$ or $5.693e^{-j9.922^\circ}$ (approx)

SOL'N: Use Euler's formula to write each complex number in rectangular form $a + jb$:

$$Ae^{j\phi} = A \cos \phi + jA \sin \phi$$

$$3e^{j32^\circ} = 3 \cos(32^\circ) + j3 \sin(32^\circ) = 2.544 + j1.590$$

$$4e^{-j40^\circ} = 4 \cos(-40^\circ) + j4 \sin(-40^\circ) = 3.064 - j2.571$$

Sum the real and imaginary parts:

$$3e^{j32^\circ} + 4e^{-j40^\circ} = 2.544 + 3.064 + j(1.590 - 2.571)$$

Our answer in rectangular form:

$$3e^{j32^\circ} + 4e^{-j40^\circ} = 5.608 - j0.981$$

Use the Pythagorean theorem to find the magnitude for polar form:

$$|3e^{j32^\circ} + 4e^{-j40^\circ}| = \sqrt{5.608^2 + 0.981^2} = 5.693$$

Use tangent of phase angle = Im/Re to find angle for polar form:

$$\angle(3e^{j32^\circ} + 4e^{-j40^\circ}) = \tan^{-1}\left(\frac{-0.981}{5.608}\right) = -9.922^\circ$$

Our answer in polar form:

$$3e^{j32^\circ} + 4e^{-j40^\circ} = 5.693e^{-j9.922^\circ}$$

EX: Rationalize $\frac{3 + j1.5}{5 - j2.6}$

ANS: $0.35 + j0.48$

SOL'N: Multiply the numerator and denominator by the denominator's complex conjugate. This turns the denominator into a real number equal to the magnitude squared of the original denominator.

$$\begin{aligned}\frac{3 + j1.5}{5 - j2.6} &= \frac{3 + j1.5}{5 - j2.6} \frac{5 + j2.6}{5 + j2.6} \\ &= \frac{(3 + j1.5)(5 + j2.6)}{5^2 + 2.6^2} \\ &= \frac{15 - 3.9 + j(7.8 + 7.5)}{31.76} \\ &= 0.35 + j0.48\end{aligned}$$

OR

$$\frac{3 + j(1.5)}{5 - j2.6} = \frac{\sqrt{9 + 1.5^2} e^{j \tan^{-1}(\frac{1.5}{3})}}{\sqrt{25 + 2.6^2} e^{-j \tan^{-1}(\frac{2.6}{5})}}$$

$$= \frac{3.354}{5.636} e^{j(26.6^\circ + 27.5^\circ)} = 0.6 e^{j(54.1^\circ)}$$

$$\begin{aligned}&= 0.6 \cos(54.1^\circ) + 0.6 \sin(54.1^\circ) j \\ &= 0.35 + 0.49 j\end{aligned}$$

EX: Evaluate $\frac{4+j3}{2-j6}$

ANS: $-\frac{1}{4} + j\frac{3}{4}$

SOL'N: Multiply the numerator and denominator by the denominator's complex conjugate. This turns the denominator into a real number equal to the magnitude squared of the original denominator. Divide the numerator's real and imaginary parts by this new denominator.

$$\begin{aligned}\frac{4+j3}{2-j6} &= \frac{4+j3}{2-j6} \frac{(2-j6)^*}{(2-j6)^*} \\ &= \frac{4+j3}{2-j6} \frac{2+j6}{2+j6} \\ &= \frac{8-18+j(6+24)}{4+36} \\ &= \frac{-10+j30}{40} \\ &= -\frac{1}{4} + j\frac{3}{4}\end{aligned}$$

OR

$$\begin{aligned}\frac{\sqrt{16+9} e^{j \tan^{-1}(\frac{3}{4})}}{\sqrt{4+36} e^{-j \tan^{-1}(\frac{6}{2})}} &= \frac{5}{6.32} e^{j(37^\circ + 72^\circ)} = 0.8 e^{j(109^\circ)} \\ &= +0.8 \cos(109^\circ) + 0.8 \sin(109^\circ)j = -26 + 0.76j\end{aligned}$$

DEF: Complex Conjugate of $a + jb = a - jb$ = complex number with imaginary part inverted

NOT'N: z^* = complex conjugate of z

TOOL: To find the complex conjugate of an expression, change each j to $-j$.

NOTE: This is equivalent to (but easier than) converting the expression to form $a + jb$ and changing it to $a - jb$.

TOOL: $(Ae^{j\phi})^* = Ae^{-j\phi}$ when A is real

TOOL: To find z^* , reflect z around the real axis. In other words, preserve the magnitude but take the negative of the phase angle.

EX: Evaluate $(2 + j3)^*$

ANS: $2 - j3$

SOL'N: This is a direct application of the definition of complex conjugate.

EX: Evaluate $(Re^{j\psi})^*$ where R is real

ANS: $Re^{-j\psi}$

SOL'N: We retain the magnitude but invert the phase angle to find the conjugate of a complex number in polar form.

EX: Evaluate $\left[\frac{(6 - j2)e^{j35^\circ}}{(4 + j5)\sin(x)} \right]^*$ where x is real

ANS: $\left[\frac{(6 + j2)e^{-j35^\circ}}{(4 - j5)\sin(x)} \right]^*$

SOL'N: Change each j to $-j$. Thus, $-j$ becomes $-(-j) = j$.

CONCEPTUAL TOOLS

By: Carl H. Durney and Neil E. Cotter

COMPLEX ANALYSIS
BASIC MATH
Re[]
EXAMPLE 1

EX: Find $\text{Re}[7e^{j25^\circ}]$, (i.e., find the real part)

ANS: $7\cos(25^\circ) \approx 6.34$

SOL'N: The answer follows directly from Euler's formula (see tools for rectangular and polar forms for complex numbers):

$$Ae^{j\phi} = A\cos\phi + jA\sin\phi$$

CONCEPTUAL TOOLS

By: Carl H. Durney and Neil E. Cotter

COMPLEX ANALYSIS
BASIC MATH
Magnitude
EXAMPLE 1

EX: Find $|2e^{j182^\circ}|$, (i.e., find the magnitude)

ANS: 2

SOL'N: The magnitude of a product is the product of the magnitudes:

$$|2e^{j182^\circ}| = 2 \cdot |e^{j182^\circ}|$$

The magnitude of a real number is the absolute value of that real number:

$$|2| = 2$$

The magnitude of e^{jx} for any real x is 1:

$$|e^{jx}| = 1$$

Thus we have:

$$|e^{j182^\circ}| = 1$$

Putting our results together gives the answer:

$$|2e^{j182^\circ}| = 2$$

EX: Find $\text{Re}[\frac{6+j3}{2-j4}e^{jx}]$, (i.e., find the real part) where "x" is real

ANS: $1.5 \cos(x + \pi/2)$

SOL'N: We may take one of several different approaches to convert the quantity inside the brackets into the form $a + jb$ (where a is our final answer). We'll take the approach of rationalizing the fraction.

$$\begin{aligned}\text{Re}[\frac{6+j3}{2-j4}e^{jx}] &= \text{Re}[\frac{6+j3}{2-j4} \frac{2+j4}{2+j4}e^{jx}] \\ &= \text{Re}[\frac{12-12+j(24+6)}{2^2+4^2}e^{jx}] \\ &= \text{Re}[\frac{j30}{20}e^{jx}]\end{aligned}$$

We now use Euler's formula to expand the complex exponential:

$$\begin{aligned}&= \text{Re}[\frac{j30}{20}\{\cos(x) + j\sin(x)\}] \\ &= \text{Re}[-1.5\sin(x) + j1.5\cos(x)]\end{aligned}$$

Our final answer is the real part, which we may express in several ways.

$$\begin{aligned}\text{Re}[\frac{6+j3}{2-j4}e^{jx}] &= -1.5\sin(x) \text{ or} \\ \text{Re}[\frac{6+j3}{2-j4}e^{jx}] &= 1.5\cos(x + \pi/2) = 1.5\cos(x + 90^\circ)\end{aligned}$$

NOTE: A curious feature of this problem is that the fraction consisting of complex numbers is purely imaginary. We now examine this symbolically.

$$k \cdot \frac{a + jb}{b - ja} = k \cdot \frac{j(b - ja)}{b - ja} = jk$$

Whenever the numerator and denominator of a fraction have the above pattern, we will find that the result is purely imaginary. Note the necessary minus sign.

CONCEPTUAL TOOLS

By: Carl H. Durney and Neil E. Cotter

COMPLEX ANALYSIS
BASIC MATH
Roots and Powers
Nth ROOTS
example

EX: Find the value (in polar form) of $(6 + j5)^{1/5}$.

ANS: $1.51e^{j7.96^\circ}$ (approx)

SOL'N: First, we convert the number being raised to a power to polar form:

$$6 + j5 = \sqrt{6^2 + 5^2} e^{j \tan^{-1} \left(\frac{5}{6} \right)} \approx 7.81e^{j39.8^\circ}$$

Now take the power inside the parentheses and use the identity

$$(ab)^n = a^n b^n$$

giving the answer:

$$(6 + j5)^{1/5} \approx 7.81^{1/5} e^{j39.8^\circ/5} \approx 1.51e^{j7.96^\circ}$$

CONCEPTUAL TOOLS

By: Carl H. Durney and Neil E. Cotter

COMPLEX ANALYSIS
BASIC MATH
Roots and Powers
POWERS
example

EX: Find the value (in polar form) of $(3e^{j40^\circ})^3$.

ANS: $27e^{j120^\circ}$

SOL'N: We take the power inside the parentheses and use the identity

$$(ab)^n = a^n b^n$$

giving the answer:

$$3^3 e^{j3 \cdot 40^\circ} = 27e^{j120^\circ}$$

EX: Find the polar form of $2.5 - j3.2$.

ANS: $4.06e^{-j52^\circ}$

SOL'N: We express $2.5 - j3.2$ in polar form $Ae^{j\phi}$.

Use the pythagorean theorem to find magnitude A :

$$A = \sqrt{2.5^2 + 3.2^2} \approx 4.06$$

Set the tangent of the phase angle equal to the side opposite (imaginary part) over the side adjacent (real part):

$$\tan \phi = \frac{\text{Im}[2.5 - j3.2]}{\text{Re}[2.5 - j3.2]} = \frac{-3.2}{2.5} = -1.28$$

$$\phi = \tan^{-1}\left(\frac{-3.2}{2.5}\right) \approx -52^\circ \text{ or } -0.9076 \text{ radians}$$

Our final answer:

$$2.5 - j3.2 \approx 4.06e^{-j52^\circ}$$

NOTE: When calculating the inverse tangent, if we use -1.28 rather than both the imaginary and real parts, we have two possible values for ϕ that differ by 180 degrees. The ratio of the imaginary and real parts is the same for $1 + j$ and $-1 - j$, for example. Thus, it is necessary to keep track of which quadrant the complex number lies in if we wish to avoid confusion about the correct value of phase angle ϕ .

EX: Find the rectangular form of $6e^{-j47^\circ}$

ANS: $4.09 - j4.39$

SOL'N: We must express $6e^{-j47^\circ}$ in rectangular form $a + jb$.

We use Euler's formula for the complex exponential:

$$6e^{-j47^\circ} = 6\cos(-47^\circ) + j6\sin(-47^\circ)$$

Applying identities, $\cos(-A) = \cos(A)$ and $\sin(-A) = -\sin(A)$, we have

$$= 6\cos(47^\circ) - j6\sin(47^\circ)$$

$$6e^{-j47^\circ} = 4.09 - j4.39$$

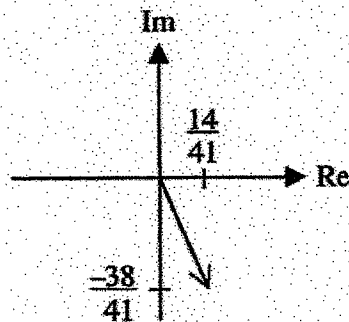
EX: Find the rectangular and polar forms of $\frac{6-j2}{4+j5}$.

ANS: $\frac{14}{41} - j\frac{38}{41}$ and $0.987e^{-j69.78^\circ}$

SOL'N:
$$\frac{6-j2}{4+j5} = \frac{6-j2}{4+j5} \frac{(4-j5)^*}{(4+j5)^*} = \frac{6-j2}{4+j5} \frac{(4-j5)}{(4+j5)}$$
$$= \frac{24-10-j(30+8)}{16+25} = \frac{14}{41} - j\frac{38}{41}$$

$$A = \sqrt{\left(\frac{14}{41}\right)^2 + \left(\frac{38}{41}\right)^2} = 0.987$$

$$\phi = \tan^{-1}\left(\frac{-38}{14}\right) = -69.78^\circ$$



TUTOR: THE PHASOR TRANSFORM

All voltages and currents in linear circuits with sinusoidal sources are described by constant-coefficient linear differential equations of the form

$$(1) \quad a_n \frac{d^n f}{dt^n} + a_{n-1} \frac{d^{n-1} f}{dt^{n-1}} + \dots + a_0 f = C \cos(\omega t + \phi)$$

where f is a function of time, the a_n are constants, C is a constant, ω is the radian frequency of the sinusoidal source, and ϕ is the phase of the sinusoidal source. In (1), f represents any voltage or current in the circuit.

A particular solution to (1) can be found by an elegant procedure called the **phasor transform** method. This supplementary material outlines the mathematical basis of the method. The phasor transform is defined by

$$(2) \quad f(t) = \operatorname{Re} [F(\omega) e^{j\omega t}]$$

where $F(\omega)$ is a function of ω called the **phasor transform** of $f(t)$, and Re means the real part of the quantity in the brackets. $F(\omega)$ is complex; it has a real and an imaginary part.

Two key mathematical relationships are used in finding a particular solution to (1). The first is

$$(3) \quad \operatorname{Re}[W] = \frac{W + W^*}{2}$$

where W is any complex number and W^* is the complex conjugate of W . Using (3) with (2) gives

$$(4) \quad f = \frac{F e^{j\omega t} + F^* e^{-j\omega t}}{2}$$

where f has been written for $f(t)$ and F for $F(\omega)$ for brevity. Note that F is not a function of time. The second relationship is

$$(5) \quad \cos(\omega t + \phi) = \frac{e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}}{2}$$

which is called Euler's formula.

Substituting (5) and (4) into (1), taking the derivatives with respect to time, and collecting terms gives

$$(6) \quad \begin{aligned} & [a_n(j\omega)^n F + a_{n-1}(j\omega)^{n-1} F + \dots + a_0 F - Ce^{j\phi}] e^{j\omega t} + \\ & [a_n(-j\omega)^n F^* + a_{n-1}(-j\omega)^{n-1} F^* + \dots + a_0 F^* - Ce^{-j\phi}] e^{-j\omega t} = 0. \end{aligned}$$

Now because $e^{j\omega t}$ and $e^{-j\omega t}$ are **linearly independent** functions (see, for example, C. R. Wylie, *Advanced Engineering Mathematics*, 3rd ed., New York: McGraw-Hill, 1966, p. 444), (6) can be true for all time only if

$$(7) \quad [a_n(j\omega)^n F + a_{n-1}(j\omega)^{n-1} F + \dots + a_0 F - Ce^{j\phi}] = 0$$

and

$$(8) \quad [a_n(-j\omega)^n F^* + a_{n-1}(-j\omega)^{n-1} F^* + \dots + a_0 F^* - Ce^{-j\phi}] = 0.$$

Equations (7) and (8) are identical because one is the complex conjugate of the other, so only one is needed. An expression for F from (7) is

$$(9) \quad F = \frac{Ce^{j\phi}}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_0}.$$

A particular solution to (1) can now be obtained from (9) and (2):

$$(10) \quad f = \operatorname{Re}[Fe^{j\omega t}] = \operatorname{Re}\left[\frac{Ce^{j\phi}e^{j\omega t}}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_0}\right].$$

Symbolically, the notation for a phasor transformation is

$$(11) \quad \mathbf{P}[f(t)] = \mathbf{F}(\omega)$$

where the bold $\mathbf{P}$ means "phasor transform of". Thus, $\mathbf{F}$ is the phasor transform of f . Taking the derivative of both sides of (2) gives

$$\frac{df}{dt} = \text{Re}[j\omega \mathbf{F}(\omega) e^{j\omega t}]$$

which corresponds to

$$\mathbf{P}\left[\frac{df}{dt}\right] = j\omega \mathbf{F}.$$

Similarly,

$$(12) \quad \mathbf{P}\frac{d^n f}{dt^n} = (j\omega)^n \mathbf{F}$$

and

$$(13) \quad \mathbf{P}[\cos(\omega t + \phi)] = e^{j\phi}$$

because

$$(14) \quad \cos(\omega t + \phi) = \text{Re}[e^{j\phi} e^{j\omega t}].$$

From the basic relation in (2) it can also be shown that

$$(15) \quad \mathbf{P}[f_1 + f_2] = \mathbf{F}_1 + \mathbf{F}_2$$

and

$$(16) \quad P[af] = aF$$

where

$$P[f_1] = F_1$$

and

$$P[f_2] = F_2$$

and "a" is a constant. The relation in (15) means that the phasor transform of a sum of functions can be found by taking the transform of each function and adding the transforms.

Equations (11), (12), (13), (15), and (16) describe phasor transforms. An inverse phasor transform relation is written as

$$(17) \quad f(t) = P^{-1}[F(\omega)].$$

Equations (11) and (17) are called a **transform pair**. Equation (11) states how to get F when f is known; (17) how to get f when F is known. Equation (2) is the inverse transform relation. The transform relation is derived as follows. $f(t)$ will always be a sinusoid, because it is a particular solution to (1). Thus $f(t)$ can be written as

$$(18) \quad f(t) = f_m \cos(\omega t + \alpha).$$

Substituting (18) into (2), using Euler's formula and (3) gives

$$\frac{f_m e^{j\omega t} e^{j\alpha} + f_m e^{-j\omega t} e^{-j\alpha}}{2} = \frac{F e^{j\omega t} + F^* e^{-j\omega t}}{2}.$$

Collecting terms and using the linear independence of $e^{j\omega t}$ and $e^{-j\omega t}$, as before, gives

$$(19) \quad F = f_m e^{j\alpha}$$

so the phasor transform of $f_m \cos(\omega t + \alpha)$ is $f_m e^{j\alpha}$. The transform pairs are thus

$$(20) \quad F = \mathbf{P}[f] = f_m e^{j\alpha}$$

and

$$(21) \quad f = \mathbf{P}^{-1}[F] = \text{Re}[F e^{j\omega t}].$$

With the phasor transform relations given in (12), (15), (16), (20), and (21), a particular solution to (1) can be found without going through the detailed derivation using (3) and linear independence. The phasor transform of (1) is taken term-by-term using (12), (13), (15), and (16) to get (7), which is then solved for F . Having found F , f is found by taking the inverse transform according to (21).

EX: Let's find a particular solution to

$$(22) \quad \frac{d^3 f}{dt^3} + 3 \frac{d^2 f}{dt^2} + 50 \frac{df}{dt} - 60f = 500 \cos(10t + \pi/3).$$

Taking the phasor transform of this equation gives

$$(j10)^3 F + 3(j10)^2 F + 50(j10)F - 60F = 500e^{j\pi/3}.$$

Solving for F ,

$$F = \frac{500e^{j\pi/3}}{-j100 - 300 + j500 - 60}.$$

Converting F to polar form gives

$$\mathbf{F} = 0.812e^{-j174.25^\circ}$$

and finding the inverse transform gives

$$f = \operatorname{Re}[\mathbf{F}e^{j\omega t}] = \operatorname{Re}[0.812e^{-j174.25^\circ}e^{j10t}] = 0.812\cos(10t - 174.25^\circ).$$

COMMENT: The phasor transform method is powerful because it transforms a differential equation (1) into an algebraic equation (7), which can be solved for the phasor $\mathbf{F}$, and then f can be found by taking the inverse transform.

Phasor voltages and currents satisfy Kirchhoff's laws, because of (15). Consequently, circuits can be transformed into the frequency domain, eliminating the need to write differential equations in the time domain and solve them by phasor transforms. The procedure for analyzing and designing circuits by transforming them into the frequency domain is summarized in the figure below. Note that impedance is defined as the ratio of a phasor voltage to a phasor current. Impedance is not defined in the time domain.