When a plane wave solution

\[ E = E_0 e^{-jkr} \]

is tried for the time harmonic wave equation

\[ \nabla^2 E = -\omega^2 \epsilon \mu E \]

a constraint on the wave-vector known as the dispersion relation is found.

\[ \mathbf{k} \cdot \mathbf{k} = k_x^2 + k_y^2 + k_z^2 = \omega^2 \epsilon \mu \]

It is worth noting that the magnitude of the wave vector is not bounded by this equation. One could have an arbitrarily large imaginary component balanced by an arbitrarily large real component. In this case the wave is said to be evanescent in the direction for which the component is imaginary. It is so named since it has a decay length and does not transfer power in that direction, as waves do in directions for which the wave vector component is real (or complex).

For the single boundary problem we need to find solutions on the two sides of the boundary that are connected through, and satisfy, the boundary conditions. It is desirable to find the simplest such solution. One begins by insisting on at least one plane wave on one side of the boundary. We choose this wave to be carrying power toward the boundary and call it the incident wave. We want to find the minimum number of additional plane waves (to carry the incident power away from the boundary) that are required to satisfy the boundary conditions. One can show, by trial and error, that in general, one will require two additional plane waves, one on the same side as the incident wave, and one on the opposite side. Since the three plane waves are connected through boundary conditions that are linear relations, the two additional plane waves will have magnitudes that are linear scalings of the incident wave. We call these scaling coefficients the reflection coefficient and the transmission coefficient for the additional wave on the same, and opposite side of the boundary, respectively.

We choose the boundary to be the \( xy \) plane. We can, without loss of generality, choose the incident wave to lie in the \( xz \) plane. Since the boundary conditions must be satisfied everywhere on the boundary, there is no way to connect plane wave solutions that have different values for the in-plane component of the wave vector. Stripes of differing periods can not match everywhere. As a result, since the incident wave has no \( y \)-component of the wave-vector, the other two waves will not either, and they all must have the same common \( x \)-component. We also note that the in-plane component must be real. This is necessary because the boundary is infinite in the \( x \)- and \( y \)-directions. If the in-plane wave-vector component had an imaginary part the, wave magnitude would diverge in some direction along the boundary. The wave-vector \( z \)-components are determined by the dispersion relation.

\[
\begin{align*}
E_i &= \hat{y}E_0e^{-jk_ir} \\
\mathbf{k}_i &= k_i \hat{x} + k_i \hat{z} \\
k_i &= \sqrt{\omega^2 \epsilon \mu - k_z^2}
\end{align*}
\]

\[
\begin{align*}
E_i' &= \hat{y}GE_0e^{-jk'r} \\
\mathbf{k}_i' &= k_i \hat{x} + k_i \hat{z} \\
k_i' &= -k_i
\end{align*}
\]

\[
\begin{align*}
E_i'' &= \hat{y}TE_0e^{-jk''r} \\
\mathbf{k}_i'' &= k_i \hat{x} + k_i \hat{z} \\
k_i'' &= \sqrt{\omega^2 \epsilon \mu - k_z^2}
\end{align*}
\]

The ambiguity of branch choice for the square root in the expression for the \( z \)-component of the wave-vectors is resolved by insisting that the incident wave carries power toward the boundary and the reflected and transmitted waves carry power away. The power flow is given by the harmonic time average Poynting vector. It is already clear though that \( z \)-component of the reflected wave-vector must be minus that of the
incident wave. Since the material properties, $\mu$ and $\varepsilon$, and the in-plane wave-vector components, are the same for both, this is the only freedom left to make the reflected wave something other than a rescaled version of the incident wave.

Now we need to find the magnetic field associated with these given electric fields. Recall (or note) that for a vector field given by

$$E = E_0 e^{-jk_r}$$

where $E_0$ is a constant, the curl is given by

$$\nabla \times E = -j k \times E_0 e^{-jk_r}.$$  

We can then use the Maxwell equation

$$\nabla \times E = -j \omega \mu H$$

to solve for the magnetic field

$$H = \frac{k \times E_0}{\omega \mu} e^{-jk_r}.$$  

For the incident plane wave we find

$$H^i = \frac{k^i \times E_0}{\omega \mu_1} e^{-jk'_r} = \left(\frac{k_x \hat{x} + k_z \hat{z}}{\omega \mu_1}\right) \times \hat{y} E_0 e^{-j(k_x + k'z)\tau} = E_0 \left(\frac{k_x \hat{z} - k_z \hat{x}}{\omega \mu_1}\right) e^{-jk_x x} e^{-jk_z z}.$$  

Similar results obtain for the reflected and transmitted wave, so that the three associated magnetic fields are given by

$$H^i = \frac{E_0}{\omega \mu_1} \left(\frac{k_x \hat{z} - k_z \hat{x}}{\omega \mu_1}\right) e^{-jk_x x} e^{-jk_z z}$$

$$H^r = \frac{1}{\omega \mu_1} \left(\frac{k_x \hat{z} + k_z \hat{x}}{\omega \mu_1}\right) e^{-jk_x x} e^{jk_z z}$$

$$H^t = \frac{T E_0}{\omega \mu_2} \left(\frac{k_x \hat{z} - k_z \hat{x}}{\omega \mu_2}\right) e^{-jk_x x} e^{-jk_z z}.$$  

Now we can calculate the harmonic time averaged Poynting vector.

$$S_{av} = \frac{1}{2} \text{Re} \left( E \times H^* \right)$$

Plugging in from the expressions above
\[
S_{av}^i = \frac{1}{2} \text{Re}(E^i \times H^i) = \frac{1}{2} \text{Re} \left( \hat{y}E_0 e^{-jk_x x} e^{-jk_z z} \times \frac{E_0^*}{\omega \mu_1} (k_x \hat{z} - k_z \hat{x}) e^{jk_x x} e^{jk_z z} \right)
\]

Evaluating the cross product, combing complex conjugate terms and noting that the real part is unchanged by complex conjugation

\[
S_{av}^i = \frac{|E_0|^2}{2\omega} \text{Re} \left( \frac{1}{\mu_1} (k_x \hat{x} + k_z \hat{z}) e^{-j(k_x - k_z') z} \right) = \frac{|E_0|^2}{2\omega} \left[ \text{Re} \left( \frac{k_x}{\mu_1} \right) \hat{x} + \text{Re} \left( \frac{k_z'}{\mu_1} \right) \hat{z} \right] e^{2\text{Im}(k_z') z}
\]

We find the Poynting vector of the other plane wave similarly

\[
S_{av}^i = \frac{|E_0|^2}{2\omega} \left[ \text{Re} \left( \frac{k_x}{\mu_1} \right) \hat{x} + \text{Re} \left( \frac{k_z'}{\mu_1} \right) \hat{z} \right] e^{2\text{Im}(k_z') z}
\]

\[
S_{av}^r = \frac{|E_0|^2}{2\omega} \left[ \text{Re} \left( \frac{k_x}{\mu_2} \right) \hat{x} - \text{Re} \left( \frac{k_z}{\mu_2} \right) \hat{z} \right] e^{-2\text{Im}(k_z') z}
\]

\[
S_{av}^t = \frac{|E_0|^2}{2\omega} \left[ \text{Re} \left( \frac{k_x}{\mu_2} \right) \hat{x} + \text{Re} \left( \frac{k_z}{\mu_2} \right) \hat{z} \right] e^{2\text{Im}(k_z') z}
\]

We have specified that the incident wave will carry power toward the boundary and the reflected and transmitted wave will carry power away. Since all factors in the above expressions are positive, except the real part of the wave-vector components divided by the permeability, we have

\[
S_{av}^i \cdot \hat{z} > 0 \quad \Rightarrow \quad \text{Re} \left( \frac{k_x}{\mu_1} \right) > 0
\]

\[
S_{av}^r \cdot \hat{z} < 0 \quad \Rightarrow \quad \text{Re} \left( \frac{k_x}{\mu_1} \right) > 0
\]

\[
S_{av}^t \cdot \hat{z} > 0 \quad \Rightarrow \quad \text{Re} \left( \frac{k_x}{\mu_2} \right) > 0
\]

These conditions allow us to unambiguously choose the branch of the square root for calculating the z-component of the wave vectors.

From the Poynting vector, we can also derive a constraint on complex valued material properties for passive (non-powered) material behavior. The magnitude of the Poynting vector must decay in the direction of power flow. A plane must have decreasing (or constant) magnitude moving away from its source. Thus we have, for example, for the incident wave

\[
\text{Re} \left( \frac{k_x}{\mu_1} \right) \geq 0 \quad \text{and} \quad \text{Im} \left( \frac{k_x}{\mu_1} \right) \leq 0
\]
Since we are working with isotropic materials, any wave direction can be used for analyzing the material constraint. We can choose \( k_x = 0 \) and note that the positive real valued \( \omega \) does not contribute to the phase, then we have (dropping the subscript since this applies to all materials)

\[
\text{Re}\left(\sqrt{\frac{\varepsilon \mu}{\mu}}\right) \geq 0 \quad \text{and} \quad \text{Im}\left(\sqrt{\frac{\varepsilon \mu}{\mu}}\right) \leq 0
\]

Where the branch of the square root must be chosen the same for both, since they derive from the same wave-vector component. This is a constraint on the argument of the complex permittivity and permeability. We can plot the allowed values of these complex arguments.

The white regions indicate allowed values. Note that there are allowed regions other than the lower left quarter (where both \( \varepsilon \) and \( \mu \) have negative imaginary parts).

If we define

\[
n \equiv \sqrt{\varepsilon \mu} \quad \text{and} \quad \eta \equiv \frac{\mu}{n}
\]

then the conditions for passive media can be written as

\[
\text{Re}(\eta) \geq 0 \quad \text{and} \quad \text{Im}(n) \leq 0
\]

where we have used the fact that the sign of the real part of complex number is the same as the sign of the real part of the the inverse of the complex number.

\[
\text{Re}\left(\frac{1}{z}\right) = \text{Re}\left(\frac{z^*}{|z|^2}\right) = \frac{1}{|z|^2}\text{Re}(z^*) = \frac{1}{|z|^2}\text{Re}(z)
\]

A situation that often arises is where just the permittivity or just the permeability has an imaginary part. Consider when just the permittivity has a non-zero imaginary part, as would be the case in a non-magnetic material. We note that if \( \mu \) is real it must also be positive, so it plays no role in the complex phase. The conditions become

\[
\text{Re}\left(\sqrt{\varepsilon_i}\right) \geq 0 \quad \text{and} \quad \text{Im}\left(\sqrt{\varepsilon_i}\right) \leq 0
\]
Thus the square root of the permittivity lies in the fourth quadrant of the complex plain, and the
permittivity must lie in lower half plane. If both the permittivity and the permeability are complex but we
require them to be independently passive the condition is state as

\[ \text{Im}(\varepsilon) \leq 0 \quad \text{Im}(\mu) \leq 0 \]

Which when plotted as above would look like

![Graph showing the allowed regions](image)

The additional allowed regions in the previous figure occur when, for example, the electric response
exhibits gain behavior but the magnetic response is sufficiently lossy that the combined effect on a wave
is attenuating.

Now we will solve the equations arising from the boundary conditions for the unknown reflection and
transmission coefficients.

The boundary conditions at \( z = 0 \) are

\[ E'_y + E''_y = E'_y \]
\[ H'_x + H'_r = H'_x \]

Plugging in our fields we find

\[ E_0 e^{-jk_z} + \Gamma E_0 e^{-jk_z} = TE_0 e^{-jk_z} \]
\[ -\frac{E_0}{\omega \mu_1} k'_z e^{-jk_z} + \frac{\Gamma E_0}{\omega \mu_1} k'_z e^{-jk_z} = -\frac{TE_0}{\omega \mu_2} k'_z e^{-jk_z} \]

Canceling common factors

\[ 1 + \Gamma = T \]
\[ \frac{k'_z}{\mu_1} + \frac{\Gamma k'_z}{\mu_1} = \frac{T k'_z}{\mu_2} \]

Solving we obtain
\[ \Gamma_\perp^E = \frac{k_i^j \mu_2 - k_i^j \mu_1}{k_i^j \mu_2 + k_i^j \mu_1} \quad T_\perp^E = \frac{2k_i^j \mu_2}{k_i^j \mu_2 + k_i^j \mu_1} \]

Where the fact that these results are for perpendicular polarization and represent ratios of the electric field magnitude are explicitly indicated by the sub- and superscripts.

To find these coefficients for the parallel polarization case, we will use a duality principle. Note what happens when we make a particular substitution in our source-free Maxwell’s equations (in linear isotropic media)

\[
\begin{align*}
\nabla \cdot (\varepsilon E) &= 0 \quad \varepsilon \rightarrow \mu \\
\nabla \cdot (\mu H) &= 0 \quad \mu \rightarrow \varepsilon \nabla \times E &= -j\omega \mu H \\
\n\nabla \times H &= j\omega \varepsilon E
\end{align*}
\]

The transformed equations are still Maxwell’s equations. So any solutions we perform this substitution on will also be solutions of Maxwell’s equations. If we perform this substitution on our perpendicular polarized plane waves we will get parallel polarized plane waves. However, we need the coefficients for magnetic field ratios, so that the transformed results will be electric field ratios. First we write the magnetic field in the following form.

\[
\begin{align*}
H^i &= H^i \hat{\sigma}^i e^{-k^i \cdot r} \\
H^r &= H^r \hat{\sigma}^r e^{-k^r \cdot r} \\
H^t &= H^t \hat{\sigma}^t e^{-k^t \cdot r}
\end{align*}
\]

Where the \( \sigma \) are unit vectors. (There is some arbitrariness in the definition of these unit vectors that was not present in the perpendicular case. There it made sense to look at ratios of the \( y \)-components of electric field. Here one could choose a different sign for the polarization vectors and get a different sign in the reflection and transmission coefficients.) The normalizing factor is the norm of the wave vector

\[ k = |k| = \sqrt{k \cdot k^*} \]

not the wave dotted with itself as appears in the dispersion relation. From these expressions we can find the reflection and transmission coefficients for the magnetic field ratios.

\[
\begin{align*}
\Gamma_\perp^H &\equiv \frac{H^r}{H^i} = -\Gamma_\perp^E = -\frac{k_i^j \mu_2 - k_i^j \mu_1}{k_i^j \mu_2 + k_i^j \mu_1} \\
T_\perp^H &\equiv \frac{H^t}{H^i} = T_\perp^E \frac{\mu_i k^i}{\mu_2 k^i} = \frac{2k_i^j \mu_2}{\mu_2 k^i \mu_2 + k_i^j \mu_1}
\end{align*}
\]
Then the reflection and transmission coefficients for the electric field ratios, and parallel polarization are found by applying the substitution.

\[
\Gamma_{\parallel}^E = -\frac{k^i e_2 - k^i e_1}{k^i e_2 + k^i e_1}, \quad T_{\parallel}^E = \frac{\varepsilon_1 k^i}{\varepsilon_2 k^i} \frac{2k^i e_2}{k^i e_2 + k^i e_1}
\]

The conditions for choosing the square-root branch for the z-component of the wave-vectors must also be transformed.

\[
\text{Re} \left( \frac{k^i}{\varepsilon_1} \right) > 0 \\
\text{Re} \left( \frac{k^i}{\varepsilon_2} \right) > 0
\]

I believe these conditions are the same as those used in the perpendicular polarization case when \( \varepsilon \) and \( \mu \) are independently passive, but not when they are only passive together. No proof offered though.

Was the duality path to these solutions easier than a direct derivation? Maybe not, but it always sounds cool to invoke duality.

Find the expressions for the reflection and transmission coefficients when the materials are lossless and evanescent waves are excluded.

\[
k = nk_0 \\
k^i = nk_0 \cos \theta \\
k_0 = \omega \sqrt{\varepsilon_0 \mu_0}
\]

\[
\Gamma_{\perp}^E = \frac{n_1 k_0 \cos \theta \mu_2 - n_2 k_0 \cos \theta \mu_1}{n_1 k_0 \cos \theta \mu_2 + n_2 k_0 \cos \theta \mu_1} \frac{1}{n_1 n_2} = \frac{\mu_2 \cos \theta_1 - \mu_1 \cos \theta_1}{n_2 \mu_2 \cos \theta_1 + n_1 \mu_1 \cos \theta_1} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_1}
\]

\[
\Gamma_{\perp}^E = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_1}, \quad T_{\perp}^E = \frac{2n_2 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_1}
\]

\[
T_{\parallel}^E = \frac{\varepsilon_1 n_2 k_0}{\varepsilon_2 n_1 k_0} \frac{2n_1 k_0 \cos \theta \varepsilon_2}{n_1 k_0 \cos \theta \varepsilon_2 + n_2 k_0 \cos \theta \varepsilon_1} = \frac{n_1}{\varepsilon_1 \varepsilon_2} \frac{2n_2 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_1}
\]

\[
\Gamma_{\parallel}^E = -\frac{\eta_1 \cos \theta_1 - \eta_2 \cos \theta_1}{\eta_1 \cos \theta_1 + \eta_2 \cos \theta_1}, \quad T_{\parallel}^E = \frac{2\eta_2 \cos \theta_1}{\eta_1 \cos \theta_1 + \eta_2 \cos \theta_1}
\]