

Numerical Methods in Electromagnetics

CLASS NOTES FOR EE553, SPRING 1998

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Introduction

Numerical techniques are exceptionally important in electromagnetics. Only a few problems can be solved analytically, and those problems that have an analytical solution are relatively simple. Real life electromagnetic problems are usually complex and, therefore, techniques to solve these problems numerically with reasonable accuracy are needed.

Most of the techniques we will present are based on some form of “discretization,” that means that the integral and differential equations that characterize the problem are solved by discretizing some of the variables in space, or frequency, or time.

These are intended to be class notes to help the student in following the topic on the textbook (Sadiku). These notes are **not** intended to substitute the book, that should remain the main student reference.

These notes are structured in a way that you can find the **project** relative to each section at the end of the section itself. Four sections have been included in these notes, and each of these concludes with a project to solve. Note that **is not a good idea** to approach the project at the end of each section (that means just before the project is due). During the course you should continuously looking at the project relative to that part of the course and solve the points you are able to with the information and the knowledge that you gained at that stage.

1. The Finite Difference (FD) Method

The Finite Difference (FD) method is extremely useful for solving a wide variety of problems. It is largely used for the solution of electrostatic and magnetostatic problems, as well as for the analysis of guiding structures, or anyway problems characterized by closed boundaries.

The Finite-Difference method is widely used for the solution of problems described by Poisson's or Laplace's equations. We will illustrate its application both for electrostatic and waveguide problems.

Electrostatic Problems

From previous EM courses, we know that Maxwell's equations can be expressed both in integral and differential form:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV \quad \text{or} \quad \nabla \cdot \mathbf{D} = \rho_v$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} = 0$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad \text{or} \quad \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S} + \int_S \mathbf{J} \cdot d\mathbf{S} \quad \text{or} \quad \nabla \times \mathbf{H} = \frac{d\mathbf{D}}{dt} + \mathbf{J}.$$

For static fields, the equations simplify in

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV \quad \text{or} \quad \nabla \cdot \mathbf{D} = \rho_v$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} = 0$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{or} \quad \nabla \times \mathbf{E} = 0$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \quad \text{or} \quad \nabla \times \mathbf{H} = \mathbf{J}.$$

It is evident, therefore, that in the static case the electric field is decoupled from the magnetic field, that means that the electric field can exist without the presence of a magnetic field, and vice versa. It is also clear that, in case of an electrostatic problem, the electric field intensity and electric flux density must satisfy the boundary conditions

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s,$$

$$\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

while, in the case of magnetostatic problem,

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s .$$

We can now focus on electrostatic problems, introducing a scalar potential ϕ that can be interpreted as the **work done by electric forces to move a unit positive charge from a certain point P to the origin O**. Since for Maxwell's equations in differential form the curl of \mathbf{E} is 0, for vector properties \mathbf{E} can be expressed as the gradient of a scalar function ϕ . Therefore, from Gauss's equation in differential form we obtain

$$\nabla \cdot \epsilon (-\nabla \phi) = \rho_v$$

that gives the Poisson's equation:

$$\nabla^2 \phi = - \frac{\rho_v}{\epsilon} ,$$

and, in case of absence of free charges,

$$\nabla^2 \phi = 0 .$$

It is worth noticing that from the Maxwell's equations in integral form, it must be satisfied the relation

$$\phi_P = \int_P^0 \mathbf{E} \cdot d\mathbf{l} = \text{const.}$$

that justify the interpretation as a work done to move a charge given above. Moreover,

$$\phi_a - \phi_b = \int_a^b \mathbf{E} \cdot d\mathbf{l} .$$

These two relations say that the potential in a certain point does not depend on the path followed to arrive to that point. Therefore, a potential difference between two points is independent from the path chosen to go from a point to another.

Now that we know the equations to solve (Laplace and Poisson), and how to relate the unknown function ϕ to the desired electric field, we can get into the details of the solution of the problem. Poisson's and Laplace's equations are differential equations and, therefore, we shall find a method to approximate the derivatives that we should calculate. Given the function sketched in fig.1, for example, we can approximate the derivative basically in three ways. Referring to the notation of fig.1, we could use a forward difference approximation defined by

$$\frac{d\phi}{dx} \approx \frac{\phi(x_0 + h) - \phi(x_0)}{h}$$

or a backward difference approximation defined by

$$\frac{d\phi}{dx} \approx \frac{\phi(x_0) - \phi(x_0 - h)}{h}$$

or a central difference approximation, defined by

$$\frac{d\phi}{dx} \approx \frac{\phi(x_0 + h) - \phi(x_0 - h)}{2h}$$

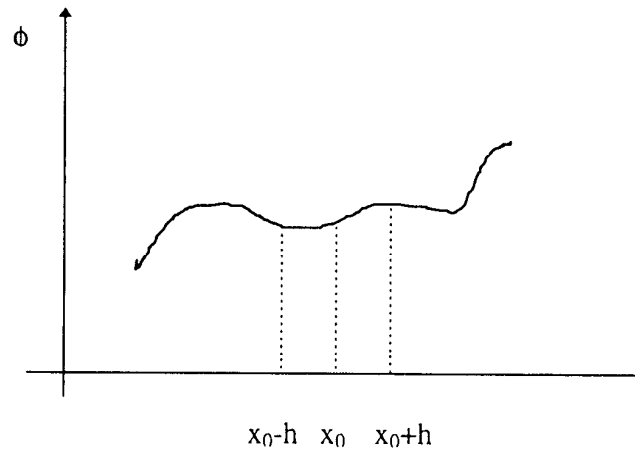


Fig.1. Example of function f for the definition of various possibility for the approximation of the derivative

It is possible to show by means of the Taylor series that *the central difference approximation is the most convenient among these three cases since it leads to a second order accurate result* (see Iskander p.301, or Sadiku p. 138).

Using the central difference method, we can therefore estimate also the second derivative of the function ϕ :

$$\begin{aligned} \frac{d^2}{dx^2} \phi &\approx \frac{\frac{d\phi(x_0 + h/2)}{dx} - \frac{d\phi(x_0 - h/2)}{dx}}{h} \approx \frac{\frac{\phi(x_0 + h) - \phi(x_0)}{h} - \frac{\phi(x_0) - \phi(x_0 - h)}{h}}{h} = \\ &= \frac{\phi(x_0 + h) + \phi(x_0 - h) - 2\phi(x_0)}{h^2} \end{aligned}$$

Consider now a two-dimensional case. The geometry to focus on is the one in fig.2. In this figure, we show the point we are trying to estimate numerically the second derivatives (ϕ_0) by using its neighbors ($\phi_1, \phi_2, \phi_3, \phi_4$).

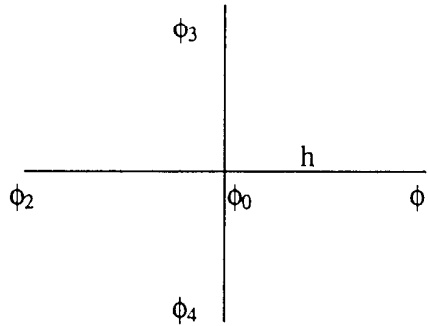


Fig.2. Disposition of the points for the calculation of $\nabla^2\phi$ in two dimension.

Similarly to what we have done for the one-dimensional case, we obtain for $\nabla^2\phi$ the following expression:

$$\nabla^2\phi \approx \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{h^2}$$

The FD method for electrostatic problems is nothing more than the solution of Laplace or Poisson's equations by using the described approximation of $\nabla^2\phi$.

To apply the method, one needs first to subdivide the region of interest into a "reasonably" fine grid. The approximation of the $\nabla^2\phi$ should then be applied to each node of the grid, obtaining therefore a system of N equations in N unknowns (the potential at each node). The final system of equation can then be solved by using a direct method (direct inversion of the matrix of the coefficients) or an iterative method like the method of successive over-relaxation (SOR) (p. 154 Sadiku).

Multiple Dielectric Region

What happens if we have different dielectrics in the region of interest? The same theoretical concept applies, but now we have to modify slightly the approximation of the differential to take into account the discontinuity in the dielectrics.

The equation to apply if the node is on the boundary between two dielectrics can be derived by using Maxwell's equation in integral form. Considering the geometry of fig.3 we obtain

$$\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S} = \oint_S \epsilon \nabla \phi \cdot d\mathbf{S} = \oint_C \epsilon \nabla \phi \cdot d\mathbf{C} = \oint_C \epsilon \frac{\partial \phi}{\partial n} \cdot d\mathbf{C} = 0$$

and, carrying out the integration,

$$\begin{aligned} \oint_C \frac{\partial \phi}{\partial n} \epsilon d\mathbf{C} = & \frac{\phi_1 - \phi_0}{h} \left(\epsilon_2 \frac{h}{2} + \epsilon_1 \frac{h}{2} \right) + \frac{\phi_3 - \phi_0}{h} (\epsilon_1 l_1) + \frac{\phi_2 - \phi_0}{h} \left(\epsilon_1 \frac{h}{2} + \epsilon_2 \frac{h}{2} \right) + \\ & + \frac{\phi_4 - \phi_0}{h} (\epsilon_2 h) \end{aligned}$$

that leads to the relation we need to use for the nodes that lie on the interface between two dielectrics:

$$2\epsilon_1 \phi_3 + 2\epsilon_2 \phi_4 + (\epsilon_1 + \epsilon_2) \phi_1 + (\epsilon_1 + \epsilon_2) \phi_2 - 4(\epsilon_1 + \epsilon_2) \phi_0 = 0.$$

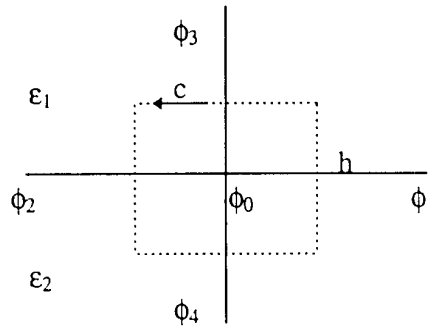


Fig.3. Disposition of the points for the calculation of $\nabla^2 \phi$ in two dimension for a two-dielectric region.

Note that we have considered always a second order approximation for the differentials. In many applications, where a better approximation is needed, other schemes are used like for example the fourth order approximation (p. 153 Sadiku).

The Final System of Equations

With the formulae we have developed, is now easy to solve the problem. The steps are the following:

- Discretize the region of interest. Make sure that, if there are multiple dielectrics, some of the nodes lie on the interface between two dielectrics.
- Write the necessary equation at each node of the mesh.
- Obtain a system of equation like

$$[A] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = [B]$$

d. Solve the system by using a direct method or an iterative method (like SOR).

Refer to p. 154 of Sadiku, and look at examples 3.3, 3.4.

One should be aware the several errors may result from the application of the method, the most common of which are round-off error and discretization error. Refer to p. 163 of Sadiku.

Symmetry Considerations

Symmetry is in real problems extremely important. It may lead to enormous savings in computer time and memory and, therefore, may also allow a very accurate discretization without the need to increase the computer resources.

The application of symmetry to the solution of the equations we are interested in is straightforward. Consider only the following case (fig. 4):

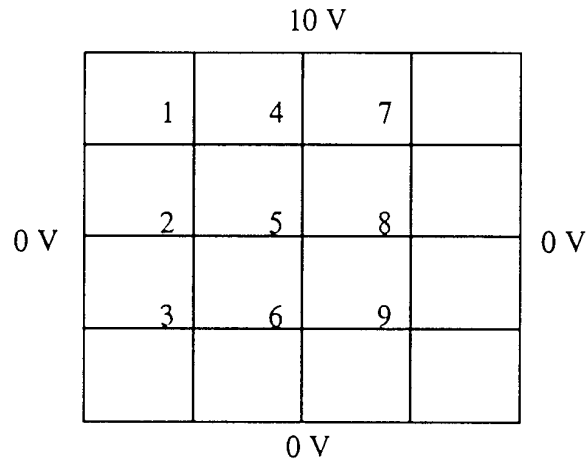


Fig.4. Example of geometry where it is convenient to use symmetry.

It is evident from fig.4 that, for symmetry consideration, the value ϕ_9 will be equal to the value ϕ_7 . Similarly, $\phi_2 = \phi_8$, and $\phi_3 = \phi_6$. Therefore, without having to solve the system that includes all these nodes, we can just use these information as a boundary condition. The equation for node 4 will be, for example:

$$\frac{\phi_1 + \phi_7 + \phi_5 + 10 - 4\phi_4}{h^2} = \frac{\phi_1 + \phi_1 + \phi_5 + 10 - 4\phi_4}{h^2} = \frac{2\phi_1 + \phi_5 + 10 - 4\phi_4}{h^2} = 0 \text{ or } \frac{\rho}{\epsilon}$$

(Laplace or Poisson eq.)

We can proceed similarly for the other symmetric nodes.

Calculation of Capacitance with FD

To calculate the capacitance of a structure, we may use Gauss's law to relate the potential with the enclosed charge. At this point, for example, in a two conductor structure the capacitance will be given by

$$C = \frac{q}{V}$$

with V potential difference between the two conductors, and q calculated by using

$$\oint_c \epsilon \frac{\partial \phi}{\partial n} dc = -q$$

Write as exercise this expression in terms of the discrete values f at the various nodes (cfr. p. 311 Iskander or p.171 Sadiku).

Analysis of TEM (or Quasi-TEM) Transmission lines and Waveguides with FD

Since TEM or quasi-TEM transmission lines obey Laplace's equation in the transverse direction, the procedure to analyze this kind of structure is identical to that explained up to now. Symmetry considerations, special treatment of dielectric boundaries, and capacitance calculation can be performed without basically any difference.

One important application is the calculation of characteristic impedance of microstrip lines. The basic structure is shown in fig. 5.

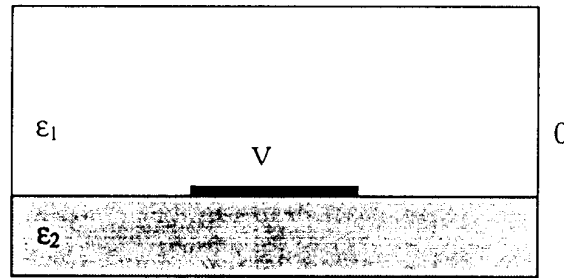


Fig.5. Geometry of a boxed microstrip line.

The problem can be analyzed by using symmetry consideration and proceeding writing a system of equations as explained for electrostatic problems. To calculate the characteristic impedance of the line, we should remember from previous EM classes that

$$Z_0 = \sqrt{\frac{L}{C}}$$

and

$$u = \sqrt{\frac{1}{LC}} .$$

In case of “unloaded” microstrip line, that means a line where no dielectric substrate is present (no ϵ_2), we could write for the characteristic impedance and the phase velocity

$$Z_{00} = \sqrt{\frac{L}{C_0}}$$

and

$$u_0 = \sqrt{\frac{1}{LC_0}} ,$$

with C_0 capacity per unity length without dielectric. Therefore, we can calculate Z_0 as

$$Z_0 = \frac{1}{u_0 \sqrt{C C_0}} ,$$

with u_0 speed of the light in free space ($u_0=3 \cdot 10^8$ m/s).

To calculate Z_0 we can just proceed as follows:

- a. Calculate the capacitance of the structure solving the problem with two dielectrics;
- b. Calculate the capacitance of the structure without substrate dielectric;
- c. Calculate Z_0 using the formula given above.

The analysis of waveguides is slightly more complicate, since for waveguides we should solve Helmholtz equation instead of Laplace equation. For waveguides, the equation to solve is, therefore,

$$\nabla^2 \phi + k^2 \phi = 0 ,$$

where $\phi=E_z$ for TM modes and $\phi=H_z$ for TE modes. From previous EM classes, you should know that, for example, rectangular waveguides do not support TEM modes, that is the mode with electric and magnetic field transverse to the direction of propagation. As a matter of fact, all single conductor structures do not support TEM modes. Structures like the rectangular waveguide support, instead, TE or TM modes. TE modes are those where only the electric field is transverse to the direction of propagation, while TM modes are those where only the magnetic field is transverse to the direction of propagation. These two modes are associated with two indices that indicate the field distribution in the cross-sectional plane of the waveguide. These topics will be covered in detail in advanced electromagnetic classes, and it is not the purpose of this course to go into details. At this time, you should just be aware that the electromagnetic field in a waveguide can be calculated by using first the Helmholtz equation (to calculate E_z or H_z for TM or TE modes respectively), and then relating all the other components to the one calculated (see, for example, Balanis book for explicit formulae that express these relations).

The real problem in these case is the determination of the wave number k^2 given by $k^2=\omega^2\mu\epsilon-\beta^2$, where β is the propagation constant.

In the case of TE modes, the boundary condition to apply at the walls of the waveguide is $\phi=0$ (since the tangential electric field to a metal is 0). Conversely, in the case of TM modes to satisfy the boundary conditions we need to apply to the walls the condition

$$d\phi / dn = 0 .$$

At this point we can solve the problem using the same technique explained for the static case with these new equation and boundary conditions. This means that we have to apply at each node of the discretizing mesh the finite difference version of the Helmholtz equation. The system of equations can be written in the form

$$([A] - (kh)^2 [I]) \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_m \end{bmatrix} = ([A] - \lambda [I]) \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_m \end{bmatrix} = 0 .$$

For this system to have a non-trivial solution, it should be satisfied the condition

$$|[A] - \lambda[I]| = 0 .$$

This condition will provide a polynomial in λ , which can be solved for the various eigenvalues λ . For each of the resulting λ , it is possible to find the corresponding vector ϕ . Alternatively, the problem can be solved by using an iterative method, as explained in Sadiku p. 174.