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# 8

# Numerical Integration

## 8.1 Introduction

Frequently, many engineering problems require the evaluation of the integral

$$I = \int_a^b f(x)\,dx, \qquad (8.1)$$

where the function $f(x)$ is called the integrand and $a$ and $b$ are called the limits of integration. If the function $f(x)$ is continuous, finite, and well behaved over the range of integration $a \le x \le b$, the integral ($I$) can be evaluated using the available mathematical techniques. If $f(x)$ denotes a simple function such as a polynomial, an exponential, or a trigonometric function, the integrals are well known from calculus. If $f(x)$ involves more complicated functions, often, standard tables of integrals can be used to evaluate the integral ($I$) in closed form. The analytical or closed form expressions for the integrals, if available, are very valuable, since they are exact and no error is involved in their evaluation. In addition, the influence of changing some physical parameter of the engineering problem on the integral can be studied easily. Finally, the closed form

### CHAPTER CONTENTS

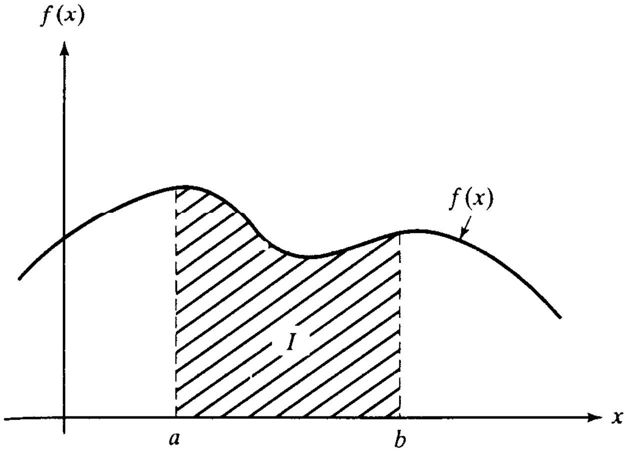


**Figure 8.1** Integral as area under the curve.

expressions of the integral ($I$) can be used to verify the accuracy of numerical integration.

On the other hand, the function $f(x)$ may be a complicated continuous function that is difficult or impossible to integrate in closed form. In some cases, $f(x)$ may not be known in analytical form; it may be known only in a tabular form, where the values of $x$ and $f(x)$ are available at a number of discrete points in the interval $a$ to $b$ (may be, from an experimental study). The limits of integration may be infinite or the function $f(x)$ may be discontinuous or may become infinite at some point in the interval $a$ to $b$. In all these cases, the integral ($I$) can be evaluated only numerically.

The integral of a function $f(x)$ between the limits $a$ and $b$ basically denotes the area under the curve of $f(x)$ between $a$ and $b$ as shown in Fig. 8.1. Integration is also known as quadrature. A simple, intuitive approach to evaluate the integral in Eq. (8.1) is to plot the function $f(x)$ on a grid or graph paper and count the number of boxes or rectangles that approximate the area under the curve of $f(x)$. (See Fig. 8.2.) The product of the number of boxes and the area of each box gives

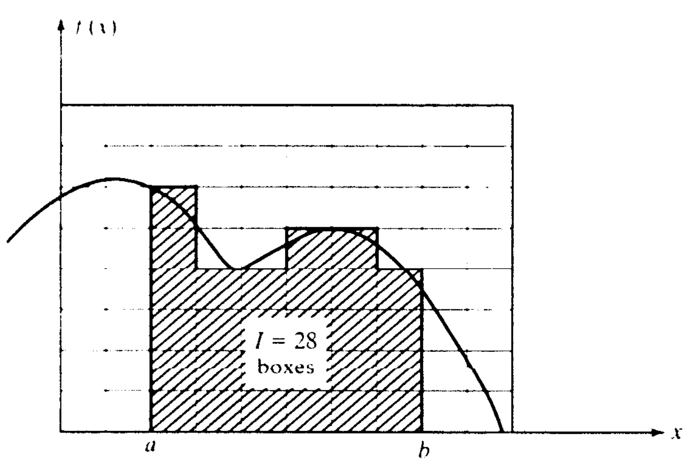


**Figure 8.2** Evaluation of an integral using a grid or graph paper.

an estimate of the total area under the curve (i.e., the integral, $I$). This estimate can be refined, if necessary, by using a finer grid. However, the method used is very impractical and inaccurate in many cases.

## 8.2 Engineering Applications

### ▶Example 8.1

A semiinfinite solid body, initially at temperature $T_i$, is suddenly exposed to a fluid at temperature $T_0$ at the face $x = 0$ as shown in Fig. 8.3. If the diffusivity of the material ($\alpha$) is constant, the unsteady state temperature distribution in the body, $T(x, t)$, is governed by the equation

$$\frac{d^2\theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0 \tag{a}$$

subject to

$$\theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{b}$$

and

$$\theta(0) = 1, \tag{c}$$

where

$$\theta = \frac{T - T_i}{T_0 - T_i} \tag{d}$$

and

$$\eta = \frac{x}{2\sqrt{\alpha t}}. \tag{e}$$

Determine the temperature distribution in the body.

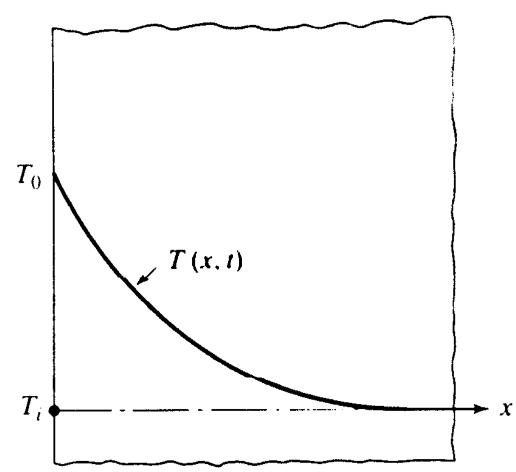


**Figure 8.3** Semi-infinite solid.

## Solution

The solution of Eq. (a) is given by

$$\theta(\eta) = c_1 + c_2 \int e^{-\eta^2} d\eta, \tag{f}$$

where the constants $c_1$ and $c_2$ can be evaluated, using Eqs. (b) and (c), as $c_1 = 1$ and $c_2 = -\frac{2}{\sqrt{\pi}}$. Thus, the solution becomes

$$\theta(\eta) = \frac{T - T_i}{T_0 - T_i} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-z^2} dz. \tag{g}$$

The right-hand side of Eq. (g) is known as the complementary error function and the integral of Eq. (g) cannot be evaluated in closed form. ◀

## ▶Example 8.2

The axial displacement ($du$) of an elemental length ($dx$) of the bar, shown in Fig. 8.4, under a load $P$ is given by

$$\frac{du}{dx} = \frac{\sigma}{E} = \frac{P}{EA}, \tag{a}$$

where $E$ is Young's modulus and $A$ is the cross-sectional area. Determine the axial displacement of the bar for the following data: $P = 5000$ lb, $l = 10$ in, $E = 30 \times 10^6(1 - 0.01x - 0.0005x^2)$ psi, and $A = A_0 e^{-0.1x} = 2e^{-0.1x}$ in$^2$.

## Solution

The axial displacement of the bar at $x = l$ can be determined by integrating Eq. (a) as

$$u = \int du = \int_0^l \frac{P}{EA} dx, \tag{b}$$

where the integral can be conveniently evaluated using a numerical integration procedure. ◀

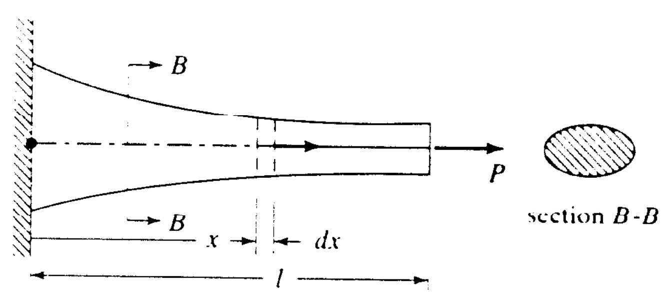


**Figure 8.4** Nonuniform bar under axial load.

## ▶Example 8.3

The turning moment developed at various positions of the crank from the inner dead center in a multicylinder internal combustion engine is shown in Fig. 8.5. If the speed of the engine is 1,500 rpm, determine the power developed.

## Solution

The power developed per cycle (one revolution of the crank) can be found as the area under the turning-moment–crank-angle diagram. The integral of the function $f(x)$ can be evaluated analytically as follows:

$$I = \int_0^{1.5} f(x)\,dx = 5.5606613. \tag{a}$$

Thus, the area under the turning moment diagram per one cycle of the engine is given by

$$5.5606613\ (100)\left(\frac{4\pi}{3}\right) = 2{,}329.249805 \text{ lb-ft/rev.}$$

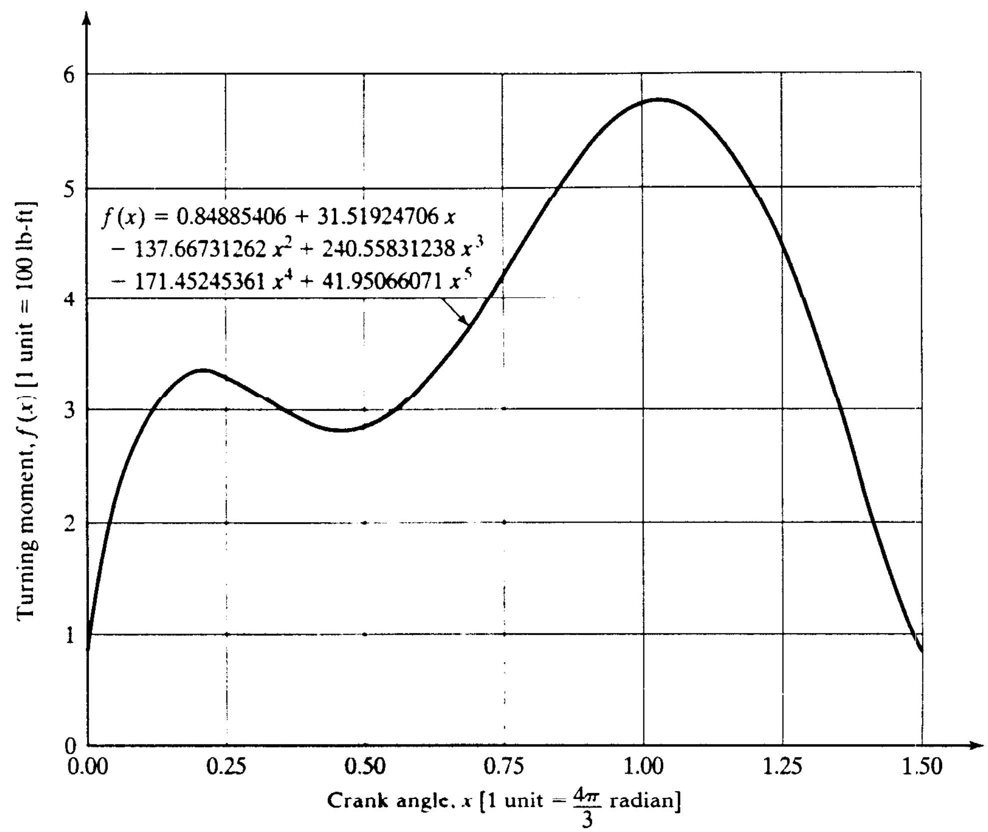


**Figure 8.5** Turning moment diagram.

The power developed by the engine ($P$) can be determined by multiplying the power per cycle and the speed of the engine. This gives

$$P = 2329.249805 \times 1500 = 3.4938747 \times 10^6 \text{ lb-ft/min}$$
$$= \frac{3.4938747 \times 10^6}{33,000} = 105.874991 \text{ hp}$$

◀

## 8.3 Newton–Cotes Formulas

The Newton–Cotes formulas are the most commonly used numerical integration methods. They are based on replacing a complicated function or tabular data by some approximating function that can be integrated easily; that is,

$$I = \int_a^b f(x)\,dx \approx \int_a^b p_m(x)\,dx, \tag{8.2}$$

where $p_m(x)$ is the approximating function, usually taken as an $m$th-degree polynomial, viz.,

$$p_m(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0, \tag{8.3}$$

where the coefficients of the polynomial (constants) $a_m, a_{m-1}, \ldots a_1, a_0$ are determined such hat $f(x)$ and $p_m(x)$ have the same values at a finite number of points. (See Section 5.3.) Figure 8.6 shows the approximation of $f(x)$ using three of the simplest polynomials, namely, a constant, a straight line, and a parabola.

### 8.3.1 Rectangular Rule

The function or data of $f(x)$ can also be approximated using a series of piecewise polynomials shown in Fig. 8.7. In this approach, the range of integration $a \le x \le b$ is first divided into a finite number ($n$) of intervals or strips such that the width of each interval is given by

$$h = \Delta x = \frac{b-a}{n}. \tag{8.4}$$

The discrete points in the range of integration are then defined as $x_0 = a, x_1, x_2, \ldots,$ $x_{n-1}$, and $x_n = b$ with

$$x_i = a + ih; i = 0, 1, 2, \ldots, n. \tag{8.5}$$

The values of the function $f(x)$ at the discrete point $x_i$ is assumed to be known as $f_i (i = 0, 1, 2, \ldots, n)$. As shown in Fig. 8.7(a), the simplest approximation to the function $f(x)$ is a piecewise polynomial of order zero (i.e., a series of constants) Clearly, from Fig. 8.7(a), the function $f(x)$ can be approximated over the interval $x_i \le x \le x_{i+1}$ either by the value of $f_i$ or $f_{i+1}$. If the values of $f_i$ are used (i.e., $f(x)$ is approximated by its values at the beginning of each interval), the area under the curve $f(x)$ in the interval $x_i \le x \le x_{i+1}$ is taken as $(f_i h)$ and hence the integral (I)

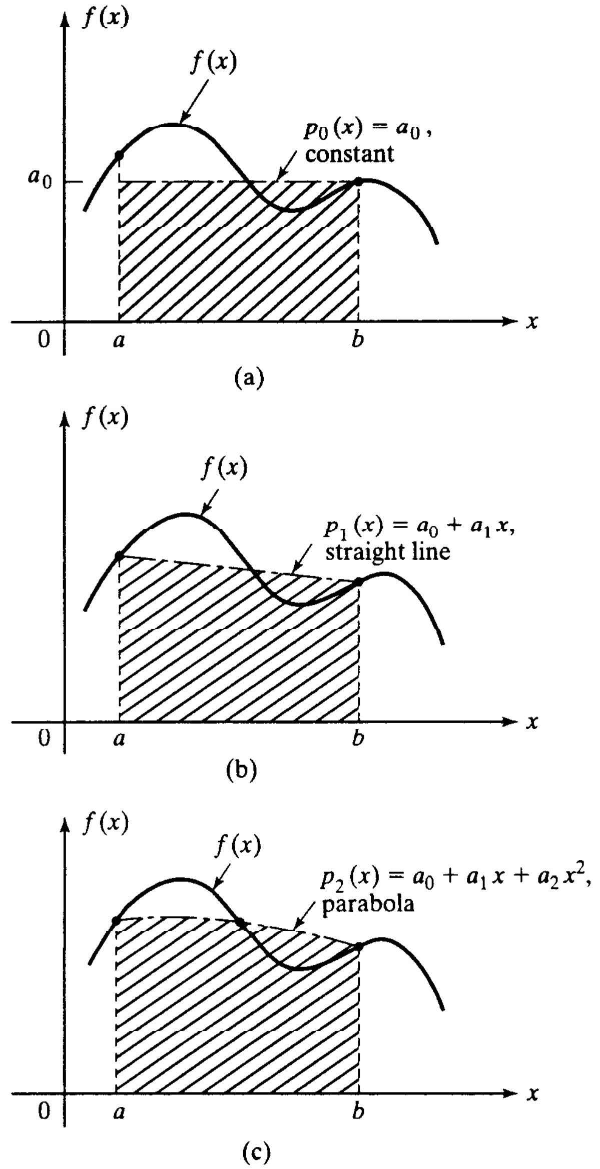


**Figure 8.6** Different types of approximation of $f(x)$.

is evaluated as

$$I = \int_a^b f(x)\,dx \approx h\left(\sum_{i=0}^{n-1} f_i\right) \tag{8.6}$$

On the other hand, if the values of $f_{i+1}$ are used (i.e., $f(x)$ is approximated by its values at the end of each interval), the area under the curve $f(x)$ in the interval $x_i \le x \le x_{i+1}$ is taken as $(f_{i+1}h)$ and hence the integral (I) is evaluated as

$$I = \int_a^b f(x)\,dx \approx h\left(\sum_{i=0}^{n-1} f_{i+1}\right) \equiv h\left(\sum_{i=1}^{n} f_i\right). \tag{8.7}$$

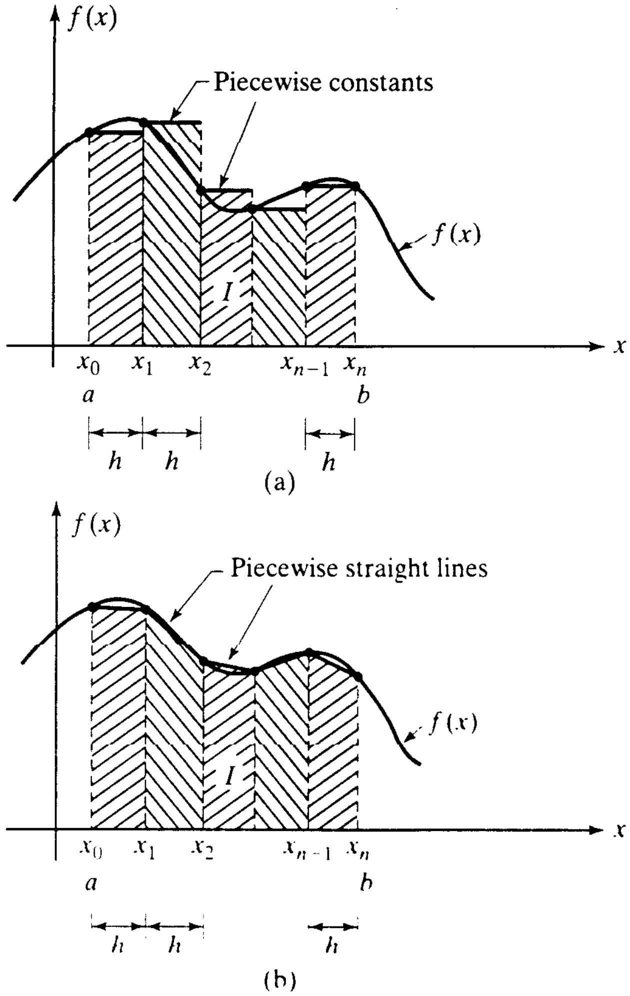


**Figure 8.7** Approximation of $f(x)$ by piecewise polynomials of degree 0 and 1.

For a monotonically increasing function, Eq. (8.6) underestimates and Eq. (8.7) overestimates the actual value of the integral. (See Fig. 8.8.) On the other hand, for a monotonically decreasing function, Eq. (8.6) overestimates and Eq. (8.7) underestimates the true value of the integral. In practice, the rectangular rule leads to large truncation errors for general nonlinear functions $f(x)$ and, hence, is not commonly used. However, the method serves to illustrate the basic concepts used in numerical integration and Newton–Cotes formulas. An improvement in accuracy of the piecewise-constant approximation (rectangular rule) can be achieved by using the average value of $f_i$ and $f_{i+1}$ in the interval $x_i \le x \le x_{i+1}$ as shown in Fig. 8.9. In this case, the integral ($I$) is evaluated as

$$I = \int_a^b f(x)\,dx \approx h \sum_{i=0}^{n-1} \left( \frac{f_i + f_{i+1}}{2} \right). \tag{8.8}$$

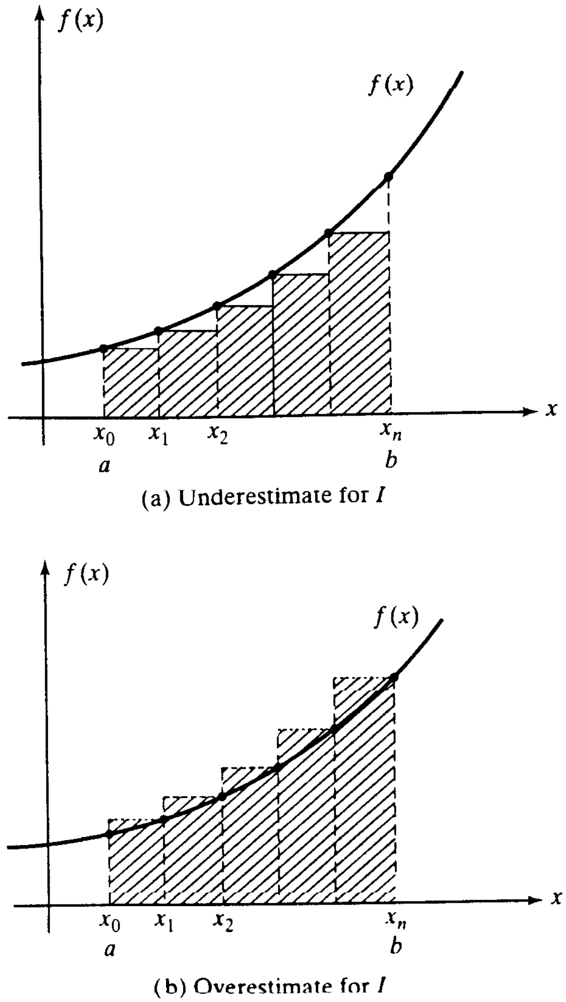


**Figure 8.8** Under- and over-estimation of $I$.

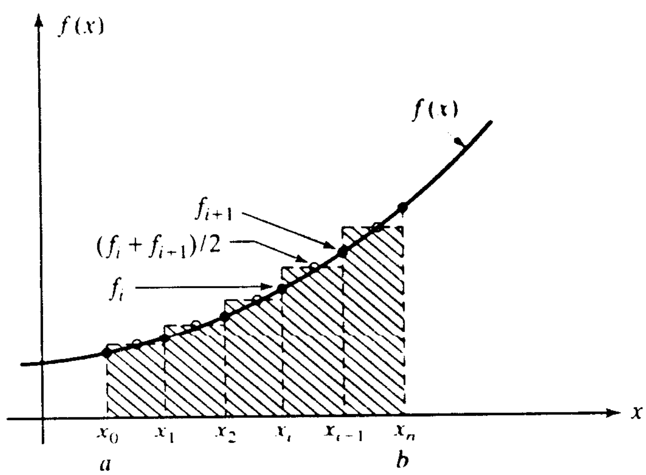


**Figure 8.9** Approximation of $f(x)$ by $(f_i + f_{i+1})/2$ in $x_i \leq x \leq x_{i+1}$.

## 8.3.2 Trapezoidal Rule

The trapezoidal rule is extensively used in engineering applications because of its simplicity in developing a computer program. The method corresponds to the approximation of $f(x)$ by piecewise polynomials of order one $[p_1(x) = c_1x + c_0]$, that is, by straight-line segments as shown in Fig. 8.7(b). In this case, the area under the curve $f(x)$ in the interval $x_i \le x \le x_{i+1}$ is equal to the area of the trapezoid; hence the name trapezoidal rule. Denoting the areas of the trapezoids as $I_1, I_2, \ldots, I_n$, we have (Fig. 8.10)

$$I_1 = \left(\frac{f_0 + f_1}{2}\right)h,\ I_2 = \left(\frac{f_1 + f_2}{2}\right)h, \ldots,$$

$$I_i = \left(\frac{f_{i-1} + f_i}{2}\right)h, \ldots, \text{ and } I_n = \left(\frac{f_{n-1} + f_n}{2}\right)h. \tag{8.9}$$

The integral can be evaluated as

$$I = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} I_i = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right). \tag{8.10}$$

## 8.3.3 Truncation Error in Trapezoidal Rule

The basic truncation error $(E)$ of the trapezoidal rule is given by

$$E = \int_a^b f(x)\,dx - \left[\frac{f(a) + f(b)}{2}\right](b - a), \tag{8.11}$$

where the first term on the right-hand side of Eq. (8.11) denotes the exact integral and the second term represents the approximate integral given by the trapezoidal rule. Note that only one segment is considered in the interval for simplicity. (See Fig. 8.11.) To derive a more convenient expression for the error, we use Taylor's

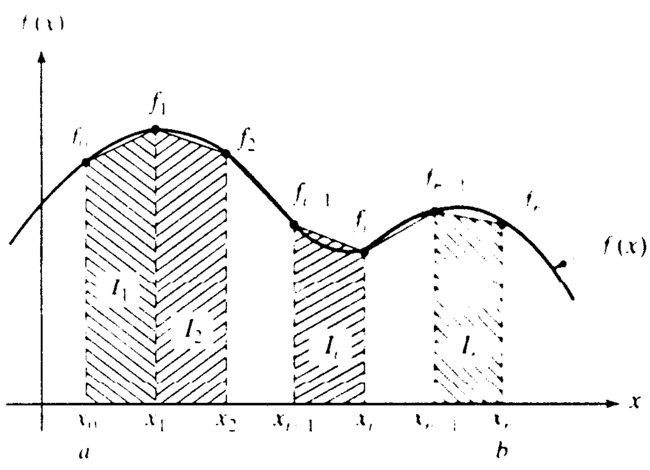


**Figure 8.10** Trapezoidal rule.

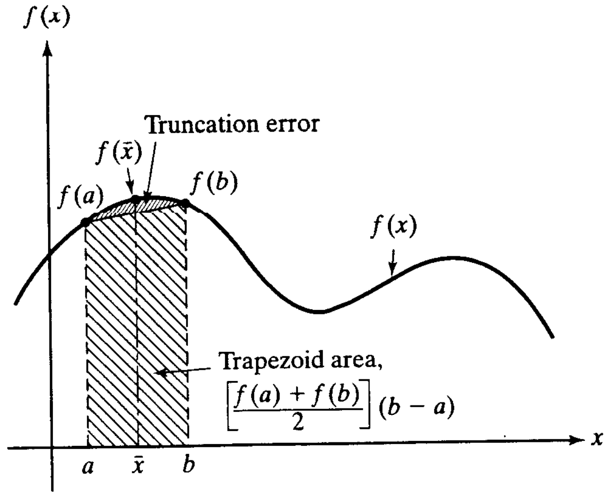


**Figure 8.11** Truncation error.

series expansion of $f(x)$ about the midpoint of the range, $\bar{x} = \frac{a+b}{2}$:

$$f(x) = f(\bar{x}) + yf'(\bar{x}) + \frac{y^2}{2!}f''(\bar{x}) + \cdots. \tag{8.12}$$

Here $y = x - \bar{x}$, a prime indicates a derivative, and the function $f(x)$ is assumed to be analytical in the interval $a \le x \le b$. Equation (8.12) can be used to express

$$\int_a^b f(x)\,dx = \int_{-h/2}^{h/2}\left\{f(\bar{x}) + yf'(\bar{x}) + \frac{y^2}{2!}f''(\bar{x}) + \cdots\right\}dy, \tag{8.13}$$

where $y = -h/2$ and $y = +h/2$ can be seen to correspond to $x = a$ and $x = b$, respectively. By carrying out the integration in Eq. (8.13), we obtain

$$\int_a^b f(x)\,dx = f(\bar{x})\,(y)\Big|_{-h/2}^{h/2} + f'(\bar{x})\left(\frac{y^2}{2}\right)\Bigg|_{-h/2}^{h/2} + \frac{1}{2}f''(\bar{x})\left(\frac{y^3}{3}\right)\Bigg|_{-h/2}^{h/2} + \cdots$$

$$= hf(\bar{x}) + \frac{1}{24}h^3 f''(\bar{x}) + \cdots. \tag{8.14}$$

Substituting $x = a$ and $x = b$ into Eq. (8.12) yields

$$f(a) = f(\bar{x}) - \frac{h}{2}f'(\bar{x}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 f''(\bar{x}) - \cdots; \tag{8.15}$$

$$f(b) = f(\bar{x}) + \frac{h}{2}f'(\bar{x}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 f''(\bar{x}) + \cdots, \tag{8.16}$$

where the values of $y$ at $x = a$ and $x = b$ are taken as $x - \bar{x} = a - \bar{x} = -\frac{h}{2}$ and $x - \bar{x} = b - \bar{x} = +\frac{h}{2}$, respectively. Noting that $(b - a) = h$, the second term on the right-hand side of Eq. (8.11) can be expressed as

$$(b-a)\left[\frac{f(a)+f(b)}{2}\right]=\frac{h}{2}\left[f(\bar{x})-\frac{h}{2}f'(\bar{x})+\frac{1}{8}h^2f''(\bar{x})-\cdots+f(\bar{x})+\frac{h}{2}f'(\bar{x})\right.$$
$$\left.+\frac{1}{8}h^2f''(\bar{x})+\cdots\right]=hf(\bar{x})+\frac{1}{8}h^3f''(\bar{x})+\cdots. \quad (8.17)$$

Substituting Eqs. (8.14) and (8.17) into Eq. (8.11) and truncating the higher order derivative terms yields

$$E=\left[hf(\bar{x})+\frac{1}{24}h^3f''(\bar{x})+\cdots\right]-\left[hf(\bar{x})+\frac{1}{8}h^3f''(\bar{x})+\cdots\right]$$
$$\approx -\frac{1}{12}h^3f''(\bar{x}). \quad (8.18)$$

This shows that the error of the trapezoidal rule (per segment or step) is proportional to $f''(\bar{x})$ and $h^3$. Thus, the error can be reduced by reducing the value of $h=b-a$.

The error in the multisegmented trapezoidal rule, Eq. (8.10), can be determined by summing the errors of the individual segments $(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n)$. Since the range of integration is divided into $n$ equal segments, we have $h=\frac{b-a}{n}$ and hence

$$E\approx -\frac{1}{12}\left(\frac{b-a}{n}\right)^3\sum_{i=1}^{n}f''(\bar{x}_i), \quad (8.19)$$

where $\bar{x}_i$ is the midpoint between $x_i$ and $x_{i+1}$. By defining an average value of the second derivative

$$\bar{f}''=\frac{1}{n}\sum_{i=1}^{n}f''(\bar{x}_i), \quad (8.20)$$

Eq. (8.19) can be written as

$$E\approx -\frac{1}{12}(b-a)\left(\frac{b-a}{n}\right)^2\bar{f}''=-\frac{1}{12}(b-a)h^2\bar{f}''=O(h^2). \quad (8.21)$$

This indicates that the error of the multisegment trapezoidal rule, Eq. (8.10), is proportional to $h^2$ (since $(b-a)$ is fixed).

### 8.3.4 Truncation Error in Rectangular Rule

The foregoing procedure can be used to evaluate the truncation error in rectangular rule. The error can be expressed, for a single segment $a\le x\le b$, as

$$E=\int_a^b f(x)\,dx-f(a)h,\ \text{for Eq. (8.6)}, \quad (8.22)$$

and

$$E=\int_a^b f(x)\,dx-f(b)h,\ \text{for Eq. (8.7)}, \quad (8.23)$$

where the first term on the right-hand sides of Eqs. (8.22) and (8.23) denotes the exact integral and the second term represents the approximate integral given by the particular rectangular rule. Taylor's series expansion of $f(x)$ about $a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots. \tag{8.24}$$

The integration of Eq. (8.24) yields

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_0^h \left[ f(a) + yf'(a) + \frac{y^2}{2!}f''(a) + \cdots \right] dy \\ &= f(a)y|_0^h + f'(a)\left.\frac{y^2}{2}\right|_0^h + f''(a)\left.\frac{y^3}{6}\right|_0^h + \cdots \\ &= f(a)h + f'(a)\frac{h^2}{2} + f''(a)\frac{h^3}{6} + \cdots,\end{aligned} \tag{8.25}$$

where $y = x - a$ and $h = b - a$. Thus, Eq. (8.22) gives

$$E = f'(a)\frac{h^2}{2} + f''(a)\frac{h^3}{6} + \cdots. \tag{8.26}$$

Similarly, Taylor's series expansion of $f(x)$ about $b$ can be expressed as

$$f(x) = f(b) - (b-x)f'(b) + \frac{(b-x)^2}{2!}f''(b) - \cdots. \tag{8.27}$$

The integration of Eq. (8.27) gives

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^b \left[ f(b) - yf'(b) + \frac{y^2}{2!}f''(b) - \cdots \right] dy \\ &= f(b)\,y|_0^h - f'(b)\left.\frac{y^2}{2}\right|_0^h + f''(b)\left.\frac{y^3}{6}\right|_0^h - \cdots \\ &= f(b)h - \frac{h^2}{2}f'(b) + \frac{h^3}{6}f''(b) - \cdots,\end{aligned} \tag{8.28}$$

where $y = b - x$ and $h = b - a$. Thus, Eq. (8.23) yields

$$E = -\frac{h^2}{2}f'(b) + \frac{h^3}{6}f''(b) - \cdots. \tag{8.29}$$

Equations (8.26) and (8.29) indicate that the error of the rectangular rule per step is proportional to $h^2$ and $f'(a)$ or $f'(b)$. By proceeding as in the case of the trapezoidal rule, the error in a multistep rectangular rule can be expressed as (see Problem 8.5)

$$E = \frac{1}{2}(b-a)\left(\frac{b-a}{n}\right)\bar{f}' = \frac{1}{2}(b-a)h\bar{f}', \text{ for Eq. (8.6)}, \tag{8.30}$$

and

$$E = -\frac{1}{2}(b-a)\left(\frac{b-a}{n}\right)\bar{f}' = -\frac{1}{2}(b-a)h\bar{f}', \text{ for Eq. (8.7)}, \tag{8.31}$$

where $\bar{f}'$ in Eqs. (8.30) and (8.31) denotes the average value of the first derivative at the discrete points $a, x_1, x_2, \ldots, x_{n-1}$ and $x_1, x_2, \ldots, x_{n-1}, b$, respectively. This shows that the error in a multistep rectangular rule, Eq. (8.6) or (8.7), is proportional to $h$ since $(b-a)$ is fixed.

## ▶Example 8.4

Determine the value of the integral $I = \int_a^b f(x)\,dx$, where

$$f(x) = 0.84885406 + 31.51924706x - 137.66731262x^2 \\ + 240.55831238x^3 - 171.45245361x^4 + 41.95066071x^5 \tag{a}$$

with $a = 0.0$ and $b = 1.5$ using trapezoidal rule with different step lengths.

### Solution

The value of the integral given by the trapezoidal rule with $n$ steps is

$$I = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right), \tag{b}$$

where $h = (b-a)/n$ and $f_i = f(x = a + ih)$. For $n = 1$ with $h = 1.5$, Eq. (b) becomes

$$I = \left(\frac{f(a) + f(b)}{2}\right)(b-a). \tag{c}$$

Since $a = 0.0$, $b = 1.5$, $f(a) = f(0.0) = 0.84885406$ and $f(b) = f(1.5) = 0.84542847$, Eq. (c) gives the value of the integral as $I = 1.2707119$. The exact value of the integral, determined analytically, is 5.5606613. Thus, the error in the one-step trapezoidal rule is 77.148186%. Using $n = 3$ with $h = 0.5$, the values of the function $f(x)$ are given by

$$f_0 = f(0.0) = 0.84885406,\ f_1 = f(a+h) = f(0.5) = 2.8566201,\ f_2 = f(a+2h) \\ = f(1.0) = 5.7573166,\ \text{and } f_3 = f(a+3h) = f(1.5) = 0.84542847.$$

The trapezoidal rule with $n = 3$ gives the value of the integral as

$$I = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + f_3) = 4.7305388. \tag{d}$$

Compared with the exact value of the integral, 5.5606613, the error in the trapezoidal rule is 14.928484%. The significance of $I$, for $n = 1$ and 3, is shown in Fig. 8.12. The results given by the trapezoidal rule with $n = 1, 2, 3, 4, 5, 6, 9, 12$, and 15 are shown in Table 8.1. ◀

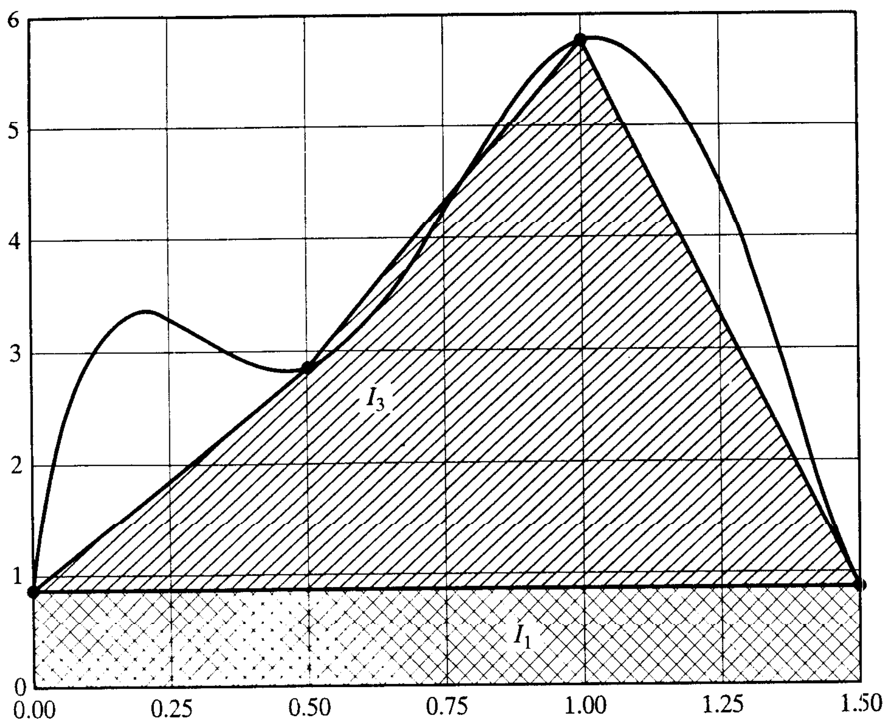


**Figure 8.12** Significance of trapezoidal rule.

**Table 8.1**

| Number of steps ($n$) | Step length ($h$) | Value of the integral ($I$) | Percent error |
|---|---|---|---|
| 1 | 1.5 | 1.2707119 | 77.148186 |
| 2 | 0.75 | 3.8171761 | 31.353920 |
| 3 | 0.5 | 4.7305388 | 14.928484 |
| 4 | 0.375 | 5.0828342 | 8.5929899 |
| 5 | 0.3 | 5.2516432 | 5.5572190 |
| 6 | 0.25 | 5.3448267 | 3.8814559 |
| 9 | 0.16666667 | 5.4640293 | 1.7377790 |
| 12 | 0.125 | 5.5061684 | 0.97997248 |
| 15 | 0.1 | 5.5257416 | 0.62797815 |

Exact value of the integral: 5.5606613

## 8.4 Simpson's Rule

The accuracy of the trapezoidal rule can be improved by reducing the step size $h$ (or increasing the number of segments $n$). However, the round-off error increases with a reduction in the step size $h$. Another way of obtaining a more accurate estimate of an integral is to use higher order polynomials for approximating the function $f(x)$. For example, we can use piecewise quadratic functions to approximate $f(x)$, which corresponds to the case of $m = 2$ in the Newton–Cotes formulas (Fig. 8.6). This method is also known as Simpson's one-third rule. As a further improvement, we can use piecewise cubic functions to approximate $f(x)$, corresponding to the case of $m = 3$ in the Newton–Cotes formulas. The method is also known as Simpson's three-eighths rule.

## 8.4.1 Simpson's One-Third Rule

As just stated, the integral

$$I = \int_a^b f(x)\,dx \tag{8.32}$$

is evaluated using a parabola or second-order polynomial for approximating $f(x)$. Assuming that $a \le x_{i-1} < x_i < x_{i+1} \le b$, the three points $(x_{i-1}, f_{i-1})$, $(x_i, f_i)$ and $(x_{i+1}, f_{i+1})$, as shown in Fig. 8.13, are used to define a second-degree polynomial, $p_2(x)$. By making the polynomial

$$p_2(x) = c_2x^2 + c_1x + c_0 \tag{8.33}$$

pass through the three points shown in Fig. 8.13, the constants $c_0, c_1$, and $c_2$ can be determined. For this, we take the origin at $x_i (x = 0 \text{ at } x_i)$ so that $x_{i-1}$ and $x_{i+1}$ correspond to $-h$ and $+h$, respectively. Such a choice of the origin does not influence the final result. By using the relations

For $x_{i-1}$,

$$p_2(x = -h) = f_{i-1} = c_2(-h)^2 + c_1(-h) + c_0 = c_2h^2 - c_1h + c_0; \tag{8.34}$$

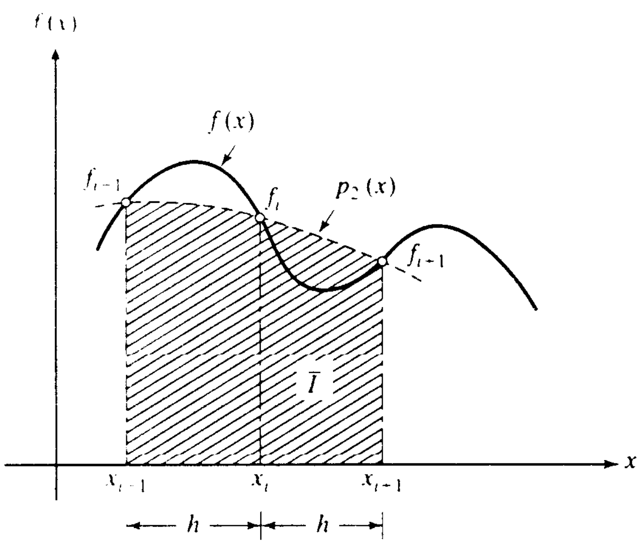


**Figure 8.13** Simpson's one-third rule.

For $x_i$,

$$p_2(x = 0) = f_i = c_2(0)^2 + c_1(0) + c_0 = c_0; \tag{8.35}$$

For $x_{i+1}$,

$$p_2(x = h) = f_{i+1} = c_2(h)^2 + c_1(h) + c_0 = c_2h^2 + c_1h + c_0 \tag{8.36}$$

the solution of Eqs. (8.34) through (8.36) can be found as

$$c_2 = \frac{f_{i-1} - 2f_i + f_{i+1}}{2h^2},\ c_1 = \frac{f_{i+1} - f_{i-1}}{2h},\ \text{and } c_0 = f_i. \tag{8.37}$$

(See Problem 8.46.) The area ($\bar{I}$) under the second-degree polynomial $p_2(x)$ between $x_{i-1}$ and $x_{i+1}$ can be determined as follows:

$$\begin{aligned}\bar{I} &= \int_{x_{i-1}}^{x_{i+1}} p_2(x)\,dx = \int_{-h}^{h} \left(c_2x^2 + c_1x + c_0\right) dx \\ &= \frac{c_2}{3}\,(x^3)\Big|_{-h}^{h} + \frac{c_1}{2}\,(x^2)\Big|_{-h}^{h} + c_0\,(x)|_{-h}^{h} \\ &= \frac{2}{3}c_2\,h^3 + 2\,c_0\,h.\end{aligned} \tag{8.38}$$

By substituting for $c_2$ and $c_0$ from Eqs. (8.37), Eq. (8.38) gives

$$\bar{I} = \frac{2}{3}h^3\left(\frac{f_{i-1} - 2\,f_i + f_{i+1}}{2\,h^2}\right) + 2\,h\,f_i = \frac{h}{3}\left(f_{i-1} + 4\,f_i + f_{i+1}\right). \tag{8.39}$$

The term "$\frac{1}{3}$" in Simpson's one-third rule refers to the presence of the factor "$\frac{1}{3}$" in Eq. (8.39). Note that two segments are used in deriving Eq. (8.39). Thus, for a multistage application of Simpson's one-third rule, we need to divide the range $a \le x \le b$ into $n$ segments of equal width $h = \frac{b-a}{n}$. The number of segments must be an even number so that Eq. (8.39) can be applied for groups of two segments. The integral in Eq. (8.32) can be evaluated as

$$I = \int_a^b f(x)\,dx \approx \sum_{j=1}^{n/2} \left(\bar{I}\right)_j, \tag{8.40}$$

where $\left(\bar{I}\right)_j$ denotes the value of $\bar{I}$ corresponding to the $j$th pair of segments and is given by Eq. (8.39) with $i = 2j - 1$. Equation (8.39) and (8.40) lead to

$$I \approx \frac{h}{3}\left[f_0 + 4\sum_{i=1,3,5,\ldots}^{n-1} f_i + 2\sum_{i=2,4,6,\ldots}^{n-2} f_i + f_n\right]. \tag{8.41}$$

## ▶Example 8.5

Determine the value of the integral described in Example 8.4 with $a = 0.0$ and $b = 1.5$ using Simpson's $\frac{1}{3}$ rule with different step sizes.

## Solution

Simpson's $\frac{1}{3}$ rule gives the value of the integral, with $n$ steps, as

$$I = \frac{h}{3}\left(f_0 + 4\sum_{i=1,3,5,\ldots}^{n-1} f_i + 2\sum_{i=2,4,6,\ldots}^{n-2} f_i + f_n\right). \tag{a}$$

For $n = 2$ and $h = 0.75$, Eq. (a) gives

$$I = \frac{h}{3}(f_0 + 4 f_1 + f_2), \tag{b}$$

where $f_0 = f(0.0) = 0.84885406$, $f_1 = f(0.75) = 4.2424269$, and $f_2 = f(1.5) = 0.84542847$. Thus, Eq. (b) gives $I = 4.66599753$. Noting that the exact value of the integral is 5.5606613, the error in Simpson's $\frac{1}{3}$ rule is 23.7064321%. For $n = 4$ and $h = 0.375$, Eq. (a) gives

$$I = \frac{h}{3}(f_0 + 4 f_1 + 2 f_2 + 4 f_3 + f_4), \tag{c}$$

where $f_0 = f(0.0) = 0.84885406$, $f_1 = f(0.375) = 2.9153550$, $f_2 = f(0.75) = 4.2424269$, $f_3 = f(1.125) = 5.5493011$, and $f_4 = f(1.5) = 0.84542847$. Thus, Eq. (c) gives $I = 5.50472009$ with an error of 1.0060172%. By proceeding in a similar manner, the value of the integral is computed for $n = 6, 8, 10$, and 12. The results are given in Table 8.2. ◀

### 8.4.2 Simpson's Three-Eighth's Rule

In this method, the integral is evaluated by approximating the function $f(x)$ by a third-degree polynomial, $p_3(x)$, as shown in Fig. 8.14. By assuming the polynomial $p_3(x)$ as

$$p_3(x) = c_3x^3 + c_2x^2 + c_1x + c_0, \tag{8.42}$$

**Table 8.2**

| Number of steps ($n$) | Step length ($h$) | Value of the integral ($I$) | Percent error |
|---|---|---|---|
| 2 | 0.75 | 4.6659975 | 16.089163 |
| 4 | 0.375 | 5.5047202 | 1.0060153 |
| 6 | 0.25 | 5.5495892 | 0.19911586 |
| 8 | 0.1875 | 5.5571246 | 0.063602172 |
| 10 | 0.15 | 5.5592151 | 0.026008544 |
| 12 | 0.125 | 5.5599494 | 0.012802755 |
| 14 | 0.10714286 | 5.5602551 | 0.0073060603 |
| 16 | 0.09375 | 5.5604043 | 0.0046220263 |

Exact value of the integral: 5.5606613

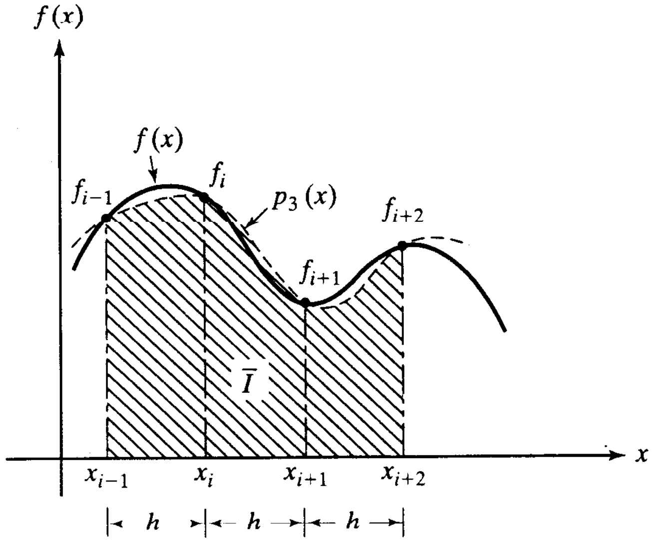


**Figure 8.14** Simpson's three-eighth's rule.

the constants $c_0, c_1, c_2$, and $c_3$ are determined by making the polynomial pass through the four points $(x_{i-1}, f_{i-1})$, $(x_i, f_i)$, $(x_{i+1}, f_{i+1})$, and $(x_{i+2}, f_{i+2})$. By taking the origin at $x_i$ ($x = 0$ at $x_i$), $x_{i-1}$, $x_{i+1}$, and $x_{i+2}$ can be assumed to correspond to $x = -h, h$, and $2\,h$, respectively. Such a choice of the origin does not influence the final result. By using the relations

For $x_{i-1}$,

$$p_3(x = -h) = f_{i-1} = -h^3\, c_3 + h^2 c_2 - h\, c_1 + c_0; \tag{8.43}$$

For $x_i$,

$$p_3(x = 0) = f_i = c_0; \tag{8.44}$$

For $x_{i+1}$,

$$p_3(x = h) = f_{i+1} = h^3\, c_3 + h^2\, c_2 + h\, c_1 + c_0; \tag{8.45}$$

For $x_{i+2}$,

$$p_3(x = 2\,h) = f_{i+2} = 8\,h^3\, c_3 + 4\,h^2\, c_2 + 2\,h\, c_1 + c_0 \tag{8.46}$$

the solution of Eqs. (8.43) through (8.46) can be determined as

$$c_0 = f_i \tag{8.47}$$

$$c_1 = \frac{1}{6\,h}(-f_{i+2} + 6\,f_{i+1} - 3 f_i - 2\,f_{i-1}); \tag{8.48}$$

$$c_2 = \frac{1}{2\,h^2}(f_{i-1} - 2\,f_i + f_{i+1}); \tag{8.49}$$

$$c_3 = \frac{1}{6\,h^3}(f_{i+2} - 3\,f_{i+1} + 3\,f_i - f_{i-1}). \tag{8.50}$$

(See Problem 8.47.) The area $(\bar{I})$ under the third-degree polynomial $p_3(x)$ between $x_{i-1}$ to $x_{i+2}$ can be found as

$$\begin{aligned}\bar{I} &= \int_{x_{i-1}}^{x_{i+2}} p_3(x)\,dx = \int_{-h}^{2h} \left(c_3x^3 + c_2x^2 + c_1x + c_0\right) dx \\ &= \frac{c_3}{4}\,(x^4)\Big|_{-h}^{2h} + \frac{c_2}{3}\,(x^3)\Big|_{-h}^{2h} + \frac{c_1}{2}\,(x^2)\Big|_{-h}^{2h} + c_0\,(x)|_{-h}^{2h} \\ &= \frac{c_3}{4}(15h^4) + \frac{c_2}{3}(9h^3) + \frac{c_1}{2}(3h^2) + c_0\,(3h)\,. \end{aligned} \tag{8.51}$$

By substituting for $c_0$ to $c_3$ from Eqs. (8.47) through (8.50), Eq. (8.51) gives

$$\begin{aligned}\bar{I} &= \frac{15h^4}{4}\left(\frac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{6h^3}\right) + 3h^3\left(\frac{f_{i+1} - 2f_i + f_{i-1}}{2h^2}\right) \\ &+ \frac{3h^2}{2}\left(\frac{-f_{i+2} + 6f_{i+1} - 3f_i \quad 2f_{i-1}}{6h}\right) + 3hf_i \\ &= \frac{3h}{8}\left[f_{i+2} + 3f_{i+1} + 3f_i + f_{i-1}\right]. \end{aligned} \tag{8.52}$$

The term "$\frac{3}{8}$" in Simpson's three-eighths rule refers to the presence of the factor "$\frac{3}{8}$" in Eq. (8.52). Note that three segments are used in deriving Eq. (8.52). Thus, for a multistage application of Simpson's three-eighths rule, we need to divide the range $a \le x \le b$ into $n$ segments of equal width $h = \frac{b-a}{n}$. The number of segments $n$ must be a multiple of 3 so that Eq. (8.52) can be applied for groups of three segments. The integral in Eq. (8.32) can be evaluated as

$$I = \int_a^b f(x)\,dx \approx \sum_{j=1}^{n/3} \left(\bar{I}\right)_j\,, \tag{8.53}$$

where $\left(\bar{I}\right)_j$ represents the value of $\bar{I}$ corresponding to the $j$th group of three segments and is given by Eq. (8.52) with $i = 3j - 2$. The use of Eqs. (8.52) and (8.53) yields

$$I \approx \frac{3h}{8}\left[f_0 + 3\sum_{i=1,4,7,\ldots}^{n-2} (f_i + f_{i+1}) + 2\sum_{i=3,6,9,\ldots}^{n-3} f_i + f_n\right]. \tag{8.54}$$

It can be shown that the truncation error in using Eq. (8.54) is of the same order as that of Simpson's one-third rule. But the use of Eq. (8.54) requires the number of segments to be a multiple of 3. Hence, Eq. (8.54) is rarely used by itself. Often both Simpson's one-third and three-eighths rules are used together so that the number of segments $n$ need not be constrained in any way. If the number of segments is even, Simpson's one-third rule can be used. On the other hand, if the number of segments is odd, Simpson's three-eighths rule can be applied, for instance, for the first three segments and Simpson's one-third rule can be used for the remaining even number of segments.

**Table 8.3**

| Number of steps ($n$) | Step length ($h$) | Value of the integral ($I$) | Percent error |
|---|---|---|---|
| 3 | 0.5 | 5.1630173 | 7.1510205 |
| 6 | 0.25 | 5.5357828 | 0.44740185 |
| 9 | 0.16666667 | 5.5557156 | 0.088941850 |
| 12 | 0.125 | 5.5590858 | 0.028332422 |
| 15 | 0.1 | 5.5600042 | 0.011816609 |
| 18 | 0.08333333 | 5.5603180 | 0.0061741355 |
| 21 | 0.071428575 | 5.5604572 | 0.003670180 |
| 24 | 0.0625 | 5.5605288 | 0.0023839022 |

Exact value of the integral: 5.5606613

## ▶Example 8.6

Determine the value of the integral described in Example 8.4 with $a = 0.0$ and $b = 1.5$ using Simpson's $\frac{3}{8}$ rule with different step sizes.

### Solution

Equation (8.54) gives the value of the integral according to Simpson's $\frac{3}{8}$ rule for $n$ steps. For $n = 3$ and $h = 0.5$, Eq. (8.54) gives

$$I = \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right], \tag{a}$$

where $f_0 = f(0.0) = 0.84885406$, $f_1 = f(0.5) = 2.8566201$, $f_2 = f(1.0) = 5.7573166$, and $f_3 = f(1.5) = 0.84542847$. Equation (a) gives $I = 5.16301737$ with an error of 7.1510187%. For $n = 6$ and $h = 0.25$, Eq. (8.54) gives

$$I = \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + f_6\right], \tag{b}$$

where $f_0 = f(0.0) = 0.84885406$, $f_1 = f(0.25) = 3.2544143$, $f_2 = f(0.5) = 2.8566201$, $f_3 = f(0.75) = 4.2424269$, $f_4 = f(1.0) = 5.7573166$, $f_5 = f(1.25) = 4.4213867$, and $f_6 = f(1.5) = 0.84542847$. Thus, Eq. (b) gives $I = 5.5357828$ with an error of 0.44740185%. By proceeding in a similar manner, the value of the integral is computed for $n = 9$, 12, 15, and 18. The results are shown in Table 8.3. ◀

### 8.4.3 Truncation Error

As in the case of trapezoidal rule, the basic truncation error ($E$) in Simpson's one-third rule, considering only two segments in the interval $a$ to $b$, is given by

$$E = \int_a^b f(x)\,dx - \left(\frac{b-a}{6}\right)\left[f(a) + 4f(x_1) + f(b)\right], \tag{8.55}$$

where the first term on the right-hand side of Eq. (8.55) denotes the exact integral, while the second term represents the approximate integral given by Simpson's one-third rule. (See Fig. 8.15.) We expand $f(x)$ using Taylor's series about the midpoint of the range, $x_1$, that is,

$$f(x) = f(x_1) + yf'(x_1) + \frac{y^2}{2!}f''(x_1) + \frac{y^3}{3!}f'''(x_1) + \frac{y^4}{4!}f''''(x_1) + \frac{y^5}{5!}f'''''(x_1) + \cdots, \tag{8.56}$$

where $y = x - x_1$. Equation (8.56) can be used to express the integral of $f(x)$ as

$$\int_a^b f(x)\,dx = \int_{-h}^{h} \left\{ f(x_1) + yf'(x_1) + \frac{y^2}{2}f''(x_1) + \frac{y^3}{6}f'''(x_1) + \frac{y^4}{24}f''''(x_1) + \frac{y^5}{120}f'''''(x_1) + \cdots \right\} dy, \tag{8.57}$$

where $y = -h$ and $y = h$ correspond to $x = a$ and $x = b$, respectively. By carrying out the integration in Eq. (8.57), we obtain

$$\begin{aligned}\int_a^b f(x)\,dx &= f(x_1)\,(y)\big|_{-h}^{h} + f'(x_1)\left(\frac{y^2}{2}\right)\Bigg|_{-h}^{h} + f''(x_1)\left(\frac{y^3}{6}\right)\Bigg|_{-h}^{h} \\ &\quad + f'''(x_1)\left(\frac{y^4}{24}\right)\Bigg|_{-h}^{h} + f''''(x_1)\left(\frac{y^5}{120}\right)\Bigg|_{-h}^{h} + f'''''(x_1)\left(\frac{y^6}{720}\right)\Bigg|_{-h}^{h} + \cdots \\ &= 2hf(x_1) + \frac{h^3}{3}f''(x_1) + \frac{h^5}{60}f''''(x_1) + \cdots. \end{aligned} \tag{8.58}$$

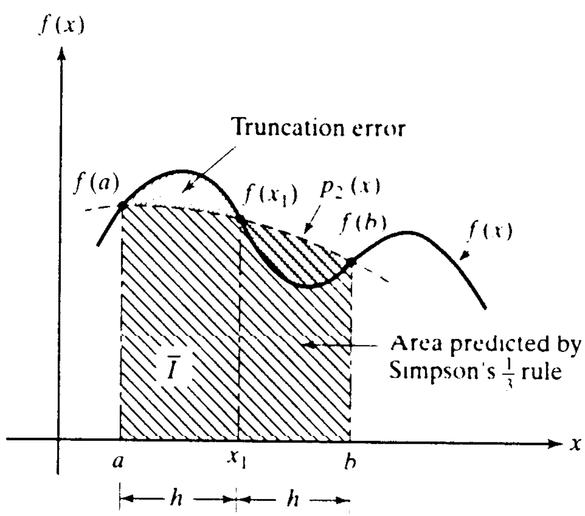


**Figure 8.15** Truncation error.

Substituting $x = a(y = -h)$, $x = x_1(y = 0)$, and $x = b(y = h)$ into Eq. (8.56) yields

$$f(a) = f(x_1) - hf'(x_1) + \frac{h^2}{2}f''(x_1) - \frac{h^3}{6}f'''(x_1) + \frac{h^4}{24}f''''(x_1) - \frac{h^5}{120}f'''''(x_1) + \cdots; \tag{8.59}$$

$$f(x_1) = f(x_1); \tag{8.60}$$

$$f(b) = f(x_1) + hf'(x_1) + \frac{h^2}{2}f''(x_1) + \frac{h^3}{6}f'''(x_1) + \frac{h^4}{24}f''''(x_1) + \frac{h^5}{120}f'''''(x_1) + \cdots. \tag{8.61}$$

Now the second term on the right-hand side of Eq. (8.55) can be expressed, using Eqs. (8.59) through (8.61), as

$$\begin{aligned}
&\left(\frac{b-a}{6}\right)\Bigg[f(x_1) - hf'(x_1) + \frac{h^2}{2}f''(x_1) - \frac{h^3}{6}f'''(x_1) \\
&+\frac{h^4}{24}f''''(x_1) - \frac{h^5}{120}f'''''(x_1) + \cdots + 4f(x_1) \\
&+f(x_1) + hf'(x_1) + \frac{h^2}{2}f''(x_1) + \frac{h^3}{6}f'''(x_1) \\
&+\frac{h^4}{24}f''''(x_1) + \frac{h^5}{120}f'''''(x_1) + \cdots\Bigg] \\
&= \left(\frac{b-a}{6}\right)\left[6f(x_1) + h^2 f''(x_1) + \frac{h^4}{12}f''''(x_1) + \cdots\right].
\end{aligned} \tag{8.62}$$

Substituting Eqs. (8.58) and (8.62) into Eq. (8.55) and truncating terms involving derivatives higher than the fifth gives

$$\begin{aligned}
E &\approx \left[2\left(\frac{b-a}{2}\right)f(x_1) + \frac{1}{3}\left(\frac{b-a}{2}\right)^3 f''(x_1) + \frac{1}{60}\left(\frac{b-a}{2}\right)^5 f''''(x_1)\right] \\
&\quad - \left[(b-a)f(x_1) + \left(\frac{b-a}{6}\right)\left(\frac{b-a}{2}\right)^2 f''(x_1) + \left(\frac{b-a}{6}\right)\frac{1}{12}\left(\frac{b-a}{2}\right)^4 f''''(x_1)\right] \\
&\approx -\frac{1}{2880}(b-a)^5 f''''(x_1) \\
&\approx -\frac{1}{90}h^5 f''''(x_1).
\end{aligned} \tag{8.63}$$

This indicates that the error of Simpson's one-third rule (per each pair of segments) is proportional to $h^5$ and $f''''(x_1)$. Thus, the error will be zero if $f(x)$ is a third-order polynomial, since $f'''' = 0$.

The error in a multisegmented Simpson's one-third rule, Eq. (8.54), can be found by summing the errors of the individual pairs of segments $(x_0, x_2)$, $(x_2, x_4)$, ..., $(x_{n-2}, x_n)$:

$$E \approx -\frac{h^5}{90} \sum_{j=1,3,5,\ldots}^{n-1} f''''(x_j). \tag{8.64}$$

By defining an average value of the fourth derivative, $\overline{f''''}$, as

$$\overline{f''''} = \frac{2}{n}\left(\sum_{j=1,3,5,\ldots}^{n-1} f''''(x_j)\right). \tag{8.65}$$

Eq. (8.64) can be expressed as follows:

$$\begin{aligned} E &\approx -\frac{1}{90}h^5\frac{n}{2}\overline{f''''} \\ &\approx -\frac{1}{180}h^4(b-a)\overline{f''''} \\ &= O\left(h^4\right). \end{aligned} \tag{8.66}$$

This indicates that the error in a multisegment Simpson's one-third rule, Eq. (8.41), is proportional to $h^4$, since $(b-a)$ is fixed.

By following a similar approach, the truncation error in a multisegment Simpson's three-eighths rule can also be shown to be proportional to $h^4$. (See Problem 8.8.)

## 8.5 General Newton–Cotes Formulas

As stated earlier, the Newton–Cotes formulas are derived by using a polynomial of order $m$ to approximate the function $f(x)$. That is,

$$\int_a^b f(x)\,dx \approx \int_a^b p_m(x)\,dx, \tag{8.67}$$

where

$$p_m(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_2x^2 + c_1x + c_0. \tag{8.68}$$

Numerical integration formulas corresponding to $m = 0$ (rectangular rule), $m = 1$ (trapezoidal rule), $m = 2$ (Simpson's one-third rule), and $m = 3$ (Simpson's three-eighths rule) have been derived in Sections 8.3 and 8.4. Formulas corresponding to higher order polynomials can also be derived. An estimate of error associated with any formula can also be derived as outlined earlier. A summary of some of

**Table 8.4**

| Value of $m$ | Name of formula | Formula $h = (b-a)/n$ | Estimate of truncation error | Number of segments of width $h$ (in each group) |
|---|---|---|---|---|
| 0 | Rectangular | $hf_i$ or $hf_{i+1}$ | $\frac{1}{2}h^2 f'$ or $-\frac{1}{2}h^2 f'$ | 0 |
| 1 | Trapezoidal | $\frac{h}{2}(f_{i-1} + f_i)$ | $-\frac{1}{12}h^3 f''$ | 1 |
| 2 | Simpson's one-third | $\frac{h}{3}(f_{i-1} + 4f_i + f_{i+1})$ | $-\frac{1}{90}h^5 f''''$ | 2 |
| 3 | Simpson's three-eighth | $\frac{3h}{8}(f_{i-1} + 3f_i + 3f_{i+1} + f_{i+2})$ | $-\frac{3}{80}h^5 f''''$ | 3 |
| 4 | Boole's | $\frac{2h}{45}(7f_{i-2} + 32f_{i-1} + 12f_i$ $+32f_{i+1} + 7f_{i+2})$ | $-\frac{8}{945}h^7 f^{vi}$ | 4 |
| 5 | — | $\frac{5h}{288}(19f_{i-2} + 75f_{i-1} + 50f_i$ $+50f_{i+1} + 75f_{i+2} + 19f_{i+3})$ | $-\frac{275}{12096}h^7 f^{vi}$ | 5 |

the Newton–Cotes formulas, along with the associated error estimates, is given in Table 8.4.

## 8.6 Richardson's Extrapolation

In many engineering problems, the integrals are to be evaluated very accurately. One possibility is to use a large number of segments ($n$) in the trapezoidal or Simpson's method to reduce the truncation error. However, beyond a certain number of segments, the round-off error begins to dominate and the accuracy of the result may suffer as shown in Fig. 8.16. Also, the computational effort required will be

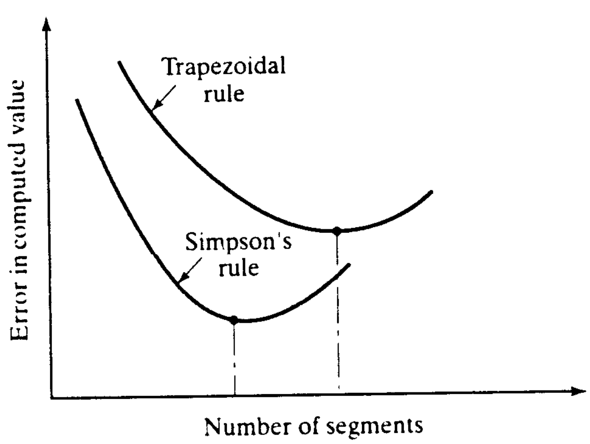


**Figure 8.16** Variation of accuracy with increasing number of segments.

more with larger number of segments. Another possibility to improve the accuracy is to use a higher order Newton–Cotes formula. Alternatively, the accuracy of the estimated integral can be improved by using a scheme known as Richardson's extrapolation, in which two numerical integral estimates are combined to obtain a third, more accurate value. The computational algorithm, which implements Richardson's extrapolation in an efficient manner, is known as Romberg integration. This is a recursive procedure that can be used to generate the value of the integral to within a prespecified error tolerance.

Richardson's extrapolation is a numerical procedure that can be used to improve the accuracy of the results obtained from another numerical method, provided an estimate of the error is available. The procedure can be used not only in numerical integration of functions, but also in other methods such as numerical integration of differential equations. In this section, we consider the application of Richardson's extrapolation procedure to trapezoidal and Simpson's rules.

## 8.6.1 Trapezoidal Rule

The truncation error in multisegmented trapezoidal rule is given by Eq. (8.21):

$$E \approx -\frac{1}{12}(b-a)h^2\overline{f''}. \tag{8.69}$$

If $I_1(h_1)$ denotes the value of the integral (approximate value) given by the trapezoidal rule and $E_1(h_1)$ indicates the truncation error with a step size $h_1$, the exact value of the integral can be expressed as

$$I \approx I_1(h_1) + E_1(h_1) \approx I_1(h_1) + ch_1^2, \tag{8.70}$$

where $c = -\frac{1}{12}(b-a)\bar{f''}$ is a constant. Similarly, if $I_2(h_2)$ denotes the value of the integral given by the trapezoidal rule with a step size $h_2$ and $E_2(h_2)$ represents the associated truncation error, we can write

$$I \approx I_2(h_2) + E_2(h_2) \approx I_2(h_2) + ch_2^2 \tag{8.71}$$

by assuming that $\overline{f''}$ is constant regardless of the step size. Equations (8.70) and (8.71) can be used to obtain

$$I_1(h_1) + ch_1^2 \approx I_2(h_2) + ch_2^2,$$

or

$$c \approx \frac{I_2(h_2) - I_1(h_1)}{h_1^2 - h_2^2}. \tag{8.72}$$

Substituting this expression of $c$ into Eq. (8.71) yields an improved estimate of the integral $(I)$ as

$$I \approx I_2(h_2) + \frac{I_2(h_2) - I_1(h_1)}{\left\{\left(\frac{h_1}{h_2}\right)^2 - 1\right\}}. \tag{8.73}$$

It can be shown [8.3] that the error of this estimate is $O(h^4)$, which means that we combined two estimates given by the trapezoidal rule, which has an error of $O(h^2)$, to yield a new estimate having an error of $O(h^4)$. This can also be seen for the special case where the interval is halved $\left(h_2 = \frac{h_1}{2}\right)$. Using $h_2 = \frac{h_1}{2}$, we find that Eq. (8.73) gives

$$\begin{aligned} I &\approx I_2(h_2) + \frac{I_2(h_2) - I_1(h_1)}{3} \\ &\approx \frac{4}{3} I_2(h_2) - \frac{1}{3} I_1(h_1). \end{aligned} \tag{8.74}$$

It can be verified that this expression is identical to the one given by Simpson's one-third rule with a step size of $h_2$. Note that the estimate given by Simpson's one-third rule has an error of $O(h^4)$.

## ▶Example 8.7

Using the results of the trapezoidal rule given in Table 8.1, find an improved estimate of the value of the integral using Richardson's extrapolation.

## Solution

Using $h_1 = 1.5, I_1 = 1.2707119, h_2 = 0.75$, and $I_2 = 3.8171761$, we find that Eq. (8.74) gives the improved estimate of

$$I = \frac{4}{3}(3.8171761) - \frac{1}{3}(1.2707119) = 4.66599750,$$

which corresponds to an error of 16.0891619%. Similarly, by using $h_1 = 0.5$, $I_1 = 4.7305388$, $h_2 = 0.375$, and $I_2 = 5.0828342$, Eq. (8.74) yields the improved estimate of $I$ as

$$I = \frac{4}{3}(5.0828342) - \frac{1}{3}(4.7305388) = 5.20026600,$$

which corresponds to an error of 6.48115899%. It can be seen that, in both the cases, the value of the integral predicted by Richardson's extrapolation is superior to the original estimates. ◀

### 8.6.2 Simpson's One-Third Rule

The truncation error in a multisegmented Simpson's one-third rule is given by Eq. (8.66):

$$E \approx -\frac{1}{180}(b-a)h^4\overline{f''''}. \tag{8.75}$$

If $I_1(h_1)$ and $I_2(h_2)$ denote the values of the integral given by Simpson's one-third rule with step sizes $h_1$ and $h_2$, and the corresponding error estimates are given by

$E_1(h_1)$ and $E_2(h_2)$, respectively, we have

$$I \approx I_1(h_1) + E_1(h_1) \approx I_1(h_1) + ch_1^4 \tag{8.76}$$

and

$$I \approx I_2(h_2) + E_2(h_2) \approx I_2(h_2) + ch_2^4. \tag{8.77}$$

These equations yield

$$c \approx \frac{I_2(h_2) - I_1(h_1)}{h_1^4 - h_2^4} \tag{8.78}$$

Substituting this expression into Eq. (8.77) gives an improved estimate of the integral ($I$) as

$$I \approx I_2(h_2) + \frac{I_2(h_2) - I_1(h_1)}{\left\{\left(\frac{h_1}{h_2}\right)^4 - 1\right\}}. \tag{8.79}$$

It can be shown that the error of this estimate is $O(h^6)$. This implies that we combined two estimates given by Simpson's one-third rule, which has an error of $O(h^4)$, to obtain a new estimate having an error of $O(h^6)$. When $h_2$ is taken as $\frac{1}{2}h_1$, Eq. (8.79) gives

$$I \approx \frac{16}{15} I_2(h_2) - \frac{1}{15} I_1(h_1). \tag{8.80}$$