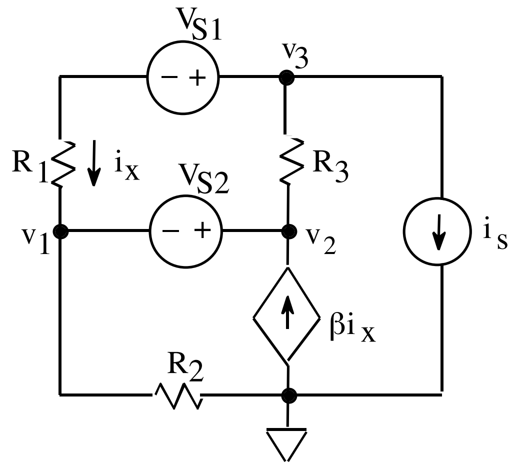
1. (a) (10 points)

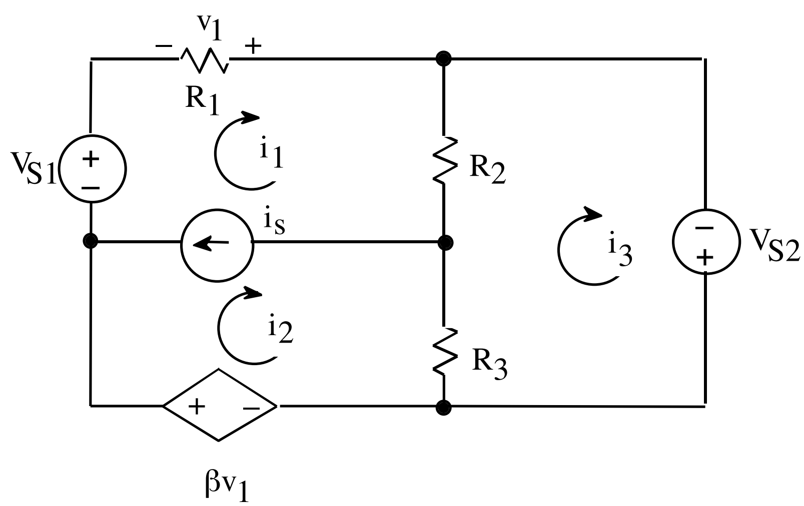


For the circuit shown, write three independent equations for the node voltages v_1, v_2, and v_3. The quantity i_x must not appear in the equations.

(b) (10 points)

Make a consistency check on your equations by setting one or more resistor values to 0 or ∞ and setting other sources and resistor to values for which v_1, v_2, and v_3 are obvious.

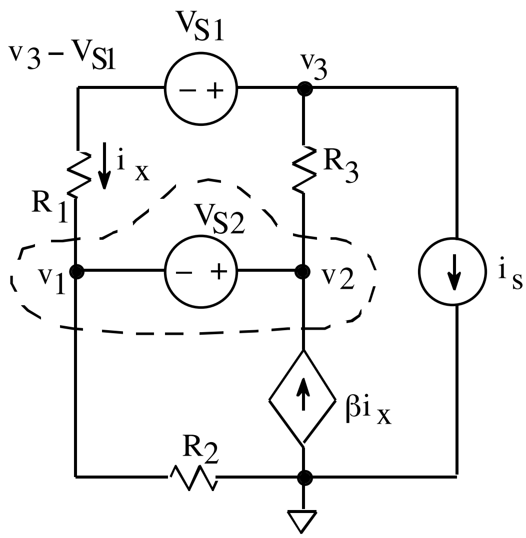
(c) (10 points)



For the circuit shown, write three independent equations for the three mesh currents i_1, i_2, and i_3. The quantity v_1 must not appear in the equations.

(d) Make a consistency check on your equations by setting one or more sources to zero and using convenient resistor and source values.

Solution:



We have a supernode, (dashed box on diagram, above), since v_1 and v_2 are connected by a V source. Sum of currents out of both nodes = 0 A:

$$\frac{v_1}{R_2} + \frac{v_1 - (v_3 - V_{S1})}{R_1} + \frac{v_2 - v_3}{R_3} - \beta i_x = 0 \text{ A}$$

v_1 and v_2 are related by V drop between them:

$$\boxed{v_2 - v_1 = V_{S2}}$$

Now we need the node-V equation for v_3:

$$\boxed{\frac{(v_3 - V_{S1}) - v_1}{R_1} + \frac{v_3 - v_2}{R_3} + i_s = 0 \text{ A}}$$

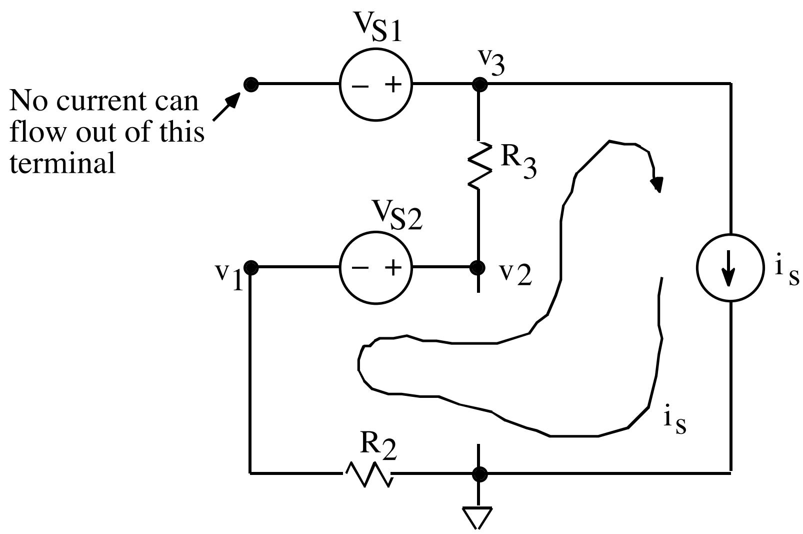
Now we eliminate i_x by expressing it in terms of node voltages:

$$i_x = \frac{(v_3 - V_{S1}) - v_1}{R_1}$$ substitute into the first equation to get:

$$\boxed{\frac{v_1}{R_2} + \frac{v_1 - (v_3 - V_{S1})}{R_1} + \frac{v_2 - v_3}{R_3} - \frac{\beta[(v_3 - V_{S1}) - v_1]}{R_1} = 0 \text{ A}}$$

(b) Consistency checks. Choose convenient source and R values, and verify that the equations are satisfied.

Check: $R_1 = \infty\ \Omega \quad \Rightarrow \quad i_x = 0,\ \beta i_x = 0,\ V_{S1}$ disconnected



We observe that i_s flows around the entire loop. Therefore,

$$v_1 = -i_s R_2$$

$$v_2 = v_1 + V_{S2} = -i_s R_2 + V_{S2}$$

$$v_3 = v_2 - i_s R_3 = -i_s(R_2 + R_3) + V_{S2}$$

Plug v's into our answer to (a):

1) $v_2 - v_1 = -i_s R_2 + V_{S2} - (-i_s R_2) = V_{S2}$ ✓

2)

$$\frac{(v_3 - V_{S1}) - v_1}{R_1} + \frac{v_3 - v_2}{R_3} + i_s = \frac{(-i_s(R_2 + R_3) + V_{S2} - V_{S1}) - (-i_s R_2)}{\infty} \to 0$$

$$+ \frac{-i_s(R_2 + R_3) + V_{S2} - (-i_s R_2 + V_{S2})}{R_3}$$

$$+ i_s$$

$$= \frac{-i_s R_3}{R_3} + i_s = 0\text{ A}$$ ✓

3)

$$\frac{v_1}{R_2}+\frac{v_1-(v_3-V_{S1})}{R_1}+\frac{v_2-v_3}{R_3}-\frac{\beta[(v_3-V_{S1})-v_1]}{R_1}$$

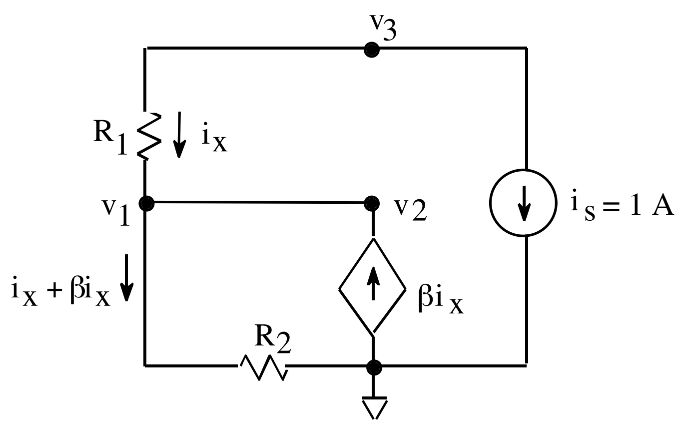
$$=\frac{-i_s\cancel{R_2}}{\cancel{R_2}}+\frac{-i_sR_2-(-i_s(R_2+R_3)+V_{S2}-V_{S1})}{\infty}\rightarrow 0$$

$$+\frac{-i_s\cancel{R_2}+\cancel{V_{S2}}-(-i_s(\cancel{R_2}+R_3)+\cancel{V_{S2}})}{R_3}$$

$$-\frac{\beta[(-i_s(R_2+R_3)+V_{S2}-V_{S1})--i_sR_2]}{\infty}\rightarrow 0$$

$$=-i_s+i_s=0\text{ A}\quad ✓$$

or **Check:** $R_3=\infty\ \Omega$ and $V_{S1}=0$ V and $V_{S2}=0$ V and $i_s=1$ A



$i_x = -i_s = -1$ A

$v_1 = i_x(1+\beta)R_2 = (-1\text{ A})(1+\beta)R_2$

$v_2 = v_1$

$v_3 = v_1 + i_xR_1 = (-1\text{ A})[(1+\beta)R_2 + R_1]$

Plug v's into our equations from (a)

1) $v_2 - v_1 = v_1 - v_1 = 0 = V_{S2}$ ✓

2)

$$\frac{(v_3-V_{S1})-v_1}{R_1}+\frac{v_3-v_2}{R_3}+i_s=\frac{([\cancel{(1+\beta)R_2}+R_1](-1\text{ A})-0)-\cancel{(1+\beta)R_2(-1\text{ A})}}{R_1}$$

$$+\frac{[(1+\beta)R_2+R_1](-1\text{ A})-(1+\beta)R_2(-1\text{ A})}{\infty}\rightarrow 0$$

$$+1\text{ A}$$

$$=-1\text{ A}+1\text{ A}=0\text{ A}\quad ✓$$

3)

$$\frac{v_1}{R_2} + \frac{v_1 - (v_3 - V_{S1})}{R_1} + \frac{v_2 - v_3}{R_1} - \frac{\beta[(v_3 - V_{S1}) - v_1]}{R_1}$$

$$= \frac{(1+\beta)\cancel{R_2}(-1\text{ A})}{\cancel{R_2}} + \frac{(1+\beta)R_2(-1\text{ A}) - (\cancel{[(1+\beta)R_2 + R_1]}(-1\text{ A}) - 0)}{\cancel{R_1}}$$

$$+ \frac{(1+\beta)R_2(-1\text{ A}) - [(1+\beta)R_2 + R_1](-1\text{ A})}{\infty} \rightarrow 0$$

$$- \frac{(\beta(\cancel{[(1+\beta)R_2 + R_1]}(-1\text{ A}) - 0) - \cancel{(1+\beta)R_2}(-1\text{ A}))}{\cancel{R_1}}$$

$$= \cancel{(1+\beta)(-1\text{ A})} + \cancel{(1+\beta)(-1\text{ A})}\frac{R_2}{R_1} - \cancel{(1+\beta)(-1\text{ A})}\frac{R_2}{R_1} - (\cancel{-1\text{ A}) - \beta(-1\text{ A}})$$

$$= 0\text{ A} \quad \checkmark$$

Note: It is probably easier to choose actual numerical values for all the components. Where possible, you may wish to use values that are easy to work with and can be uniquely associated with a component.

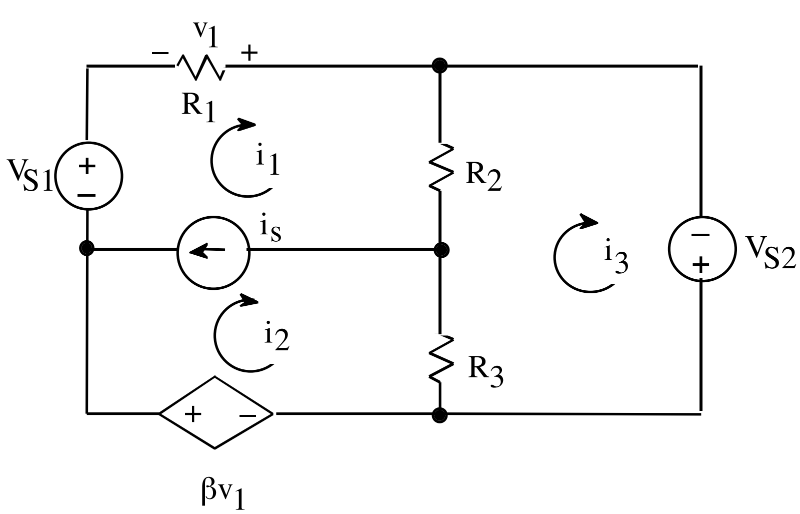
$R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 3\ \Omega$, $i_s = 4$ A, $V_{S1} = 5$ V, $V_{S2} = 10$ V, for example.

We might choose ß = 9 so we get 1 + ß = 10, for convenience.

We might also choose to avoid multiplying numbers, and instead try to cancel out terms as in the above solution.

P.S.: I found an error in my solution to (a) from the consistency checks performed above. It works!

(c)



We have a supermesh where the i_s source is located. We go around the outer loop containing i_s.

$$V_{S1} - i_1R_1 - (i_1 - i_3)R_2 - (i_2 - i_3)R_3 + \beta v_1 = 0\ \text{V}$$

We observe that we have constraint equation $v_1 = -i_1R_1$. Substitute this constraint into the first equation to eliminate v_1:

$$\boxed{V_{S1} - i_1R_1 - (i_1 - i_3)R_2 - (i_2 - i_3)R_3 - \beta i_1R_1 = 0\ \text{V}}$$

We need one more constraint equation. It arises from the current source we left out of the supermesh equation

$$\boxed{i_s = i_1 - i_2}$$

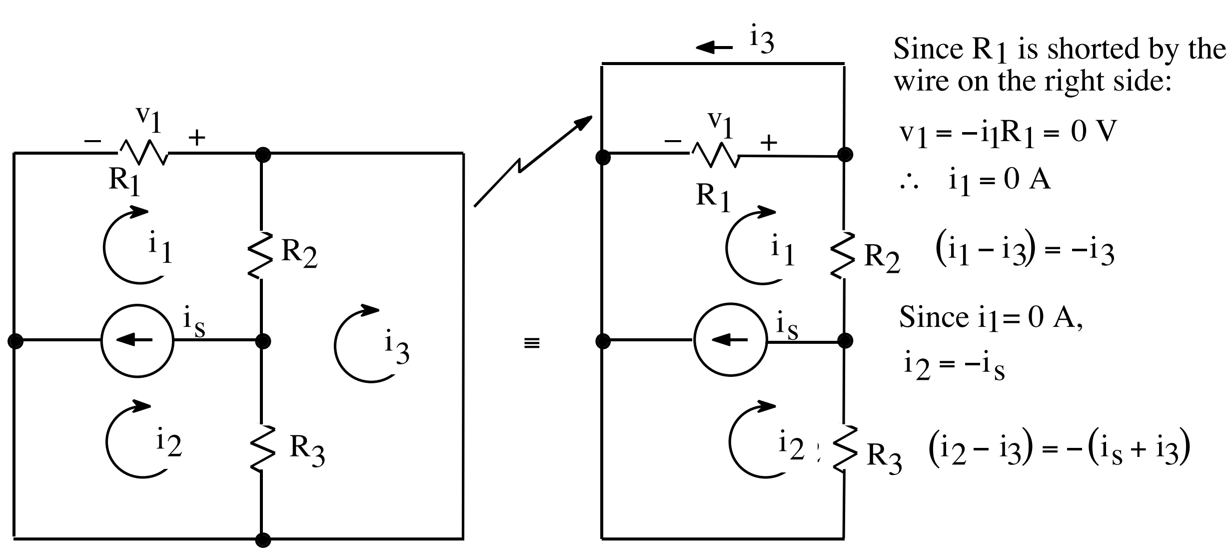
For i_3 mesh loop, we have:

$$\boxed{V_{S2} - (i_3 - i_2)R_3 - (i_3 - i_1)R_2 = 0\ \text{V}}$$

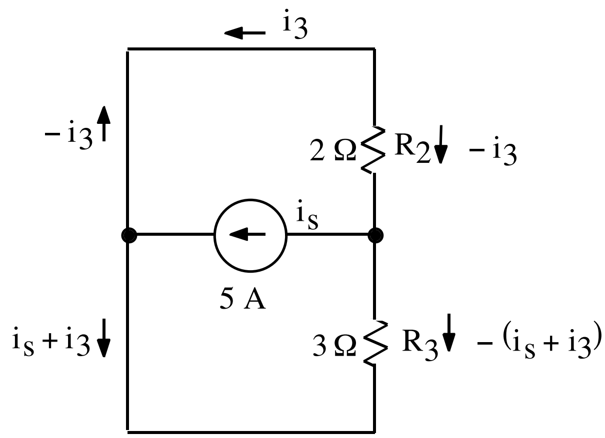
1. (d) Consistency checks.

Check: $V_{S1} = 0$ and $V_{S2} = 0$ and $ß = 0$ and $R_1 = 1\ \Omega$, $R_2 = 1\ \Omega$, $R_3 = 3\ \Omega$ and $i_s = 5$ A.

We may redraw the circuit R_1 shorted by the wire on the right. When drawn this way, i_3 flows up on the right outside edge.



Only i_1 flows through R_1, and $i_1 = 0$. Therefore, we may remove R_1.



We have a current divider.

$$i_3 = \frac{R_3}{R_2 + R_3} \cdot (-i_s) = \frac{3\ \Omega}{2\ \Omega + 3\ \Omega}(-5\ \text{A}) = -3\ \text{A}$$

$$i_2 = -i_s = -5\ \text{A}, \qquad i_1 = 0\ \text{A}$$

$$i_1 - i_2 = i_s \quad \checkmark$$

Plug i's and R's into our answer to (c):

1)

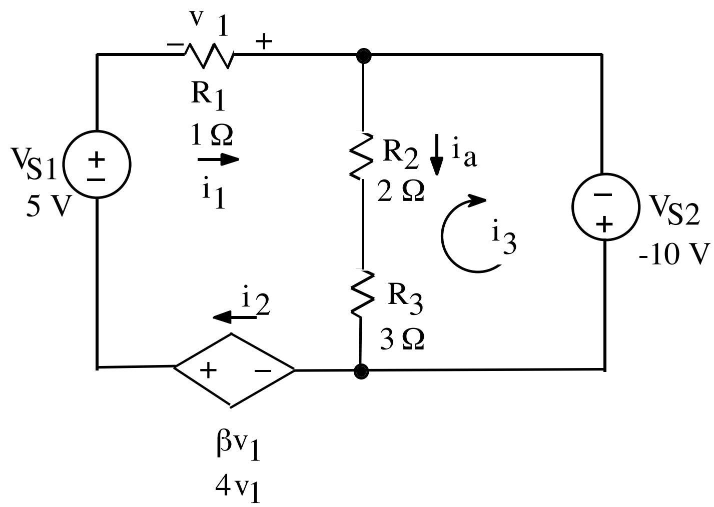
$$\begin{aligned} &V_{S1} - i_1 R_1 - (i_1 - i_3)R_2 - (i_2 - i_3)R_3 - \beta i_1 R_1 \\ &\quad = 0 - 0 \cdot 1 - (0 - -3) \cdot 2 - (-5 - -3) \cdot 3 - 0 \cdot 0 \cdot 1 \\ &\quad = 0 - 0 - 6 + 6 = 0\ \text{V} \quad \checkmark \end{aligned}$$

2)

$$\begin{aligned} &V_{S2} - (i_3 - i_2)R_3 - (i_3 - i_1)R_2 \\ &\quad = 0 - (-3 - -5)3 - (-3 - 0) \cdot 2 \\ &\quad = 0 - 6 + 6 = 0\ \text{V} \quad \checkmark \end{aligned}$$

3) $\quad i_s = 5\ \text{A} = 0 - -5 = i_1 - i_2 \quad \checkmark$

Or check: $i_s = 0$, $R_1 = 1\ \Omega$. $R_2 = 2\ \Omega$. $R_3 = 3\ \Omega$. $V_{S2} = -10$ V, $V_{S1} = 5$ V, $\beta = 4$



10 V across $R_2 + R_3 = 5\ \Omega$

$$\therefore \quad i_a = \frac{10\ \text{V}}{5\ \Omega} = 2\ \text{A} = i_1 - i_3$$

$$i_1 = i_2 = -\frac{v_1}{R_1}$$

But

$$\beta v_1 + V_{S1} + v_1 + V_{S2} = 0\ \text{V}$$

$$4v_1 + 5\ \text{V} + v_1 + -10\ \text{V} = 0\ \text{V}$$

or

$$5v_1 = 5\ \text{V} \Rightarrow v_1 = 1\ \text{V}$$

$$\therefore \quad i_1 = -\frac{1\ \text{V}}{1\ \Omega} = -1\ \text{A}, \qquad i_2 = -1\ \text{A}$$

$$i_3 = i_1 - i_a = -1 - 2 = -3\ \text{A}$$

Plug into equations from (c)

1)

$$V_{S1} - i_1 R_1 - (i_1 - i_3)R_2 - (i_2 - i_3)R_3 - \beta i_1 R_1$$
$$= 5 - (-1)\cdot 1 - (-1 - -3)\cdot 2 - (-1 - -3)\cdot 3 - 4\cdot(-1)\cdot 1$$
$$= 5 + 1 - 4 - 6 + 4 = 0\ \text{V} \checkmark$$

2)

$$\begin{aligned} V_{S2} - (i_3 - i_2)R_3 - (i_3 - i_1)R_2 \\ = -10 - (-3 - -1) \cdot 3 - (-3 - -1) \cdot 2 \\ = -10 + 6 + 4 = 0 \text{ V} \end{aligned}$$

3)

$$i_s = 0, \qquad i_1 - i_2 = -1 - -1 = 0 = i_s$$

Note: We have to be very careful about consistency checks that change the circuit topology. One helpful fact is that if a mesh loop has one side that is on the edge of the circuit diagram—so only one mesh current flows in it—then we are safe in saying the current in that component is equal to that mesh loop current. Thus, we look for components on the perimeter of the circuit when we do consistency checks.