Small Loop Antenna

For the small square loop antenna, we approximate the loop by four ideal dipoles located at the centers of the side of the loop

$$\mathbf{r}_1' = -\frac{l}{2}\hat{\mathbf{y}} \qquad \mathbf{r}_2' = \frac{l}{2}\hat{\mathbf{x}} \qquad \mathbf{r}_3' = \frac{l}{2}\hat{\mathbf{y}} \qquad \mathbf{r}_4' = -\frac{l}{2}\hat{\mathbf{x}}$$

with current magnitudes and directions given by

$$\mathbf{I}_1 = I\hat{\mathbf{x}} \qquad \mathbf{I}_2 = I\hat{\mathbf{y}} \qquad \mathbf{I}_3 = -I\hat{\mathbf{x}} \qquad \mathbf{I}_4 = -I\hat{\mathbf{y}}$$

The vector potential for a z-oriented, ideal dipole, located at the origin is

$$\mathbf{A} = \frac{\mu}{4\pi}(\beta\Delta z)\frac{e^{-j\beta r}}{\beta r}I\hat{\mathbf{z}}$$

The four dipoles, indexed by i, have distance from current element to field point, R_i, current, $\mathbf{I}_i$, and length, l

$$\mathbf{A}_i = \frac{\mu}{4\pi}(\beta l)\frac{e^{-j\beta R_i}}{\beta R_i}\mathbf{I}_i$$

For the far-field we can approximate R_i in the exponential with the usual first order expansion. (In the denominator we will use the zeroth order expansion, r.)

$$R_1 = r - \hat{\mathbf{r}}\cdot\mathbf{r}_1' = r + \frac{l}{2}\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = r + \frac{l}{2}\sin\theta\sin\phi$$
$$R_2 = r - \hat{\mathbf{r}}\cdot\mathbf{r}_2' = r - \frac{l}{2}\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = r - \frac{l}{2}\sin\theta\cos\phi$$
$$R_3 = r - \hat{\mathbf{r}}\cdot\mathbf{r}_3' = r - \frac{l}{2}\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = r - \frac{l}{2}\sin\theta\sin\phi$$
$$R_4 = r - \hat{\mathbf{r}}\cdot\mathbf{r}_4' = r + \frac{l}{2}\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = r + \frac{l}{2}\sin\theta\cos\phi$$

where in the last equality we have used the expression for the radial unit vector in cartesian coordinates

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

The total vector potential is

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 = \frac{\mu}{4\pi}(\beta l)\left[\frac{e^{-j\beta R_1}}{\beta R_1}\mathbf{I}_1 + \frac{e^{-j\beta R_2}}{\beta R_2}\mathbf{I}_2 + \frac{e^{-j\beta R_3}}{\beta R_3}\mathbf{I}_3 + \frac{e^{-j\beta R_4}}{\beta R_4}\mathbf{I}_4\right]$$

Plugging in from above and simplifying

$$
\begin{aligned}
\mathbf{A} &\simeq \frac{\mu}{4\pi}(\beta l)\left[\frac{e^{-j\beta\left(r+\frac{l}{2}\sin\theta\sin\phi\right)}}{\beta r}I\hat{\mathbf{x}}+\frac{e^{-j\beta\left(r-\frac{l}{2}\sin\theta\cos\phi\right)}}{\beta r}I\hat{\mathbf{y}}-\frac{e^{-j\beta\left(r-\frac{l}{2}\sin\theta\sin\phi\right)}}{\beta r}I\hat{\mathbf{x}}-\frac{e^{-j\beta\left(r+\frac{l}{2}\sin\theta\cos\phi\right)}}{\beta r}I\hat{\mathbf{y}}\right] \\
&\simeq \frac{\mu I}{4\pi}(\beta l)\frac{e^{-j\beta r}}{\beta r}\left[-\left(e^{j\beta\frac{l}{2}\sin\theta\sin\phi}-e^{-j\beta\frac{l}{2}\sin\theta\sin\phi}\right)\hat{\mathbf{x}}+\left(e^{j\beta\frac{l}{2}\sin\theta\cos\phi}-e^{-j\beta\frac{l}{2}\sin\theta\cos\phi}\right)\hat{\mathbf{y}}\right] \\
&\simeq 2j\frac{\mu I}{4\pi}(\beta l)\frac{e^{-j\beta r}}{\beta r}\left[-\sin\left(\beta\frac{l}{2}\sin\theta\sin\phi\right)\hat{\mathbf{x}}+\sin\left(\beta\frac{l}{2}\sin\theta\cos\phi\right)\hat{\mathbf{y}}\right] \\
&\simeq 2j\frac{\mu I}{4\pi}(\beta l)\frac{e^{-j\beta r}}{\beta r}\left[-\beta\frac{l}{2}\sin\theta\sin\phi\hat{\mathbf{x}}+\beta\frac{l}{2}\sin\theta\cos\phi\hat{\mathbf{y}}\right] \\
&\simeq j\frac{\mu I}{4\pi}(\beta l)^2\frac{e^{-j\beta r}}{\beta r}\sin\theta\left[-\sin\phi\hat{\mathbf{x}}+\cos\phi\hat{\mathbf{y}}\right] \\
&\simeq j\frac{\mu I}{4\pi}(\beta^2 S)\frac{e^{-j\beta r}}{\beta r}\sin\theta\hat{\boldsymbol{\phi}}
\end{aligned}
$$

in the third step we approximate the sine function by its argument, which we can do because $\beta l \ll 1$. We also replace the square of the side length, l, by the area, S. Though derived for the square loop, the resulting expression is correct for a loop of any shape with area S. The electric far-field is simple related to the transverse component of the vector potential

$$
\mathbf{E} \simeq -j\omega\mathbf{A}_{\perp} \simeq \frac{\omega\mu I}{4\pi}(\beta^2 S)\frac{e^{-j\beta r}}{\beta r}\sin\theta\hat{\boldsymbol{\phi}} \simeq \frac{\eta I\beta}{4\pi}(\beta^2 S)\frac{e^{-j\beta r}}{\beta r}\sin\theta\hat{\boldsymbol{\phi}}
$$

The electric field units are obviously correct in final expression.