Lorentz Relation

The Lorentz relation provides the difference between the macro (averaged) electric field, $\mathbf{E}$, and the micro (local) electric field, $\mathbf{E}_{\text{loc}}$, in a polarized medium. It is derived under certain conditions: the discrete model of the polarized medium is periodic with cubic symmetry and the electric field and polarization do not vary appreciably over many lattice spacings. The Lorentz relation may be approximately true even when the first condition is not true, as in disordered media or even non-cubic crystals. The latter condition is the the usual effective medium limit, which is ideally satisfied by metamaterials. Frequently, the Lorentz relation is derived for certain finite material bodies, e.g. ellipsoids, spheres or slabs, but one need only consider a region within a material body that satisfies the two conditions above. To find the relationship between the averaged and local field at a given point, we replace a small volume of the continuum body surrounding that point with an equivalent cubic lattice of dipole moments. Further out the discreteness will not be apparent. The contribution to the electric field at the given point due to the discrete dipole lattice for this volume, $\mathbf{E}_{\text{dip}}$, will be different than the contribution due to the corresponding volume of continuum polarization, $\mathbf{E}_{\sigma}$. Thus we have

$$\mathbf{E}_{\text{loc}} - \mathbf{E} = \mathbf{E}_{\text{dip}} - \mathbf{E}_{\sigma}$$

First we consider the contribution to the electric field of continuum polarization, $\mathbf{E}_{\sigma}$. The volume can be any shape with cubic symmetry, such as a sphere or cube. The contribution to electric field arises from unpaired, surface charge, σ, on the exterior of the volume. This unpaired surface charge is just the dot product of the polarization and the outward pointing normal.

$$\sigma = \mathbf{P}\cdot\hat{\mathbf{n}}$$

The electric field at the origin due to surface charge on some surface, S, is

$$\mathbf{E}_{\sigma} = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma\, dA}{r^2}(-\hat{\mathbf{r}})$$

where the minus sign in the direction appears because the vector $\mathbf{r}$ points from the observation point to the charge location, instead of the usual, opposite direction. Substituting from above

$$\mathbf{E}_{\sigma} = -\frac{1}{4\pi\varepsilon_0}\oint_S \mathbf{P}\cdot\hat{\mathbf{n}}\frac{\mathbf{r}}{r^3}dA$$

Without loss of generality, we can assign the z-axis to the direction of polarization. Then using spherical coordinates and a spherical volume we have

$$\mathbf{P} = P\hat{\mathbf{z}} \qquad \hat{\mathbf{n}} = \hat{\mathbf{r}} \qquad \mathbf{r} = a\sin\theta\cos\phi\hat{\mathbf{x}} + a\sin\theta\sin\phi\hat{\mathbf{y}} + a\cos\theta\hat{\mathbf{z}}$$

Plugging this into the integral

$$\mathbf{E}_{\sigma,\text{sphere}} = -\frac{1}{4\pi\varepsilon_0}\int_{-\pi}^{\pi}\int_{0}^{\pi}(P\hat{\mathbf{z}})\cdot(\hat{\mathbf{r}})\frac{a\sin\theta\cos\phi\hat{\mathbf{x}} + a\sin\theta\sin\phi\hat{\mathbf{y}} + a\cos\theta\hat{\mathbf{z}}}{a^3}a^2\sin\theta\, d\theta\, d\phi$$

Performing the ϕ integral, we find the x- and y-components are zero (since it is just the integral of a sinusoid over a full period).

$$\mathbf{E}_{\sigma,\text{sphere}} = -\frac{P\hat{\mathbf{z}}}{2\varepsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$$

By substitution, the remaining θ integral is just 2/3

$$\mathbf{E}_{\sigma,\text{sphere}} = -\frac{\mathbf{P}}{3\varepsilon_0}$$

We could also use a cubic volume, of side $2a$, for which the only contributions come from the z-oriented surfaces (since **P** is normal to the other surface normals).

$$\mathbf{P} = P\hat{\mathbf{z}} \qquad \hat{\mathbf{n}} = \pm\hat{\mathbf{z}} \qquad \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \pm a\hat{\mathbf{z}}$$

The two z-oriented surfaces have opposite charge and opposite normal, so they contribute equally.

$$\mathbf{E}_{\sigma,\text{cube}} = -2\frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\int_{-a}^{a}(P\hat{\mathbf{z}})\cdot(\hat{\mathbf{z}})\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + a\hat{\mathbf{z}}}{\left(x^2+y^2+a^2\right)^{-3/2}}dx\,dy$$

The x- and y-components are zero (since the integrand is antisymmetric and the integration range is symmetric).

$$\mathbf{E}_{\sigma,\text{cube}} = -\frac{P\hat{\mathbf{z}}a}{2\pi\varepsilon_0}\int_{-a}^{a}\int_{-a}^{a}\left(x^2+y^2+a^2\right)^{-3/2}dx\,dy$$

Using a substitution we un-dimensionalize the integral.

$$\mathbf{E}_{\sigma,\text{cube}} = -\frac{P}{2\pi\varepsilon_0}\hat{\mathbf{z}}\int_{-1}^{1}\int_{-1}^{1}\left(u^2+v^2+1\right)^{-3/2}du\,dv$$

This integral is $2\pi/3$, and we get the same result as for the sphere

$$\mathbf{E}_{\sigma,\text{cube}} = -\frac{\mathbf{P}}{3\varepsilon_0}$$

Now consider the electric field contribution from the same volume, but due to discrete dipoles. The electric field at the origin due to a point dipole at **r** is

$$\mathbf{E} = \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{p}}{4\pi\varepsilon r^5}$$

This is the same as the electric field at **r** due to point dipole at the origin. (Substitute **-r** for **r** to check this.) The field at the origin due to an array of dipoles is then

$$\mathbf{E}_{\text{dip}} = \sum_i \frac{3(\mathbf{p}_i\cdot\mathbf{r}_i)\mathbf{r}_i - r_i^2\mathbf{p}_i}{4\pi\varepsilon r_i^5}$$

Again we use uniform z-polarization

$$\mathbf{p}_i = p\hat{\mathbf{z}}$$

Then we have

$$\mathbf{E}_{\text{dip}} = \frac{p}{4\pi\varepsilon}\sum_i \frac{3(\hat{\mathbf{z}}\cdot\mathbf{r}_i)\mathbf{r}_i - r_i^2\hat{\mathbf{z}}}{r_i^5} = \frac{p}{4\pi\varepsilon}\sum_i \frac{3z_i\left(x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}} + z_i\hat{\mathbf{z}}\right) - \left(x_i^2 + y_i^2 + z_i^2\right)\hat{\mathbf{z}}}{r_i^5}$$

If the volume and the lattice have cubic symmetry, then the set of lattice points are symmetric in x and y. Summing an antisymmetric function over a symmetric domain yields zero.

$$\sum_i \frac{z_i x_i}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}} = \sum_i \frac{z_i y_i}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}} = 0$$

Note again that both a sphere and a cube have cubic symmetry. Then we have

$$\mathbf{E}_{\text{dip}} = \frac{p}{4\pi\varepsilon}\hat{\mathbf{z}}\sum_i \frac{3z_i^2 - \left(x_i^2 + y_i^2 + z_i^2\right)}{r_i^5}$$

Also, if the region and the lattice have cubic symmetry, then all of the cartesian coordinates are equivalent

$$\sum_i \frac{x_i^2}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}} = \sum_i \frac{y_i^2}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}} = \sum_i \frac{z_i^2}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}}$$

So that the terms in the numerator cancel and

$$\mathbf{E}_{\text{dip}} = 0$$

So the continuum model of our polarized volume exhibits a *depolarization field* and the discrete, cubic, array does not. Why not just use this discrete result for the whole polarized body? You can, if it's small enough, appropriately shaped (e.g. ellipsoidal) and excited uniformly along a principle axis, so as to satisfy the uniform polarization assumption. You will still need to find the continuum polarized body's electric field contribution. And, you will get the same result.

Substituting these results into our original equation we find the Lorentz relation.

$$\mathbf{E}_{\text{loc}} = \mathbf{E} + \mathbf{E}_{\text{dip}} - \mathbf{E}_\sigma = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}$$

Clausius-Mossotti

The Clausius-Mossotti relation applies to a uniformly distributed collection of electrically polarizable objects (i.e. meta-atoms or unit cells). It establishes a connection between the effective medium permittivity, ε_{eff}, of the collection and the the polarizability of an individual object, α. We begin with the relationship between electric field and polarization for a linear medium.

$$\mathbf{P} = \varepsilon_0 \chi_{\text{eff}} \mathbf{E} = \varepsilon_0 \left(\varepsilon_{\text{eff}} - 1 \right) \mathbf{E}$$

We use the subscript "eff" to distinguish the properties of the averaged, effective medium, from the properties of components that make up the unit cells. Next we note that the polarization is, by definition, the dipole moment per unit volume,

$$\mathbf{P} = \frac{\mathbf{p}}{V} = \frac{\alpha \mathbf{E}_{\text{loc}}}{V}$$

where **p** is the dipole moment of the unit cell that arises due to an applied, local electric field, $\mathbf{E}_{\text{loc}}$. The unit cell volume is V, and α is the polarizability of the unit cell. Now we use the Lorentz relation for the local electric field

$$\mathbf{E}_{\text{loc}} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}$$

Plugging in for the polarization from above we find the local electric field just in terms of the average or macroscopic electric field, another form of the Lorentz relation.

$$\mathbf{E}_{\text{loc}} = \mathbf{E} + \frac{\varepsilon_0 \left(\varepsilon_{\text{eff}} - 1 \right) \mathbf{E}}{3\varepsilon_0} = \frac{1}{3} \left(\varepsilon_{\text{eff}} + 2 \right) \mathbf{E}$$

Equating the two polarization expressions above and using this local field expression we obtain

$$\varepsilon_0 \left(\varepsilon_{\text{eff}} - 1 \right) \mathbf{E} = \frac{\alpha}{V} \frac{1}{3} \left(\varepsilon_{\text{eff}} + 2 \right) \mathbf{E}$$

Equating the coefficients of the electric field, we find the Clausius-Mossotti relation

$$\frac{\varepsilon_{\text{eff}} - 1}{\varepsilon_{\text{eff}} + 2} = \frac{\alpha}{3V\varepsilon_0}$$

Maxwell-Garnett

The Maxwell-Garnett relation makes a connection between the effective medium permittivity, ε_{eff}, and geometric and material properties of a unit-cell comprised of a spherical dielectric inclusion. We find this relation by combining the Clausius-Mossotti relation

$$\frac{\varepsilon_{\text{eff}} - 1}{\varepsilon_{\text{eff}} + 2} = \frac{\alpha}{3V\varepsilon_0}$$

with the polarizability of a dielectric sphere

$$\alpha = 4\pi\varepsilon_0 \frac{\varepsilon_i - 1}{\varepsilon_i + 2} a^3$$

to obtain

$$\frac{\varepsilon_{\text{eff}} - 1}{\varepsilon_{\text{eff}} + 2} = \frac{\frac{4}{3}\pi a^3}{V} \frac{\varepsilon_i - 1}{\varepsilon_i + 2}$$

recognizing the volume fraction of the sphere to the unit cell, δ_i, we find the usual form

$$\frac{\varepsilon_{\text{eff}} - 1}{\varepsilon_{\text{eff}} + 2} = \delta_i \frac{\varepsilon_i - 1}{\varepsilon_i + 2}$$

We can find a more general expression, with a background (matrix) dielectric, ε_m. First multiply the top and bottom of the rational expressions by the vacuum permittivity to find Maxwell-Garnett relation in terms of the absolute permittivities.

$$\frac{\varepsilon_0\varepsilon_{\text{eff}} - \varepsilon_0}{\varepsilon_0\varepsilon_{\text{eff}} + 2\varepsilon_0} = \delta_i \frac{\varepsilon_0\varepsilon_i - \varepsilon_0}{\varepsilon_0\varepsilon_i + 2\varepsilon_0}$$

Now it is more obvious where the *background* permittivity appears in the expression, and we can modify the expression with a background different from the vacuum permittivity.

$$\frac{\varepsilon_0\varepsilon_{\text{eff}} - \varepsilon_0\varepsilon_m}{\varepsilon_0\varepsilon_{\text{eff}} + 2\varepsilon_0\varepsilon_m} = \delta_i \frac{\varepsilon_0\varepsilon_i - \varepsilon_0\varepsilon_m}{\varepsilon_0\varepsilon_i + 2\varepsilon_0\varepsilon_m}$$

or in relative form

$$\frac{\varepsilon_{\text{eff}} - \varepsilon_m}{\varepsilon_{\text{eff}} + 2\varepsilon_m} = \delta_i \frac{\varepsilon_i - \varepsilon_m}{\varepsilon_i + 2\varepsilon_m}$$