A New Course on Computational Methods in Electromagnetics

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Abstract—With the proliferation of computers in university campuses, there is a significant need to introduce more courses on "computer methods" in the engineering curriculum. Furthermore, working electromagnetics engineers need a course on the "computational methods in electromagnetics" to better prepare them for the new challenges that face the microwave industry. This paper summarizes our three-year experience with such a new course which we introduced as an elective for seniors. Main features of the new course include: 1) the inclusion of the finite difference method of solving engineering problems formulated in terms of partial differential equations, 2) The use of the finite difference method to solve modern engineering problems such as the design of microstrip transmission lines as well as eigenvalue problems including the propagation characteristics in waveguides of arbitrary cross sections, and 3) the use of the method of moments to solve dynamic fields problems including radiation from linear antennas and scattering from two-dimensional inhomogeneous dielectric objects. Key steps in developing the course will be discussed and results from the various computer programs written by the students will be presented.

I. Introduction

When we examined the literature in search of an engineering text that deals with computer solutions of engineering problems formulated in terms of differential and integral equations, we did not find any. We did not find any that describes numerical solutions of both types of formulations in sufficient details for a one-semester course on the subject. Some mathematical books are available, but none with emphasis on simulation and application to engineering problems. This is also true for books in the electromagnetic fields area where books specializing in the solution of differential equations [1] or integral equations [2], [3] are available. In addition, texts with examples dealing with modern engineering design problems are required. It is also anticipated that some of the students may not be familiar with methods for solving simultaneous systems of equations, tradeoffs between direct and iterative methods of solution, and most importantly, with techniques that may be used to examine the stability and convergence of the obtained results. For this purpose, we developed a new course that deals with "computational methods in electromagnetics" and prepared an extensive set of notes that covers these topics. In addition to covering the introductory material needed to prepare the students for such a course, the notes included many exciting examples of engineering problems formulated in terms of integral and differential equations. Over the last three years, a comprehensive package of computer programs was developed and used by the students to simulate and solve several engineering problems. The course was taught twice at the University of Utah and once at Harvey Mudd College.

This paper describes the structure of the course and presents some of the design examples used to illustrate the solution procedures. Key formulation steps will be emphasized and sample results will be shown.

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II. Preliminaries

To insure smooth flow of the course materials, we decided to start the course with a brief review of methods often used to solve simultaneous sets of equations and procedures to check the stability and convergence of the solution [4]. Many mathematical books may provide reference for such materials and students were familiarized with both direct and iterative methods of solution. The Gauss elimination method may be used as an example of the direct methods, while the Gauss–Seidal procedure may be used to illustrate iterative methods. The fact that iterative methods are particularly attractive and will converge faster for diagonally dominated systems of equations, such as those resulting from the finite difference method, should be emphasized. Otherwise, reasonably accurate initial assumptions of the variables may be required. If such a good initial approximation is not available, direct methods of solution are advisable. Introduction of acceleration factors to speed up the convergence was also described and illustrated by examples. The inclusion of the LU decomposition method was also useful in this introductory section. It is also important that the students be aware and know how to check the stability and the convergence of the numerical solution. Among the procedures that may be implemented for such purposes which do not require considerable classroom time for their introduction are the following:

1) Calculation of the determinant of the coefficient matrix. Unreasonably small values of the determinant may

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give an indication that the system of equations is ill-conditioned.

2) Examination of the sensitivity of the obtained solution to small variation, one to two percent, in some of the elements of the coefficient matrix. Excessive sensitivity of the solution to such small changes may also give an indication that the system of equations is ill-conditioned.

3) Determination of the condition number. Condition number is given by $\|A\| \, \|A^{-1}\|$ where $\|A\|$ denotes the norm of A, and A^{-1} is the inverse of the matrix A. For engineering purposes, the condition number often provides a good indication of the stability of the system of equations. It is, however, time consuming to calculate because it requires the calculation of the inverse of the coefficient matrix A and then the calculations of the norm of A and its inverse. One way to calculate the norm of A is

$$\|A\|_2 = \left[\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \right]^{1/2}$$

where a_{ij} are the elements of the coefficient matrix.

The only remaining topic in this preliminary material is the introduction of the power method to solve eigenvalue problems [5]. This topic may be included in this introductory section, or may be delayed until needed in applying the finite difference method for solving eigenvalue problems. In any case this introductory section should not consume more than two to three weeks of the course. Every effort should be made to avoid making the course have the mathematical class type of flavor and instead the fact that these methods will be utilized to simulate and solve engineering problems should be emphasized.

III. Finite Difference Method

Many engineering problems in electromagnetics may be formulated in terms of partial differential equations. This may include electrostatic problems formulated in terms of Laplace's and Poisson's equations:

$$\nabla^2 \phi = 0 \qquad \nabla^2 \phi = -\frac{\rho}{\epsilon},$$

respectively, and dynamic fields problems formulated in terms of the Helmholtz equation $(\nabla^2 + k^2)\bar{E} = j\omega \bar{J}$ where k is the wave number, and the current source $\bar{J}$ may or may not be zero. In electromagnetic scattering and radiation problems, $(\bar{J} \neq 0)$ and the propagation constant k is known. In waveguide problems, on the other hand, $\bar{J} = 0$, and k is unknown and represents the eigenvalues (characteristic modes) propagating in the waveguide. It is also important to emphasize that although solutions for Laplace's and Poisson's equations would provide the variation of the scalar quantity ϕ (representing electric potential), many quantities of engineering interest such as the characteristic impedance and the velocity of propagation of a microstrip transmission line may be subsequently obtained from ϕ. We found it most efficient to introduce the finite difference method in the following sequence:

1) Derivation of the Difference Equation: This derivation is available in many texts [6] and it will only be briefly outlined here. For the function $\phi(x)$ shown in Fig. 1, the first order derivative $\partial\phi/\partial x$ may be expressed in terms of its discrete values, ϕ_1, ϕ_2, ϕ_3, at $x_0 - h$, x_0, and $x_0 + h$, respectively, by

$$\left.\frac{\partial \phi}{\partial x}\right|_{x0} \approx \frac{\phi_3 - \phi_2}{h} \quad \text{forward difference method}$$

$$\left.\frac{\partial \phi}{\partial s}\right|_{x0} \approx \frac{\phi_2 - \phi_1}{h} \quad \text{backward difference method}$$

and

$$\left.\frac{\partial \phi}{\partial x}\right|_{x0} \approx \frac{\phi_3 - \phi_1}{2h} \quad \text{central difference method.}$$

The central difference method may be interpreted as the average between the forward and backward difference equations. It may be worth emphasizing that based on Taylor series expansion of ϕ at x_0, the error in both the forward and backward methods is on the order of (h), while the error encountered in the central difference method approximation is on the order of (h^2) [6]. Hence, for small values of the incremental change in x (i.e., h), it is more advantageous to use the central difference method. From Fig. 1, and using the central difference method, the following equations may be obtained

$$\left.\frac{\partial \phi}{\partial x}\right|_{x0+h/2} = \frac{\phi_3 - \phi_2}{h}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x0-h/2} = \frac{\phi_2 - \phi_1}{h}$$

$$\rightarrow \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x0} = \frac{\left.\frac{\partial \phi}{\partial x}\right|_{x0+h/2} - \left.\frac{\partial \phi}{\partial x}\right|_{x0-h/2}}{h}$$

$$= \frac{\phi_3 - 2\phi_2 + \phi_1}{h^2}. \tag{1}$$

From (1), and if ϕ is now a function of two independent variables x and y, we obtain using the central difference method

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{h^2} \right) \tag{2}$$

where the various ϕ's are illustrated in Fig. 2. Equation (2) is the five-point equal arm difference equation. The finite difference solution procedure of Poisson's equation may then be summarized as follows:

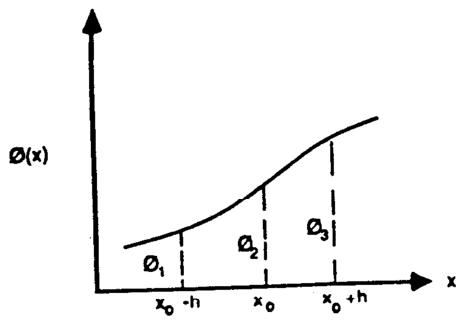


Fig. 1. Geometry utilized in the derivation of the difference equation.

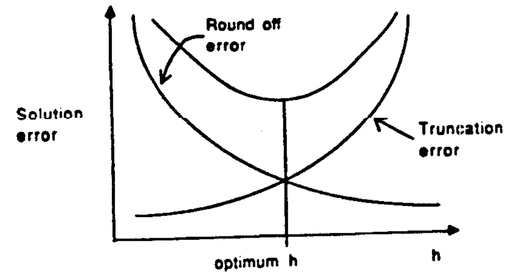


Fig. 3. Tradeoff between truncation and roundoff errors.

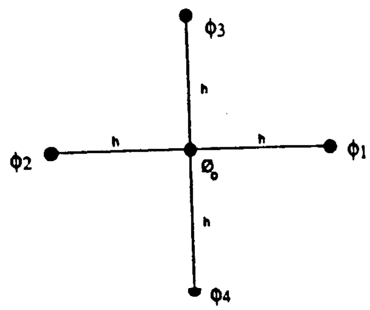


Fig. 2. Geometry illustrating the five-point star used in the two dimensional difference equation.

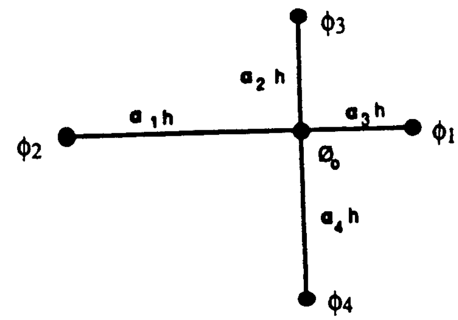


Fig. 4. Geometry of the five-point unequal arm star for the difference equation.

a) Divide the domain of interest into suitably find grid. Instead of the continuous variation of $\phi(x, y)$ for a given charge distribution $\rho(x, y)$, the finite difference solution will provide discrete numbers of ϕ at the "nodes" of the established grid. b) Apply the difference (2) at each node of the grid to obtain N equations in the N unknown node potentials. c) Solve the resulting system of equations either iteratively or using a direct method as described in Section II.

2) Tradeoffs in Choosing the Grid Size h: It is important to point out that although h should be chosen sufficiently small to allow for a most accurate discrete representation of ϕ in the domain of interest, exceedingly small values of h may cause significant roundoff errors. Fig. 3 shows the tradeoff between the truncation (at nodes) of a continuous function ϕ and the roundoff error that increases upon dealing with differences of similar numbers. Hence, in establishing the solution grid, moderate values of h (specific values of h depend on the dimension and geometry of the engineering problem to be solved) should be chosen, and the convergence of the solution should be checked with a change in the grid size.

3) Difference Equation for an Unequal Arm Grid: In certain applications, the boundary of the domain of interest, in which a solution for ϕ is desired, does not fit the regular mesh structure required by the equal arm five-point star formula. In this case, a very fine mesh may be utilized to minimize distortion of the shape of the boundary or a different finite difference formula for a five-point star with unequal arms should be used. For the geometry of Fig. 4, the difference representation of the Laplacian operator is given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{2}{h^2}\left[\frac{\phi_1}{\alpha_1(\alpha_1+\alpha_3)} + \frac{\phi_2}{\alpha_1(\alpha_1+\alpha_3)} + \frac{\phi_3}{\alpha_2(\alpha_2+\alpha_4)} + \frac{\phi_4}{\alpha_4(\alpha_2+\alpha_4)}\right] - \left(\frac{1}{\alpha_1\alpha_3} + \frac{1}{\alpha_2\alpha_4}\right)\phi_0.$$

4) Derivative Boundary Conditions: In our discussion of the solution procedures thus far, we assumed that it is desired to solve Laplace's or Poisson's equation subject to a known value of ϕ on the boundary of the domain of interest. This boundary condition is known as "Dirichlet boundary condition" and is often used in many engineering problems. Referring to Fig. 5, if ϕ is known on the boundary, we apply the difference (2) only at the interior nodes. If some of the points in the equal arm five-point star coincide with the boundary, the left hand point A in Fig. 5, the boundary value of ϕ at that point will be directly substituted in the difference equation in this case. Hence,

$$\frac{\phi_1 + \phi_3 + \phi_{B0}^1 + \phi_{B0}^2 - 4\phi_0}{h^2} = F(x_0, y_0)$$

where $F(x, y)$ is a known distribution evaluated at x_0, y_0, and ϕ_{B0}^1 and ϕ_{B0}^2 are known boundary values of ϕ at two boundaries 1 and 2, respectively. If the value of ϕ on the boundary is not known, on the other hand, and instead its normal derivative is specified, "Neuman boundary condition," the difference equation should then be applied at

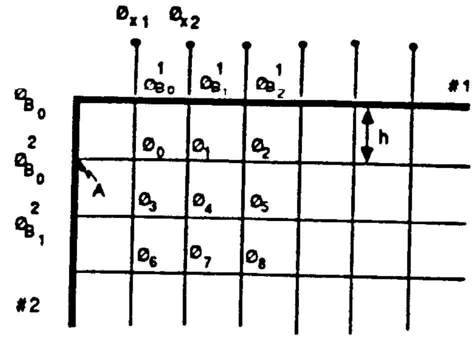


Fig. 5. Accounting for derivative boundary conditions in the finite difference method.

the boundary points to obtain additional equations for the unknown ϕ at the boundary. Let us assume that ϕ_{B0}^1, $\phi_{B1}^1, \cdots$ etc. on boundary 1 are unknown. Applying the difference equations at these points will require values of ϕ above the boundary ϕ_{x1} to complete the five-point equal arm star centered at the boundary point. Hence from (2) we have

$$\frac{\phi_{x1} + \phi_0 + \phi_{B1}^1 + \phi_{B0} - 4\phi_{B0}^1}{h^2} = F(x_0, y_0). \quad (3)$$

In (3), ϕ_{B0}^1 and ϕ_{B1}^1 are unknown values of ϕ on the boundary, $\phi_{B0} = (\phi_{B0}^1 + \phi_{B0}^2)/2$ is the average ϕ on the corner (unknown), and ϕ_{x1} is the value of ϕ on the extended mesh outside the boundary. This extended "fictitious" mesh is utilized to complete the five-point star difference equation for points on the boundary. These new unknown values of ϕ's, ϕ_{x1}, ϕ_{x2}, should not be solved for the additional unknowns, but instead they should be related to values of ϕ inside the boundary by the specified values of the boundary conditions. Hence,

$$\frac{\phi_{x1} - \phi_0}{2h} = \left.\frac{\partial \phi}{\partial x}\right|_{B0}^{1}$$

which is the specified value of the derivative

$$\left.\frac{\partial \phi}{\partial x}\right|_{B0}$$

on boundary 1 (i.e., $\partial\phi/\partial x|_{B0}^1$). If an iterative procedure such as Liebmann's method [1] is being implemented to solve the ϕ distribution, the derivative boundary conditions case may be treated most simply by assigning a new value for

$$\phi_{x1} = \phi_0 + 2h \left.\frac{\partial \phi}{\partial x}\right|_{B0}^{1} \quad (4)$$

upon the calculation of a new value of ϕ_0. In this way, when the iteration reaches the top boundary value ϕ_{B0}^1, the value of ϕ_{x1} on the fictitious mesh point will be ready for implementation in the five-point equal arm star. If Liebmann's method is not used and instead a simultaneous system of equations for the unknown node potentials is developed, the potential ϕ_{x1}, ϕ_{x2}, etc. should not be considered as new unknown quantities and instead should be eliminated by substituting the appropriate value of internal ϕ's and the specified value of the derivative boundary condition such as given in (4). In any case, it is clear that dealing with derivative boundary conditions requires establishing fictitious grids to facilitate routine writing of the difference equation at the boundary nodes. The value of ϕ's at the nodes of the fictitious grid, however, should not be considered additional unknowns and instead should be substituted for through the knowledge of the normal derivative boundary conditions.

5) Difference Equation at an Interface Between Two Dielectric Media: Interfaces between two different media are encountered primarily in engineering electromagnetics applications. This includes the microstrip transmission lines and partially filled waveguides. For this we derive a special case difference equation that should be satisfied by nodes at the interface between the two dielectrics. Fig. 6 illustrates the geometry of the problem, and the difference equation in this case may be obtained from Gauss' law for the electric field

$$\oint_s \epsilon \bar{E} \cdot d\bar{s} = q = 0. \quad (5)$$

$q = 0$ in (5) because there is no "free" charge enclosed by the surface s. Substituting $\bar{E} = -\nabla\phi$ in (5)

$$\oint_c \epsilon \nabla\phi \cdot d\bar{c} = 0$$

or

$$\oint_c \epsilon \frac{\partial \phi}{\partial n} dc = 0 \quad (6)$$

where $\partial\phi/\partial n$ denotes the derivative of ϕ on the contour c. Detailed substitution of $\partial\phi/\partial n$ in (6) yields

$$\epsilon_1 \left(\frac{\phi_3 - \phi_0}{h}\right) h + \epsilon_1 \left(\frac{\phi_2 - \phi_0}{h}\right) \frac{h}{2} + \epsilon_2 \left(\frac{\phi_2 - \phi_0}{h}\right) \frac{h}{2} + \epsilon_2 \left(\frac{\phi_4 - \phi_0}{h}\right) h + \epsilon_2 \left(\frac{\phi_1 - \phi_0}{h}\right) \frac{h}{2} + \epsilon_1 \left(\frac{\phi_1 - \phi_0}{h}\right) \frac{h}{2} = 0.$$

Rearranging the terms we obtain

$$2\phi_3 + 2\epsilon_r\phi_4 + (\epsilon_r + 1)\phi_1 + (\epsilon_r + 1)\phi_2 - 4(\epsilon_r + 1)\phi_0 = 0.$$

Actually, a similar derivation may be utilized to obtain a difference equation at an interface beween anisotropic dielectric surfaces [7]. Interested instructors may include such a derivation in a homework assignment for the students and the results should be verified from [7].

6) Application to Microstrip Transmission Lines: The formulated difference equation with its special cases described in the previous sections may be applied to a wide

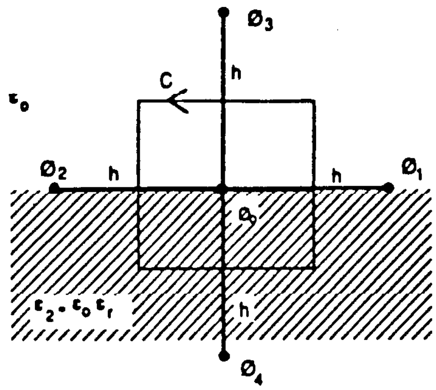
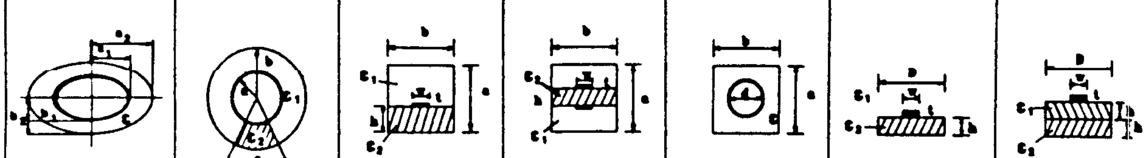


Fig. 6. Geometry of grid nodes at the interface between medium 1 of ϵ_1 and medium 2 of ϵ_2.

TABLE I
CHARACTERISTIC IMPEDANCE OF VARIOUS GEOMETRIES OF STRIP, MICROSTRIP, AND PARTIALLY FILLED TRANSMISSION LINES. THE RESULTS OBTAINED BY THE STUDENTS ARE COMPARED TO THOSE PUBLISHED IN [8]



a = 1.26 a = 2.00 b = 0.76 b = 1.732 ε = ε₀	a = 3.5 b = 8.00 θ = 36 ε₁ = ε₀ ε₂ = 30 ε₀	a = 2.02 b = 7.0 h = 1.00 w = 1.00 t = 0.01 ε₁ = ε₀ ε₂ = 9.6 ε₀	a = 6.00 b = 5.00 h = 1.00 w = 2.00 t = 0.001 ε₁ = ε₀ ε₂ = 2.35 ε₀	a = 1.0 b = 1.25 d = 0.61 ε = ε₀	w = 1.00 t = 0.002 h = 1.00 ε₁ = ε₀ ε₂ = 9.6 ε₀ D = 12.0	w = 1.0 t = 0.02 h = 0.60 h = 0.40 ε₁ = 9.6 ε₀ ε₂ = 2.65 ε₀ D = 12.0
37.74*	46.68*	46.04*	66.02*	60.43*	51.62*	63.06*
37.07**	44.88**	42.78**	62.31**	49.68**	47.41**	58.39**
* (8) Naiheng & Harrington, 1986 **Student's results						

variety of engineering problems. Besides the routine exercises and homework assignments that the instructor might assign for students to practice, we found the application of the method to a general engineering design problem, such as that shown in Table I, to be the most attractive and challenging to students. Table I was actually reproduced from a recent paper [8] where the authors used a variational technique to calculate the characteristic impedance and velocity of propagation in microstrips or partially filled coaxial transmission lines. In this kind of general problem, the student will use difference equations, developed for both equal and unequal arms, at an interface between different dielectrics, and may even utilize derivative boundary conditions if symmetry is considered in the solution. For simplicity, the microstrip structures with open boundaries may be treated with Dirichlet boundary conditions at artificial boundaries placed sufficiently far. Clearly, there is a tradeoff for how far such a boundary should be. Through several trials, students will quickly learn that excessively far boundaries require very large matrices when a mesh of reasonable size is used while very close boundaries are inaccurate unless a reasonable value of ϕ (instead of the assumed zero value) is known on these boundaries (which is not the case). Hence, a sufficiently far boundary to justify the assumption of $\phi = 0$ on it while maintaining a reasonably sized matrix is desired.

The only remaining question to be expected from students is how to relate the calculated values of the node potential to the engineering quantities of interest, such as the characteristic impedance Z_0 and the velocity of propagation V_p in the guiding structures of Table I. Assuming TEM mode of propagation in these structures, these values are given by

$$V_p = \frac{1}{\sqrt{LC}} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

where L and C are the inductance and capacitance per unit length, respectively. If the dielectric loading is assumed to have no effect on the value of L, then

$$Z_0 = \frac{\sqrt{LC_0}}{\sqrt{CC_0}} = \frac{1}{v_0\sqrt{CC_0}}$$

$$V_p = \frac{1}{\sqrt{LC_0}}\sqrt{\frac{C_0}{C}} = v_0\sqrt{\frac{C_0}{C}} \tag{7}$$

where v_0 is the velocity of light in air and C_0 is the capacitance per unit length for air filled transmission line. From (7) it is clear that Z_0 and V_p may be obtained by calculating C_0 (the capacitance per unit length of an air filled transmission line) and C (the capacitance per unit length of partially dielectric filled transmission line). Calculation of the capacitance from the obtained potential distribution may be done through Gauss' law; hence,

$$\oint_s \epsilon \bar{E} \cdot d\bar{s} = q = -\oint_c \epsilon \nabla\phi \cdot d\bar{c}$$

$$= -\oint_c \epsilon \frac{\partial \phi}{\partial n} dc = q \tag{8}$$

where the two-dimensional closed surface s in (8) was replaced by the closed contour c. This results in a charge q in coulombs per unit length. Evaluating (8) by utilizing the discrete node values of ϕ (see Fig. 7) we obtain

$$\epsilon_0\left(\frac{\phi_1 - \phi_3}{2h}\right)h + \epsilon_0\left(\frac{\phi_5 - \phi_6}{2h}\right)h$$
$$+ \epsilon_0\left(\frac{\phi_2 - \phi_4}{2h}\right)\frac{h}{2} + \epsilon_1\left(\frac{\phi_2 - \phi_4}{2h}\right)\frac{h}{2}$$
$$+ \text{contributions from other sides of the contour}$$
$$= -q. \tag{9}$$

Hence,

$$C = \frac{q}{V}$$

where V is the initially assumed potential difference between the center conductor and the ground.

The procedure in (9) is then repeated with the transmission line completely filled with air (i.e., the dielectric is removed) to calculate C_0. Values of C and C_0 are then used to calculate the characteristic impedance Z_0 and the velocity of propagation as given in (7).

The last row in Table I shows some of the results obtained by the students. They were also interested in examining the variation of the characteristic impedance and the velocity of propagation with changes in the transmission line parameters, including the width of the center strip and the thickness and type of substrate. To confirm the accuracy of their computer code, they compared their results with those published by Green [9] as shown in Fig. 8 and Table II. At this point the students were excited,

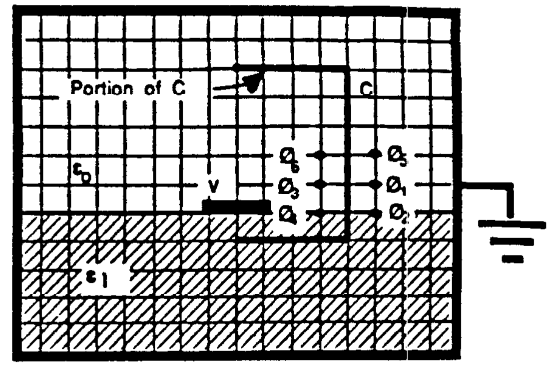


Fig. 7. Geometry of contour c utilized in calculating the capacitance.

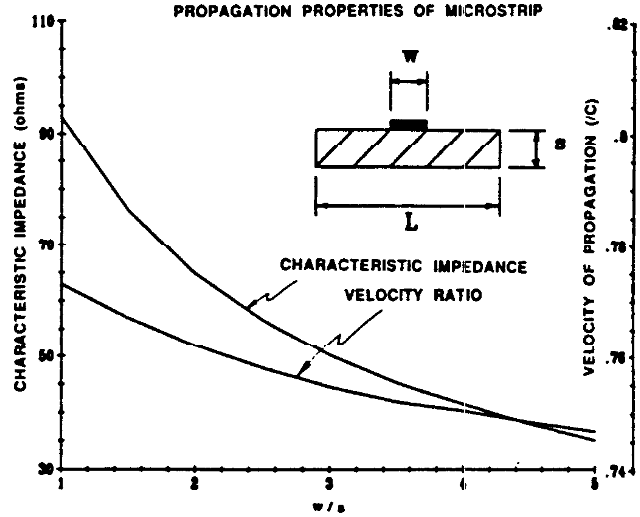
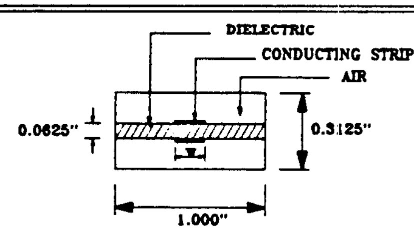


Fig. 8. Characteristic impedance and the velocity of propagation ratio V_p/V_0 of a microstrip line with PTFE dielectric ($\epsilon_r = 2.05$).

TABLE II
CHARACTERISTIC IMPEDANCE Z AND VELOCITY OF PROPAGATION V_r OF STRIP TRANSMISSION LINE. RESULTS FOR VARIOUS DIELECTRIC SUBSTRATES ARE COMPARED WITH THOSE PUBLISHED BY GREEN [9]



STRIP LINE GEOMETRY

Strip Width	Teflon ϵ_r =2.05	Rexolite ϵ_r =2.65	Air
0.2188"	Z =58.75 (58.10)	Z =57.38 (56.72)	Z =61.81 (61.19)
	Vr=0.9506 (0.9506)	Vr=0.9285 (0.9282)	Vr=1.000
0.2813"	Z =50.79 (50.56)	Z =49.74 (49.50)	Z =53.07 (52.92)
	Vr=0.9569 (0.9567)	Vr=0.9372 (0.9366)	Vr=1.000
0.3438"	Z =44.72 (44.76)	Z =43.89 (43.91)	Z =46.50 (46.62)
	Vr=0.9617 (0.9613)	Vr=0.9348 (0.9430)	Vr=1.000

( ) = STUDENT'S RESULTS

first, because of their understanding of the useful finite difference method numerical technique, second, because of their working computer code, and finally, because of the understanding they developed of the microstrip structures commonly used in industry.

7) Application to Eigenvalue Problems: In the previous sections, attention was focused on solving partial differential equations where the scalar function ϕ was the only unknown. In eigenvalue problems, including Helmholtz equations $(\nabla^2 + k^2)\phi = 0$, ϕ is not the only unknown, but instead both k and ϕ are to be determined. For each value of the eigenvalue k_i there is a solution for ϕ_i that represents the corresponding eigenfunction. In waveguide problems, for example, there is a ϕ_i distribution (field configuration of a propagating mode) for each value of the cutoff wave number k_i. In this case, by discretizing the cross section of the waveguide by a suitable square mesh and applying the finite difference representation of the Helmholz equation at each node, we obtain the following matrix equation

$$(A - \lambda I)\Phi = 0 \tag{10}$$

where A is the coefficient matrix that results from applying the difference equation at each node, $\lambda = (4 - h^2k^2)$ is the unknown eigenvalues, and I is an identity matrix. In (10), both the eigenvalue λ and the eigenvector Φ are unknowns and must be determined. There are several ways of determining λ's and the corresponding value of Φ's. The following is a summary of these options:

1) First (10) can be satisfied only if det $[A - \lambda I] = 0$. Hence calculating det $[A - \lambda I] = 0$ will result in a polynomial in λ, which can then be solved for the various eigenvalues λ's. For each of these eigenvalues, the corresponding eigenfunction Φ may be obtained from (10) [10].

2) The second alternative is to use the power method for solving eigenvalue problems. In this iterative method we search for an eigenfunction Φ that satisfies the following equation:

$$A\Phi = \lambda\Phi \tag{11}$$

i.e., when A is multiplied by Φ, the result will be constant multiplied by the same Φ. Hence, the iterative procedure starts by assuming the vector Φ (contains the value of ϕ at the various nodes) and through the repeated multiplications of Φ with A, the solution will converge to a vector that satisfies (11) multiplied by λ^m where m is the number of repeated multiplications needed for the solution to converge. Detailed examination of this method will prove its convergence for an arbitrary choice of the initial assumption of the vector Φ as described elsewhere [5]. The power method, however, provides the eigenvector Φ with the largest eigenvalue λ. Since, in waveguide problems, we are interested in solutions with the smallest eigenvalues, i.e., modes of lowest cutoff frequencies, the power method should be applied on the A^{-1}, the inverse of the matrix A. From (11) we have

$$A^{-1}A\Phi = \lambda A^{-1}\Phi$$

and since $A^{-1}A = I$ we obtain

$$A^{-1}\Phi = \frac{1}{\lambda}\Phi.$$

Hence, applying the power method on A^{-1} results in the Φ solution with largest $1/\lambda$ value which corresponds to the smallest eigenvalue λ.

3) The other available option is to use the double iterative procedure based on Liebmann's method [1]. The iterative procedure starts with assuming a value for the eigenvalue $\lambda = 4 - h^2k^2$. The potential ϕ_{ij}^{k+1} at the (i, j)th node in the $(k + 1)$th iteration is obtained from its known value in the (k)th iteration by

$$\phi_{ij}^{(k+1)} = \phi_{ij}^{(k)} + \frac{\omega R_{ij}}{(4 - h^2k^2)} \tag{12}$$

where ω is the acceleration factor $1 < \omega < 2$ that may be used to speed up the convergence of the solution and R_{ij} is the residual at the (i, j)th node calculated from

$$R_{ij} = \phi_{i,j+1} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i-1,j} - (4 - h^2k^2)\phi_{ij}.$$

After a few iterations using (12) to improve the initial assumption for the ϕ's values, the value of the eigenvalue $\lambda = 4 - h^2k^2$ should be updated using Rayleigh formula [6], [11]:

$$k^2 = \frac{\int\int_s \phi\nabla^2\phi \, ds}{\int\int_s \phi^2 \, ds}. \tag{13}$$

Replacing $\nabla^2\phi$ in (13) by its difference equation and carrying out the integration in (13) using the discrete values of ϕ, we obtain

$$k^2h^2 = \frac{\sum_{i=1}^{N}\sum_{j=1}^{M} \phi_{i,j}(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{ij})}{\sum_{i=1}^{N}\sum_{j=1}^{M} \phi_{i,j}^2} \tag{14}$$

where the summation is carried out over all points in the domain of interest. The iterative procedure involves carrying out (12) for a few iterations and then updating the eigenvalue using (14). The ϕ distribution from (12) should continue until a convergent solution is obtained.

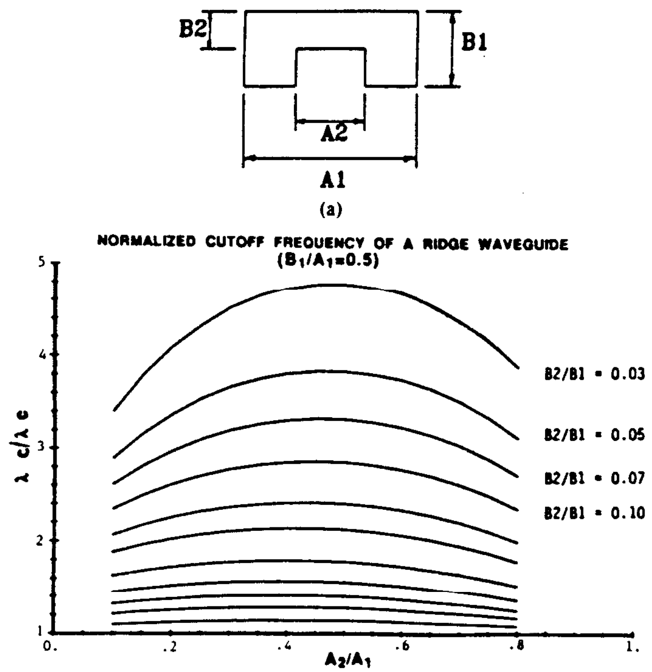


Fig. 9. (a) Schematic illustrating the geometry and the calculation parameters of a single-ridge waveguide. (b) Ratio between cutoff wavelength of the ridged waveguide λ_c' and the same guide without ridge λ_c.

Students were asked to write a computer program to solve Helmholtz's equation for the ridged waveguide shown in Fig. 9(a). To check the accuracy of their results they were asked to examine the variation of the cutoff wavelength of the dominant TE mode with the data published many years ago by Cohn [12]. Some of the obtained results are shown in Fig. 9(b) together with the data from [12]. The reward of this exercise was clearly exciting because students discovered that by changing the shape of the cross section of the waveguide, the cutoff frequency of the dominant mode may be reduced by a factor of 5. In all cases, the axial magnetic field distribution in the waveguide cross section (ϕ in this case) was plotted to illustrate the effect of changing the geometry on the mode configuration in the waveguide. In solving for TE modes, however, students should be warned about the possibility of obtaining a false convergence, the trivial TE_{00} mode, in which ϕ distribution is uniform throughout the cross section of the waveguide. Avoidance of such a possibility may be achieved by imposing other constraints on the variation of ϕ. A typical possibility is to impose the condition $\nabla \cdot B = 0$, ($B = 0$) on the plane of symmetry [1].

This concludes the first section of the course which dealt with the introduction and application of the finite difference method to the solution of engineering problems in electromagnetics.

IV. Method of Moments for Solving Integral Equations

Many engineering problems are formulated and quantified in terms of integral equations. This includes applications in the antenna, signal processing, and the image restoration areas. In addition, several electrostatic problems and microstrip transmission line designs may be formulated alternatively in terms of integral equations. It is, therefore, the objective of this section to introduce the method of moments which provide an important and very commonly used procedure for solving integral equations.

A very attractive approach for introducing this method of solution is to start from the easier to formulate electrostatic problems [13]. In the unique paper on method of moments for undergraduates [13], the authors started by trying to calculate the charge distribution on a cylindrical conductor of radius a and length $2L$. The integral equation for the known potential on the conductor is first formulated and the charge distribution is then determined using the method of moment. Since delta functions (point charges) were used as basic functions to expand the unknown charge distribution (not piecewise constant function as shown in Fig. 1 of [13]) and point matching was used to obtain the desired N equations in N unknowns, singular terms were encountered on the diagonal elements of the resulting $N \times N$ coefficient matrix. The early exposure of students to this kind of problem and the need for a special treatment of the diagonal elements are important aspects of the method of moments. Hence, we followed this introductory material in our course [13] and the reader is referred to this paper [13] for detailed description of this introductory section.

At this point, the students had the feeling that diagonal elements were treated on an ad hoc basis and that they will be required to do so arbitrarily in each problems. To remedy this situation, we continued with the electrostatic problems and as an example we tried to determine the capacitance of a square parallel plate capacitor as a function of the seperation distance between the plates [2], [3]. The point here was that by showing the students that the value of the capacitance may be three times its expected parallel plate value (due to fringing effects) at $d/W = 1$ where d is the separation distance and W is the side length of the square plate, they will be sufficiently motivated to listen carefully to another example illustrating the use of the method of moments. This example is formulated as follows:

For the parallel plate capacitor shown in Fig. 10, the potential at an observation point $\bar{r}$ due to the charge distribution on the two conductors is given by

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_s \frac{\rho(\bar{r}')}{|\bar{r} - \bar{r}'|} \, ds' \tag{15}$$

where s is the area of the two parallel plates. Following the procedure of example 1 of the cylindrical rod [13], we divide each of the parallel plates into small subareas Δs. The resulting number of subareas will then be $2N$ as shown in Fig. 10. These subareas will be chosen sufficiently small so that the charge density ρ may be assumed constant on each. In essence, we expand the unknown charge distribution in terms of piecewise constant, i.e.,

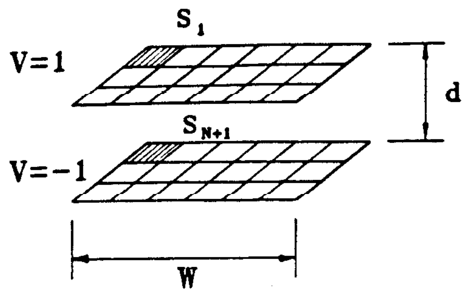


Fig. 10. Schematic illustrating the geometry of the parallel plate capacitor and its subdivision for the method of moments solution.

pulse, expansion functions

$$\rho(\overline{r'}) = \sum_{n=1}^{2N} \alpha_n f_n \qquad f_n = \begin{cases} 1 & \Delta s_n \\ 0 & \text{elsewhere} \end{cases} \tag{16}$$

where Δs_n is the area of the nth section and α_n is the unknown magnitude of each pulse. Substituting (16) into (15) we obtain

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{2N} \alpha_n \int_{\Delta s_n} \frac{ds'}{|\bar{r} - \bar{r}'|}. \tag{17}$$

The integration over s in (15) is replaced by a sum $\Sigma_{n=1}^{2N}$ and each includes an integration over only one section Δs_n. This is because the expansion functions are nonzero only over one subsection as given in (16). If we choose to determine the unknown expansion coefficients by utilizing the $2N$ known values of the potential V at the centers of the $2N$ subsections, we obtain

$$\begin{aligned} 1 &= \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{2N} \alpha_n \int_{\Delta s_n} \frac{ds'}{|\bar{r}_1 - \bar{r}'|} \\ 1 &= \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{2N} \alpha_n \int_{\Delta s_n} \frac{ds'}{|\bar{r}_2 - \bar{r}'|} \\ &\vdots \\ -1 &= \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{2N} \alpha_n \int_{\Delta s_n} \frac{ds'}{|\bar{r}_{2N} - \bar{r}'|} \end{aligned} \tag{18}$$

in (18) $V = 1$ was used at the centers of the N sections on the top plate ($\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_N$) while $V = -1$ was used on the lower plate ($\bar{r}_{N+1}, \cdots, \bar{r}_{2N}$). Equation (18) provides $2N$ equations in the $2N$ unknown expansion coefficients α_n. At this point, a decision should be made on how to handle the integration in each of these equations. The following are the options:

1) Instead of assuming uniformly distributed charge over each subsection Δs_n, we may assume that such a charge is localized $\alpha_n \Delta s_n = \alpha_n'$ at the center of each subsection. In this case, α_n' represents the yet to be determined unknown point charges at the center of the nth subsection. In this case, (18) reduces to

$$\begin{aligned} 1 &= \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{2N} \frac{\alpha_n'}{|\bar{r}_1 - \bar{r}_n'|} \\ &\vdots \\ 1 &= \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{2N} \frac{\alpha_n'}{|\bar{r}_{2N} - \bar{r}_n'|} \end{aligned} \tag{19}$$

As expected from the previous example of the charged cylindrical rod [13], (19) has the problem of blowing up along the diagonal of the coefficient matrix (i.e., self terms involving calculating the potential at the center of a subarea due to its own charge also assumed to be located at the center). To overcome this problem we once again resort to a special treatment of the diagonal elements. This involves approximating each subarea by an equivalent circular one of the same area and use values of the potential equal to that calculated at the center of a circular area [2], [3]. For a circular area of radius a, the potential at a point at its center due to its own uniform charge distribution is given by

$$V_n = \frac{1}{4\pi\epsilon_0} \int_0^a \int_0^{2\pi} \frac{\alpha_n}{\rho}\, d\rho\, d\phi = \frac{\alpha_n}{4\pi\epsilon_0} 2\pi a = \frac{\alpha_n}{2\epsilon_0} \sqrt{\frac{\Delta s}{\pi}} \tag{20}$$

where Δs is the area of the equivalent circular subarea. We may now use (20) in (19) for the diagonal (self) terms in the coefficient matrix.

2) If such an ad hoc treatment of the diagonal elements is not desirable, we may utilize analytical expressions developed for the integral in (18) as described elsewhere [3]. It should be emphasized that by carrying out the integration in (18) we basically utilized piecewise pulse expansion functions and point matching for testing. In other words, in this case there will be no singularity problem with the diagonal terms in the coefficient matrix.

3) The final option available to students is to actually carry out the integration in (18) numerically. Once again, in this case no singularity problems will be encountered in the solution.

With these options in mind, students are now prepared for a formal introduction to the method of moment. Before we do this in the next section, however, tradeoffs between the three options 1), 2), and 3) should be emphasized. Option 1) is clearly the most efficient while option 3) is the least efficient from the computational time point of view. In option 2), we substitute some of the computational effort and time by an analytical one which, depending on the problem, may or may not be possible in all cases. Furthermore, to achieve a specific degree of accuracy, the required size of the matrix ($2N \times 2N$) in option 1) may be relatively larger. These, as well as other tradeoffs, should be mentioned so as to enhance the student's depth of understanding. To practice the basic formulation of the method of moment, students were asked to write a program that utilizes the method of moment to calculate the capacitance of a parallel plate capacitor (including the fringing effects). They were also asked to compare the obtained results for two different expansion functions and/or testing functions. The obtained results from some of the computer codes written by students are shown in Fig. 11 where it is clear that the total capacitance may be three times larger than its parallel plate value with the increase in the separation distance between the plates [2]. In this homework problem, most of the students used the following two choices of the expansion and testing functions:

1) delta expansion and delta testing functions with special treatment of the diagonal elements,

2) piecewise pulse expansion and delta testing functions. The integration over each square subsection was performed either analytically or numerically using one of the known integration methods [3].

The results shown in Fig. 11 agree well with data published in the literature [2], [3]. After this practice homework problem, the students were formally introduced to the method of moments solution procedure. The following section describes the essential steps we used in the formal introduction of the method of moments [2].

1) Integral Equation Formulation: As indicated earlier, the method of moments is a computational procedure which is often used to solve engineering problems formulated in terms of integral equations. The following is a general form of a type of integral equation:

$$\int_{v} U(\bar{r}')\, G(\bar{r}/\bar{r}')\, dv' = f(\bar{r}) \qquad (21)$$

where $U(\bar{r}')$ is the unknown quantity to be determined (e.g., the charge distribution on the plates of the capacitor). $f(\bar{r})$ is a known force function (e.g., the voltage on the plates of the capacitor) and $G(\bar{r}/\bar{r}')$ is a known kernel that relates the unknown function $U(\bar{r}')$ to the known (or measured) function $f(\bar{r})$. The kernel G is often referred to as the Green's function. The derivation of this integral equation is an important part of formulating the engineering problem. Throughout the following examples, bases for deriving these integral equations will be pointed out.

2) Expansion of the Unknown Quantity $U(\bar{r}')$ in Terms of Known Basis Functions: To help us determine $U(\bar{r}')$, we expand it in terms of simpler basis functions with unknown expansion coefficients. Hence,

$$U(\bar{r}') = \sum_{n=1}^{N} \alpha_n f_n \qquad (22)$$

f_n's are the basis functions, α_n's the unknown expansion coefficients. Examples of f_n often used in the literature may include piecewise constant (pulse) basis function, the very simple delta function (point sources), or the triangular basis functions [2], [14]. Substituting $U(\bar{r}')$ in (22)

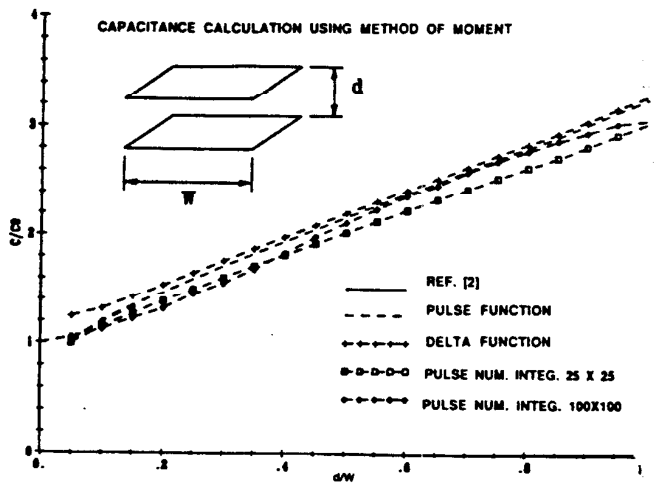


Fig. 11. Normalized capacitance C/C_0 of a parallel plate capacitor as a function of d/W.

we obtain

$$\sum_{n=1}^{N} \alpha_n \int_{\Delta v'} f_n G(\bar{r}/\bar{r}')\, dv' \approx f(\bar{r}). \qquad (23)$$

From (23) it is clear that the basis function must be chosen to accurately represent the source in (22) and at the same time be computationally efficient or analytically possible to integrate in (23). The tradeoff between accuracy of representation and efficiency of computation are the main tradeoffs in choosing the basis functions.

3) Development of the Necessary N Equations to Solve for the N Values of the Unknown Coefficients α_n: In (23), basically, we replaced the unknown function $U(\bar{r}')$ with N unknown expansion coefficients α_n, $n = 1, \cdots, N$. To determine these unknown coefficients, we once again define N different, but known, testing functions T_n, $n = 1, \cdots, N$ and perform the inner product of each of these T_n's with (23). Such a procedure results in N equations, defined as follows:

$$\alpha_1 \int_{\Delta} \int_{\Delta v'} f_1 T_1 G(\bar{r}/\bar{r}')\, dv'\, d\Delta + \cdots + \alpha_n \int_{\Delta} \int_{\Delta v'} f_n T_1 G(\bar{r}/\bar{r}')\, dv'\, d\Delta = \int_{\Delta} T_1 f(\bar{r})\, d\Delta$$

$$\alpha_2 \int_{\Delta} \int_{\Delta v'} f_1 T_2 G(\bar{r}/\bar{r}')\, dv'\, d\Delta + \cdots + \alpha_n \int_{\Delta} \int_{\Delta v'} f_n T_2 G(\bar{r}/\bar{r}')\, dv'\, d\Delta = \int_{\Delta} T_2 f(\bar{r})\, d\Delta$$

$$\vdots \qquad \vdots$$

$$\alpha_n \int_\Delta \int_{\Delta v'} f_1 T_n G(\bar{r}/\bar{r}')\, dv'\, d\Delta$$
$$+ \cdots + \alpha_n \int_\Delta \int_{\Delta v'} f_n T_n G(\bar{r}/\bar{r}')\, dv'\, d\Delta$$
$$= \int_\Delta T_n f(\bar{r})\, d\Delta \tag{24}$$

Each one of the equations in (24) was obtained by performing the inner product of one of the testing functions with (23). The inner product is defined as

$$\langle T_i, f_j \rangle = \int_\Delta T_i \int_{\Delta v'} f_j G(\bar{r}/\bar{r}')\, dv'\, d\Delta \tag{25}$$

where T_i is the testing function and Δ is its domain. From (25) it is clear that by performing such an inner product we were able to obtain the required N equations in the N unknown α_n values. It is often useful to point out the two most popular choices of the testing functions T_n. First, the testing function may be chosen to be the delta function and the procedure is this case is known as the "point matching technique." The other popular testing procedure utilized is when the testing functions are taken to be the same as the bases expansion functions. In this case, the procedure is known as the Galerkin method. It is interesting to students to note that the first choice of the testing function basically involves enforcing (24) at N points (corresponding to the location of the various delta functions) while in the other the obtained solution effectively minimizes the error in enforcing (24) throughout the domain v' in a least square error sense [15]. The following are some examples which we used to illustrate the procedure:

Example 1) Review of Electrostatic Problems: It is very helpful to students that we go back at this point to the two electrostatic examples used before the formal introduction of the method and emphasize the following:

1) In the cylindrical rode problem described in [13], delta function was used as the expansion function as well as the testing function. For this, special treatment was needed for the diagonal elements.

2) In the parallel plate capacitor example, options 2) and 3) pulse basis functions were used for expanding the unknown charge distribution while delta function was used for testing. In option 1), it is clear that delta function was used for both expansion and testing.

Example 2) Linear Wire Antenna: This example is used to illustrate the complete solution procedure using the method of moments, including the derivation of the integral equation. We started with the derivation of Hallen's integral equation [16] to emphasize the fact that $G(\bar{r}/\bar{r}')$ in (21) represents, in this case, the radiation from a building block point-source along the antenna. The total radiation is, hence, obtained by integrating an unknown current distribution $J(z')$ multiplied by $G(\bar{r}/\bar{r}') = Ke^{-jkR}/R$ where K is a proportionality constant over the antenna length. Hence,

$$\int_{-L/2}^{L/2} \frac{J(z')\, e^{-jkR}}{R}\, dz' = -\frac{j}{\eta}(c_1 \cos kz + c_2 \sin k|z|)$$

where $\eta = \sqrt{\mu_0/\epsilon_0}$ and $c_2 = V_0/2$ where V_0 is the applied voltage across the terminals of the antenna. The solution procedure at this stage is straightforward and involves expanding $J(z')$ in terms of known expansion functions (with unknown amplitude coefficients) and use, say, a point matching technique to solve for these unknown coefficients. We found it adequate at this point to point out the possibility of using an entire domain basis function. In this case, the expansion functions are nonzero throughout the entire length of the antenna and may be used because a collection of sinusoidal currents are "known" to provide a "good" guess for the current distribution on the antenna. The results from a computer program written by the students are shown in Fig. 12. The input impedance calculations are obtained from

$$Z_{\text{in}} = \frac{V_0}{I_{\text{in}}}$$

where V_0 is assumed to be 1 V at the input port of the antenna and I_{in} is the value of the current evaluated at the input port ($z = 0$). The input impedance calculations compare favorably with Hallen's curves available in the literature [17].

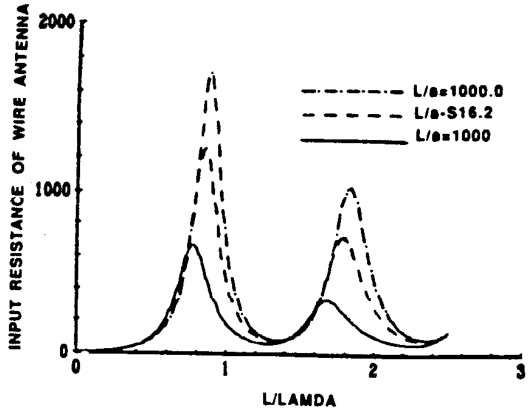
Example 3) Calculation of Two-Dimensional Scattering (TM Case): In formulating this problem, we once again utilized the two-dimensional Green's functions which basically relates the radiation from a line source to the source intensity. Without completely solving the wave equation in the cylindrical coordinate system, we proposed a solution in the form $\Psi = (I/4j)H_0^{(2)}(k\rho)$ where $H_0^{(2)}$ is the Hankel function of second kind and zeroth order [18]. We emphasized the reason for choosing the second kind because its asymptotic value for large argument $k\rho \to \infty$ together with the $e^{j\omega t}$ time harmonic dependance, assumed throughout this paper, has the correct behavior of an out going wave. It was also indicated that if the line source is located at $\bar{\rho}'$ and the observation point is located at $\bar{\rho}$ (see Fig. 13), the form of the solution would be

$$\Psi = \frac{I}{4j} H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|) \tag{26}$$

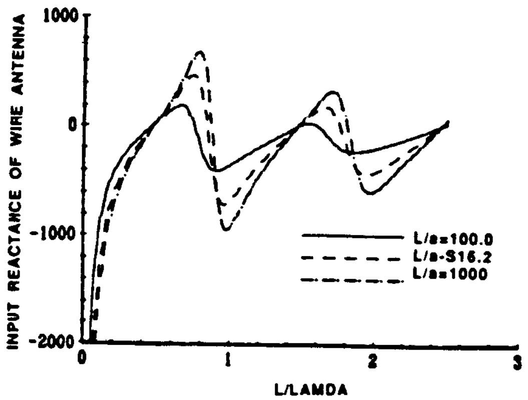
where $|\bar{\rho} - \bar{\rho}'|$ is the distance between the source and the observation point. Utilizing (26) and keeping in mind that the scattered fields are basically generated by an equivalent current distribution given by $\bar{J}_{eq}$ distributed throughout the dielectric scatterer we obtain

$$\bar{E}^s = -\frac{k\eta}{4} \int_s \bar{J}_{eq} H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|)\, ds'.$$

In other words, once again Green's function $G(\bar{r}/\bar{r}') = H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|)$ was considered to be the building block and, hence, multiplied by a current distribution $\bar{J}_{eq}$ and



(a)



(b)

Fig. 12. Input impedance of linear wire antenna. L is the half length, and a is the radius.

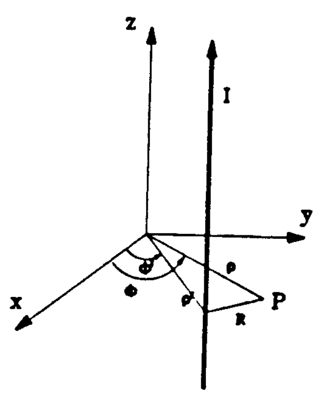


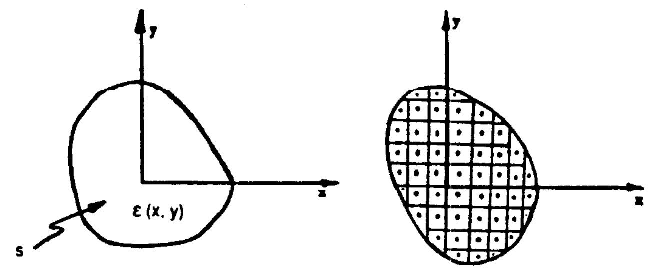
Fig. 13. Schematic illustrating the cylindrical coordinates of a line source located at $\bar\rho'(\rho', \phi')$ and an observation point P at $\bar\rho(\rho, \phi)$.

integrated over the cross section of the dielectric object. The total electric field $\bar E$ at any point in space is given by

$$\bar E = \bar E^s + \bar E^i$$
$$= \bar E^i - \frac{k\eta}{4}\int_s \bar J_{eq} H_0^{(2)}(k|\bar\rho - \bar\rho'|)\, ds'.$$

Substituting $\bar J_{eq}$ in terms of the total electric field

$$\bar J_{eq} = j\omega(\epsilon - \epsilon_0)\bar E$$



(a) (b)

Fig. 14. (a) Cross section of dielectric cylinder. (b) Its digitization into N small (approximately square) subareas.

we obtain for TM case

$$E_z = E_z^i - \frac{k\eta}{4}\int_s j\omega(\epsilon - \epsilon_0) E_z H_0^{(2)}(k|\bar\rho - \bar\rho'|)\, ds. \tag{27}$$

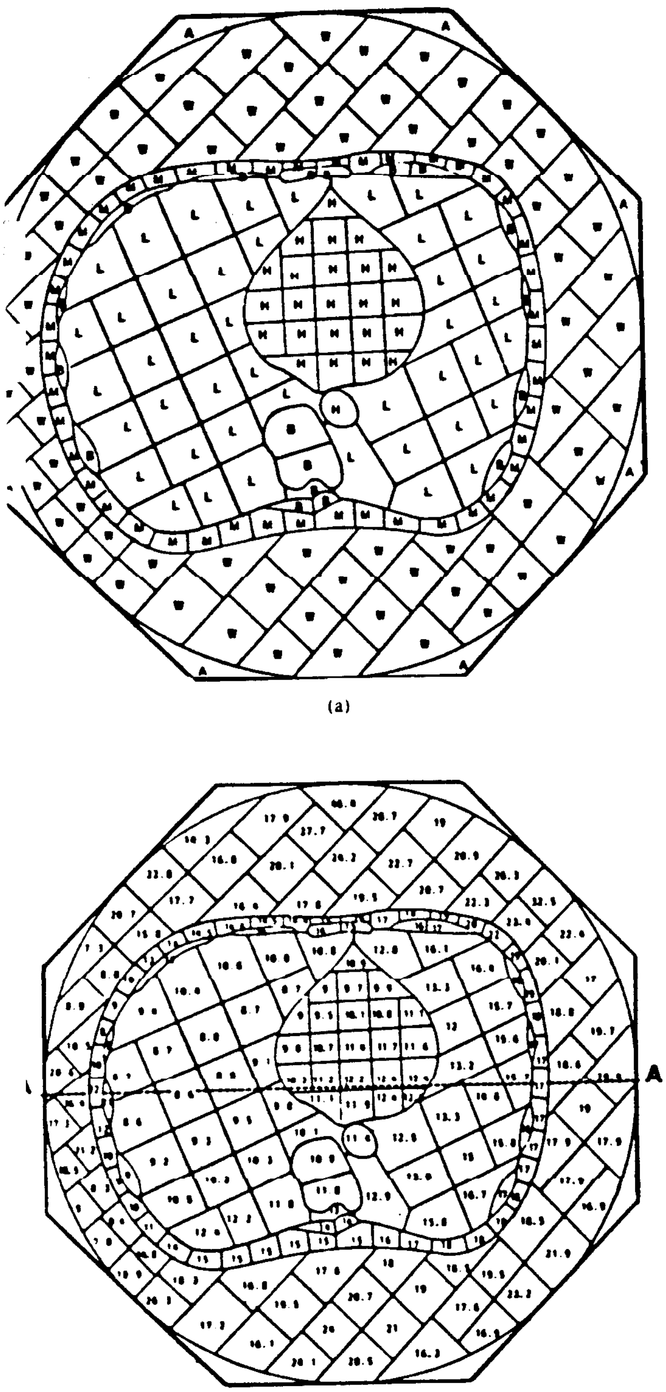
Equation (27) is the required integral equation for the total field E_z. The solution procedure involves expanding E_z in terms of the piecewise pulse basis function and using the point-matching technique to determine the unknown expansions coefficients. This is equivalent to dividing the dielectric scatterer of interest into N small mathematical cells (see Fig. 14) and assuming the electric field E_z to be constant within each. The amplitude distribution of E_z in the cross section is obtained using the point matching. Utilizing the pulse basis functions and the point-matching technique, (27) reduces to

$$E_{zm}^i = E_{zm} - \frac{jk}{4}\sum_{n=1}^{N}(\epsilon_n - 1)E_{zn} \cdot \iint_{\text{cell}} H_0^{(2)}(k\rho_m)\, dx'\, dy' \qquad m = 1, \cdots, N \tag{28}$$

where ϵ_n and E_{zn} are the relative complex permittivity and the electric field, respectively, in the nth cell. E_{zm}^i and E_{zm} are the incident and total electric fields in the mth cell, ρ_m is represented by

$$\rho_m = \sqrt{(x' - x_m)^2 + (y' - y_m)^2}.$$

Students were asked to run an already developed computer program to calculate the electric field distribution within a dielectric scatterer [19], [20]. They first checked their ability to run the program by reproducing the results published by Richmond [21]. They were then able to reproduce some of the more interesting results that may involve the calculation of the electromagnetic power deposition in the human body. Clearly, analytical calculations of such results is not possible and because of the practical importance of the problem in hand, the students had certainly a great deal of appreciation of the usefulness of numerical techniques. Some of the results reproduced by the students using a TM excitations computer program [19] are shown in Fig. 15.



(a)

(b)

Fig. 15. (a) Cross section of the human body exposed to the radiation fields of an eight-source annular phase array [19]. The body is surrounded by a water bolus W and the entire cross section (i.e., the body and the bolus) is divided into 201 mathematical cells. The symbols indicate the different tissue types. L is lung, H is heart, M for muscle, and B for bone. (b) Magnitude of the electric fields distribution in the cross section of the human body at 10 MHz.

TABLE III

COURSE CONTENTS AND NUMBER OF LECTURES SPENT ON EACH TOPIC

Subject	Number of Lectures
1. Preliminaries:	
a. Solution of a linear system of equations (direct, iterative, and semi-iterative methods).	4
b. Discussion of methods to check stability of the system of equations and the convergence of obtained results.	2
2. Finite Difference:	
a. Difference equation, unequal arm grid, derivative boundary conditions.	3
b. Examples (Iterative and direct methods for solving Laplace's and Poisson's equations).	2
c. Solution of propagation characteristics of strip and microstrip transmission lines (derivation of difference equation characteristic parameters from calculated potential at dielectric interface and application of Gauss' law).	3
3. First midterm examination.	Take Home
4. Finite Difference (continued)	
d. Propagation characteristics in waveguides, resolution of eigenvalue problems, power methods.	3
e. Solution of eigenvalue problems, double iterative Libmann's method.	1
5. Method of moments.	
a. Solution of charge distribution on a conducting rod, procedures for dealing with diagonal elements.	2
b. Formal introduction of the method of moments, example of parallel plate capacitor.	2
6. Second midterm examination	Take home
7. Method of moments (continued).	
c. Radiation characteristics of linear wire antennas, derivations and use of the method of moments solution.	3
d. Calculation of two-dimensional scattering, derivation of TM case, method of moments solution, description of computer program.	3
8. Final Examination	Take home
9. Introduction of Finite Element Method.	2

V. CONCLUDING REMARKS

In this paper, we summarized our experience with the introduction of a new course on computational methods in electromagnetics for seniors and first-year graduate students. The course combines available techniques for solving both partial differential and integral equations. Table III describes the course content and the number of lectures spent on each topic. Examples from a large variety of practical applications were utilized throughout the course. Problems in microstrip transmission lines, waveguides of arbitrary cross sections, wire antennas, and two-dimensional TM scattering were included as examples. Table IV includes a list of the computer programs used in this course. Copies of these programs are available and may be furnished upon request from the authors. Although there exist other numerical techniques which are often used to solve electromagnetics problems, efforts were focused in this course on the detailed discussion of the finite difference and the method of moments. This provides examples of solutions to engineering problems formulated in terms of differential and integral equations. Other techniques, such as the finite element method, were briefly outlined at the end of the course. In this regard, emphases were placed on derivation of variational expressions from differential equations describing the engineering problem, and also on the finite element procedure of converting the variational expression to a simultaneous system of linear equations.

TABLE IV
LIST OF COMPUTER PROGRAMS USED IN THE COURSE

Subject	Program
1. Solution of N-linear equations	a. Gauss elimination method. b. Gauss-Seidel iteration method. c. LU decomposition method. d. Conjugate gradient semi-iterative method. e. Power method to solve eigenvalue problems.
2. Finite difference method.	a. Electric potential distribution for a conducting strip placed in a square duct. i. Using the relaxation method. ii. Using Liebmann's overrelaxation method. b. Solution for the propagation characteristics of strip and microstrip transmission lines. c. Solution for cut-off frequencies and field distribution in waveguides (ridged waveguides). i. TM modes. ii. TE modes.
3. Method of moments	a. Calculation of capacitance of a rectangular parallel plate capacitor. b. Calculation of charge distribution on a conducting strip of given electric potential distribution. c. Radiation characteristics of wire antennas (Hallen's equation). d. Calculation of two-dimensional scattering by inhomogeneous cylindrical structures.

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