Causal response of a magnetic metamaterial

For simplicity, assume a cubic, non-interacting, unit cell with a square loop encompassing the entire unit cell cross-section. (More realistic interactions and filling fractions will not change the basic form of the response.) With the engineers' assumed time dependence, $e^{j\omega t}$, the magnetic field of the incident wave is

$$H = H_0 e^{-jkz}$$

The flux threading a loop lying in the *x*-*y* plane is given by

$$\psi_m = \iint_A B\,dA = \mu_0 H_0 a \int_{-a/2}^{a/2} e^{-jkz}\,dz = \mu_0 H_0 a \left[\frac{e^{-jkz}}{-jk}\right]_{-a/2}^{a/2} = \frac{2\mu_0 H_0 a}{k}\sin(ka/2)$$

where a is the edge length of the unit cell cube. From Faraday's law, the emf driving current in the loop is

$$V = -j\omega\psi_m = -j\omega\frac{2\mu_0 H_0 a}{k}\sin(ka/2) = -2j\eta_0 H_0 a \sin\left(\frac{\omega}{\omega_a}\right)$$

where η_0 is the impedance of free space and we define a frequency

$$\omega_a = \frac{2c}{a}$$

The impedance of the loop is given by

$$Z = \frac{1}{j\omega C} + R + j\omega L = -j\frac{\omega_0^2 L}{\omega}\frac{\omega_0^2 + j\omega\gamma - \omega^2}{\omega_0^2}$$

with the usual definitions

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \gamma = R/L$$

The magnetization is given by the current, I, times the loop area divided by the unit cell volume

$$M = \frac{Ia^2}{a^3} = \frac{V}{Za} = \frac{-2j\eta_0 H_0}{-j\frac{\omega_0^2 L}{\omega}}\frac{\omega_0^2}{\omega_0^2 + j\omega\gamma - \omega^2}\sin\left(\frac{\omega}{\omega_a}\right) = \alpha\frac{\omega_0^2}{\omega_0^2 + j\omega\gamma - \omega^2}\frac{\omega\omega_a}{\omega_0^2}\sin\left(\frac{\omega}{\omega_a}\right)H_0$$

where the constant α is defined by

$$\alpha = \frac{2\eta_0}{\omega_a L} = \frac{a}{L / \mu_0}$$

The relative permeability is given by

$$\frac{\mu}{\mu_0} = 1 + \chi_m = 1 + \frac{M}{H_0}$$

Inserting the result for the magnetization yields the final form of the permeability

$$\frac{\mu}{\mu_0} = 1 + \alpha \frac{\omega_0^2}{\omega_0^2 + j\omega\gamma - \omega^2} \frac{\omega\omega_a}{\omega_0^2} \sin\left(\frac{\omega}{\omega_a}\right)$$

where the Lorentzian part is shown in blue. When the material response is a moment associated with an induced current in a finite sized loop, an additional factor appears, shown in red. The permeability, rather unfortunately, approaches unity below resonance, precluding the broad-band, low-loss response available with with strictly Lorentzian systems well below their resonance. Note that this expression can have a positive imaginary part (corresponding to gain for the choice of time dependence) when ω_a is less than ω_0, i.e. when the resonance is at a frequency outside the effective medium limit. It's the usual problem of representing a strongly spatially dispersive medium with a non-spaitially dispersive response function.

In the low frequency limit, (i.e. when one operates deep in the effective medium limit) the sine function is equal to its argument and we arrive at the commonly used expression

$$\frac{\mu}{\mu_0} = 1 + \alpha \frac{\omega^2}{\omega_0^2 + j\omega\gamma - \omega^2}$$

This expression is unsatisfyingly non-causal since the permeability does not approach one at high frequency.