2.2-3 (a) Away from the origin, (2-55) becomes (2-57), and the Laplacian of ψ exists. First calculate the Laplacian of ψ. In spherical coordinates we use (C-36) and neglect the θ and ϕ derivatives because ψ depends only on r.

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) \qquad \psi = C\frac{e^{-j\beta r}}{r}$$

$$\frac{\partial\psi}{\partial r} = -Ce^{-j\beta r}\frac{1}{r^2} - j\beta Ce^{-j\beta r}\frac{1}{r}$$

$$r^2\frac{\partial\psi}{\partial r} = -Ce^{-j\beta r}\left(1+j\beta r\right)$$

$$\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) = -Ce^{-j\beta r}j\beta + j\beta Ce^{-j\beta r}\left(1+j\beta r\right) = -\beta^2 Cre^{-j\beta r}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) = -\beta^2 C\frac{e^{-j\beta r}}{r} = -\beta^2\psi$$

Which proves that

$$\boxed{\nabla^2\psi + \beta^2\psi = 0}$$

(b) Starting with (2-55)

$$\nabla^2\psi + \beta^2\psi = -\delta(x)\delta(y)\delta(z)$$

we integrate over a spherical volume centered on the origin, and take the limit as the radius of the sphere, ε, goes to zero

$$\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon}\nabla^2\psi\,dv + \beta^2\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon}\psi\,dv = -\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon}\delta(x)\delta(y)\delta(z)\,dv$$

Since the radius is small in the limit, we can approximate ψ as follows

$$\psi = C\frac{e^{-j\beta r}}{r} \approx C\frac{1}{r}$$

Since the Laplacian is the divergence of the gradient we can use the divergence theorem to convert the first volume integral to a surface integral and escape the problem of having to calculate the Laplacian at the singular point

$$\iiint_{V_\varepsilon}\nabla^2\psi dv = \iiint_{V_\varepsilon}\nabla\cdot(\nabla\psi)dv = \iint_{S_\varepsilon}\nabla\psi\cdot d\mathbf{s}$$

where S_ε is the bounding spherical surface of radius ε. Calculating the gradient

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} \simeq -C\frac{1}{r^2}\hat{\mathbf{r}}$$

The directed element of surface area is

$$d\mathbf{s} = \hat{\mathbf{r}}\,ds$$

So the integral becomes

$$\iint_{S_\varepsilon} \nabla\psi \cdot d\mathbf{s} = \iint_{S_\varepsilon} \left(-C\frac{1}{\varepsilon^2}\hat{\mathbf{r}}\right)\cdot \hat{\mathbf{r}}\,ds = -C\frac{1}{\varepsilon^2}\iint_{S_\varepsilon} ds = -C\frac{1}{\varepsilon^2}4\pi\varepsilon^2 = -C4\pi$$

where we have evaluated the gradient at the radius of the surface and pulled it out of the integral since it is constant on the surface. Then the surface integral just becomes the area of the sphere. Thus

$$\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon} \nabla^2\psi\,dv = -4\pi C$$

For the second integral, we don't have to worry about taking any derivatives at the singular point, so we can just proceed straightforwardly with the volume integral using spherical shells.

$$\iiint_{V_\varepsilon} \psi\,dv \approx \int_0^\varepsilon \left(C\frac{1}{r}\right)4\pi r^2\,dr = 4\pi C\int_0^\varepsilon r\,dr = 4\pi C\left[\frac{r^2}{2}\right]_0^\varepsilon = 2\pi C\varepsilon^2$$

So that

$$\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon} \psi\,dv = \lim_{\varepsilon\to 0} 2\pi C\varepsilon^2 = 0$$

Finally for the last integral we can switch to an integration volume that is a cube of edge 2ε, since the integrand is zero everywhere these volumes differ.

$$\iiint_{V_\varepsilon} \delta(x)\delta(y)\delta(z)dv = \iiint_{\text{cube}_\varepsilon} \delta(x)\delta(y)\delta(z)dv = \int_{-\varepsilon}^{\varepsilon}\delta(x)dx\int_{-\varepsilon}^{\varepsilon}\delta(y)dy\int_{-\varepsilon}^{\varepsilon}\delta(z)dz = 1^3 = 1$$

So that

$$\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon} \delta(x)\delta(y)\delta(z)dv = 1$$

Putting everything together

$$-4\pi C + \beta^2\cdot 0 = -1$$

and

$$\boxed{C = \frac{1}{4\pi}}$$

Physicists consider the following to be a definition of the delta function (though I am not sure mathematicians would agree).

$$\delta^3(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z) \equiv -\nabla^2\frac{1}{4\pi r}$$