**Array of arbitrarily oriented and positioned, but identical elements**

The array far-field vector potential for the total current, **J**, is

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \iiint_{V'} \mathbf{J}(\mathbf{r}') e^{j\beta \hat{\mathbf{r}}\cdot\mathbf{r}'} \, dv'$$

The total current is a sum of identical current elements, $\mathbf{J}_1$, each of which has its own position, $\mathbf{r}'_n$, orientation (i.e. rotation matrix), $\mathbf{R}_n$, and complex amplitude, $\alpha_n$

$$\mathbf{J}(\mathbf{r}') = \sum_n \alpha_n \mathbf{R}_n \mathbf{J}_1\left(\mathbf{R}_n^{-1}\left(\mathbf{r}' - \mathbf{r}'_n\right)\right)$$

Substituting in to the vector potential

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \iiint_{V'} \sum_n \alpha_n \mathbf{R}_n \mathbf{J}_1\left(\mathbf{R}_n^{-1}\left(\mathbf{r}' - \mathbf{r}'_n\right)\right) e^{j\beta \hat{\mathbf{r}}\cdot\mathbf{r}'} \, dv'$$

For finite sums, we can always exchange the order of summation and integration

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \sum_n \alpha_n \iiint_{V'} \mathbf{R}_n \mathbf{J}_1\left(\mathbf{R}_n^{-1}\left(\mathbf{r}' - \mathbf{r}'_n\right)\right) e^{j\beta \hat{\mathbf{r}}\cdot\mathbf{r}'} \, dv'$$

We define a new variable, $\mathbf{r}''$

$$\mathbf{r}'' \equiv \mathbf{R}_n^{-1}\left(\mathbf{r}' - \mathbf{r}'_n\right) \quad \Rightarrow \quad \mathbf{r}' = \mathbf{R}_n \mathbf{r}'' + \mathbf{r}'_n$$

Make a change of variables for the integral. The infinite volume of integration is unchanged

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \sum_n \alpha_n \iiint_{V''} \mathbf{R}_n \mathbf{J}_1(\mathbf{r}'') e^{j\beta \hat{\mathbf{r}}\cdot\left(\mathbf{R}_n \mathbf{r}'' + \mathbf{r}'_n\right)} \, dv''$$

We can exchange the order of any linear operation, such as rotation, and integration. We also pull out the constant phase factor associated with element position.

$$\mathbf{A} = \mu \sum_n \alpha_n e^{j\beta \hat{\mathbf{r}}\cdot\mathbf{r}'_n} \mathbf{R}_n \frac{e^{-j\beta r}}{4\pi r} \iiint_{V''} \mathbf{J}_1(\mathbf{r}'') e^{j\beta \mathbf{R}_n^{-1}\hat{\mathbf{r}}\cdot\mathbf{r}''} \, dv''$$

We define a current function to represent the phased current density integral in a particular direction, given by the position unit vector.

$$\mathbf{I}(\hat{\mathbf{r}}) \equiv \iiint_{V'} \mathbf{J}_1(\mathbf{r}') e^{j\beta \hat{\mathbf{r}}\cdot\mathbf{r}'} \, dv'$$

Now we can write the vector potential in terms of this function

$$\mathbf{A} = \mu \sum_n \alpha_n e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'_n} \mathbf{R}_n \frac{e^{-j\beta r}}{4\pi r} \mathbf{I}\left(\mathbf{R}_n^{-1} \hat{\mathbf{r}}\right)$$

As usual, the magnetic and electric fields are found from the vector potential

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

$$\mathbf{E} = \frac{1}{j\omega\varepsilon} \nabla \times \mathbf{H} = \frac{1}{j\omega\varepsilon\mu} \nabla \times \nabla \times \mathbf{A}$$

Using the vector potential above

$$\mathbf{E} = \frac{1}{j4\pi\omega\varepsilon} \sum_n \alpha_n e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'_n} \mathbf{R}_n \nabla \times \nabla \times \left[ \frac{e^{-j\beta r}}{r} \mathbf{I}\left(\mathbf{R}_n^{-1} \hat{\mathbf{r}}\right) \right]$$

The usual approximation for far-field yields

$$\nabla \times \nabla \times \left[ \frac{e^{-j\beta r}}{r} \mathbf{I}(\hat{\mathbf{r}}) \right] \simeq \beta^2 \frac{e^{-j\beta r}}{r} \mathbf{I}_\perp(\hat{\mathbf{r}})$$

where

$$\mathbf{I}_\perp(\hat{\mathbf{r}}) = \mathbf{I}(\hat{\mathbf{r}}) - \left(\mathbf{I}(\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}}\right) \hat{\mathbf{r}}$$

so that the electric field is

$$\mathbf{E} = \frac{\beta^2}{j4\pi\omega\varepsilon} \frac{e^{-j\beta r}}{r} \sum_n \alpha_n e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'_n} \mathbf{R}_n \mathbf{I}_\perp\left(\mathbf{R}_n^{-1} \hat{\mathbf{r}}\right)$$

and we see that the un-normalized, far-field antenna pattern of the array is finally

$$\boxed{\mathbf{F}(\hat{\mathbf{r}}) = \sum_n \alpha_n e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'_n} \mathbf{R}_n \mathbf{f}\left(\mathbf{R}_n^{-1} \hat{\mathbf{r}}\right)}$$

where we have identified the perpendicular component of the current density integral as the un-normalized, far-field antenna pattern of a single element.

$$\mathbf{I}_\perp(\hat{\mathbf{r}}) = \mathbf{f}(\hat{\mathbf{r}})$$

The single element pattern, **f**, is just a vector field, and is rotated in the usual manner, by rotating the field vector, and inversely rotating the field argument.

$$\mathbf{R}\mathbf{f}\left(\mathbf{R}^{-1} \hat{\mathbf{r}}\right)$$