Numerical Analysis

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C.5. The Bessel function of order zero is defined by

$$J_0(x) - \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta$$

Use Exercise C.2 to find $J_0(0.1)$, $J_0(0.5)$, and $J_0(1.0)$.

C.6. The **Frensel integral** is defined by

$$C(x) = \int_0^{x} \cos\left(\frac{\pi}{2} t^2\right) dt$$

Use Exercise C.2 to find $C(\frac{1}{2})$, $C(1)$, and $C(2)$.

6.6 GAUSSIAN INTEGRATION

All integration formulas developed are of the form

$$\int_a^b f(x)\,dx \approx a_0 f(x_0) + a_1 f(x_1) + \dots + a_N f(x_N) \qquad (6$$

The nodes $x_0, x_1, \dots, x_N$ have been specified to be equally spaced so there is no c in the selection of the base points. To integrate a function that is in an equally-sp tabulated form, these methods are clearly preferable. However, if a function f is k analytically, there is no need to require equally-spaced nodes for the integrating fo las. If $x_0, x_1, \dots, x_N$ are not fixed in advance and if there are no other restrictio them, then there are $2N + 2$ unknowns or parameters in Equation (6.6.1) that s satisfy $2N + 2$ equations. Thus it seems reasonable to expect that we can obt formula that is exact whenever f is a polynomial of degree $k \leq 2N + 1$. Gauss sh that by selecting $x_0, x_1, \dots, x_N$ properly it is possible to construct formulas far accurate than the corresponding Newton–Cotes formulas. The formulas based o principle are called **Gaussian integration formulas**.

Let us determine the parameters in the case of two points. It is convenie determine the parameters if the integral involved is of the form $\int_{-1}^{1} f(x)\,dx$. L determine four parameters a_0, a_1, x_0, and x_1 such that

$$\int_{-1}^{1} f(x)\,dx \approx a_0 f(x_0) + a_1 f(x_1) \qquad ($$

This formula gives the exact value whenever f is a polynomial of degree three o In order to get four equations, let $f(x) = 1, x, x^2$, and x^3 in Equation (6.6.2). consider $f(x) = 1$. Then $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} 1\,dx = x|_{-1}^{1} = 2$ and $a_0 f(x_0) + a_1 f($ $a_0 \cdot 1 + a_1 \cdot 1$. Thus we have $a_0 + a_1 = 2$. Let $f(x) = x$. Then $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x$ $x^2/2|_{-1}^{1} = 0$, and $a_0 f(x_0) + a_1 f(x_1) = a_0 x_0 + a_1 x_1$. Hence $a_0 x_0 + a_1 x_1 = 0$. Sim considering $f(x) = x^2$ and x^3, we get $a_0 x_0^2 + a_1 x_1^2 = \frac{2}{3}$ and $a_0 x_0^3 + a_1 x_1^3 = 0$. We h

$$\begin{aligned} a_0 + a_1 &= 2 \\ a_0 x_0 + a_1 x_1 &= 0 \\ a_0 x_0^2 + a_1 x_1^2 &= 2/3 \\ a_0 x_0^3 + a_1 x_1^3 &= 0 \end{aligned} \qquad ($$

Solving these four nonlinear equations,

$$a_0 = a_1 = 1 \qquad x_0 = -x_1 = -\sqrt{3}/3$$

Therefore the integration formula is given by

$$\int_{-1}^{1} f(x)\,dx \approx f(-\sqrt{3}/3) + f(\sqrt{3}/3) \tag{6.6.4}$$

This is called the two-point Gaussian integration formula. It is remarkable that by adding two values, we get the exact value of an integral of any polynomial of degree three or less.

In order to use Equation (6.6.4), write $\int_a^b f(x)\,dx$ in the form $\int_{-1}^{1} F(t)\,dt$. Let $x = \alpha t + \beta$ where α and β are to be determined so that when $t = 1$, $x = b$ and when $t = -1$, $x = a$. Thus we have $\alpha + \beta = b$ and $-\alpha + \beta = a$. Solving these two equations

$$\alpha = \frac{(b - a)}{2} \quad \text{and} \quad \beta = \frac{(b + a)}{2}$$

Thus if

$$x = \frac{(b - a)t + (b + a)}{2} \tag{6.6.5}$$

then

$$\int_a^b f(x)\,dx = \frac{(b - a)}{2}\int_{-1}^{1} f\left(\frac{(b - a)t + (a + b)}{2}\right) dt$$

EXAMPLE 6.6.1 Integrate $\int_0^1 e^{2x}\,dx$ using the two-point Gaussian integration formula.

$$\int_0^1 e^{2x}\,dx = \frac{1}{2}\int_{-1}^{1} e^{t+1}\,dt \approx \frac{1}{2}[e^{(-\sqrt{3}/3)+1} + e^{(\sqrt{3}/3)+1}] \approx 3.18405$$

The exact value to five places is

$$\int_0^1 e^{2x}\,dx = \left.\frac{e^{2x}}{2}\right|_0^1 = \frac{1}{2}(e^2 - 1) = 3.19453$$

The error is 0.1048×10^{-1}. ■ ■ ■

Formulas containing more terms can be derived using the same technique that was used to derive Equation (6.6.4). The solutions to the corresponding nonlinear systems are difficult to obtain, so we present an alternative derivation of these formulas. Since x_i are unknowns, use the Lagrange interpolating polynomial, which allows arbitrarily-spaced base points. Using the Lagrange Polynomial Approximation Theorem 4.3.1,

$$f(x) = \sum_{j=0}^{N} f(x_j)\,L_j(x) + \frac{f^{(N+1)}(\xi(x))}{(N + 1)!}\prod_{j=0}^{N}(x - x_j) \tag{6.6.6}$$

where

$$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{(x - x_i)}{(x_j - x_i)} \quad \text{and} \quad -1 < \xi < 1$$

For Equation (6.6.1), $a_0, a_1, \ldots, a_N, x_0, x_1, \ldots, x_N$ are $2N + 2$ unknowns, and therefore Equation (6.6.1) gives the exact value whenever $f(x)$ is a polynomial of degree $2N + 1$ or less. Now if we assume that $f(x)$ is a polynomial of degree $2N + 1$, then the term $f^{(N+1)}(\xi(x))/(N + 1)!$ is a polynomial of degree N at most. Let

$$\frac{f^{(N+1)}(\xi(x))}{(N + 1)!} = q_N(x) \tag{6.6.7}$$

where $q_N(x)$ is a polynomial of degree N.

Substituting Equation (6.6.7) in Equation (6.6.6) and integrating between -1 and 1 result in

$$\int_{-1}^{1} f(x)\,dx = \sum_{j=0}^{N} f(x_j) \int_{-1}^{1} L_j(x)\,dx + \int_{-1}^{1} q_N(x) \prod_{j=0}^{N} (x - x_j)\,dx \tag{6.6.8}$$

We want to select x_j in such a way that the error term in Equation (6.6.8) vanishes since $f(x)$ is a polynomial of degree $\leq 2N + 1$. We want

$$\int_{-1}^{1} q_N(x) \prod_{j=0}^{N} (x - x_j)\,dx = 0 \tag{6.6.9}$$

$\Pi_{j=0}^{N} (x - x_j)$ is a polynomial of degree $N + 1$ and $q_N(x)$ is a polynomial of degree N or less; therefore, Equation (6.6.9) is satisfied if we choose polynomial $\Pi_{j=0}^{N} (x - x_j)$ of degree $N + 1$ orthogonal to all polynomials of degree N or less on the interval $[-1, 1]$.

The **Legendre polynomials** defined by

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_i(x) = \frac{1}{i}[(2i - 1)\,x\,P_{i-1}(x) - (i - 1)\,P_{i-2}(x)] \quad \text{for } i = 2, 3, \ldots \tag{6.6.10}$$

are orthogonal polynomials over $[-1, 1]$ with respect to the weight function $w(x) = 1$. Orthogonal polynomials are also linearly independent (Appendix E) and, therefore, $q_N(x)$ in Equation (6.6.9) can be written as a linear combination of Legendre polynomials $P_i(x)$, $i = 0, 1, \ldots, N$. If we pick x_j, $j = 0, 1, \ldots, N$ of $\Pi_{j=0}^{N} (x - x_j)$ as the zeros of the $(N + 1)$th degree Legendre polynomial $P_{N+1}(x)$, then Equation (6.6.9) will be satisfied. Also, it is known that the zeros of the Legendre polynomial of any degree ≥ 1 are all real and distinct (Appendix E). By selecting the zeros of Legendre polynomial P_{N+1} as the nodes for Equation (6.6.1), Equation (6.6.8) reduces to

$$\int_{-1}^{1} f(x)\,dx = \sum_{j=0}^{N} f(x_j) \int_{-1}^{1} L_j(x)\,dx \tag{6.6.11}$$

whenever $f(x)$ is a polynomial of degree $2N + 1$ or less. Therefore

$$\int_{-1}^{1} f(x)\,dx = \sum_{j=0}^{N} f(x_j) \int_{-1}^{1} L_j(x)\,dx + \frac{f^{(2N+2)}(\eta)}{(2N+2)!} \int_{-1}^{1} \sum_{i=0}^{N} (x - x_i)^2\,dx \quad \textbf{(6.6.12)}$$

Thus

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=0}^{N} a_j f(x_j) \quad \textbf{(6.6.13)}$$

where $a_j = \int_{-1}^{1} L_j(x)\,dx$ and x_j, $j = 0, 1, \ldots, N$ are the zeros of a Legendre polynomial $P_{N+1}(x)$.

EXAMPLE 6.6.2

Find a_0, a_1, and a_2 for $N = 2$.

The zeros of $P_3(x) = 5(x^3 - (3/5)x)/2$ are $-\sqrt{3/5}$, 0, and $\sqrt{3/5}$. Since $x_0 = -\sqrt{3/5}$, $x_1 = 0$, and $x_2 = \sqrt{3/5}$,

$$L_0(x) = \frac{(x-0)(x-\sqrt{3/5})}{(-\sqrt{3/5}-0)(-\sqrt{3/5}-\sqrt{3/5})} = \frac{5}{6}x(x - \sqrt{3/5})$$

$$L_1(x) = \frac{5}{3}(x + \sqrt{3/5})(x - \sqrt{3/5})$$

$$L_2(x) = \frac{5}{6}x(x + \sqrt{3/5})$$

Hence

$$a_0 = \int_{-1}^{1} L_0(x)\,dx = \frac{5}{6}\int_{-1}^{1} [x^2 - (\sqrt{3/5})x]\,dx = \frac{5}{9}$$

Similarly, it can be verified that $a_1 = \int_{-1}^{1} L_1(x)\,dx = 8/9$ and $a_2 = \int_{-1}^{1} L_2\,dx = 5/9$. ■■■

A short table of Legendre polynomials, zeros of Legendre polynomials, and the values of the coefficients a_j are given in Table 6.6.1.

EXAMPLE 6.6.3

Use Gaussian quadrature with $N = 3$ and 4 to evaluate $\int_0^1 e^{2x}\,dx$.

Using Equation (6.6.5), we get $\int_0^1 e^{2x}\,dx = \frac{1}{2}\int_{-1}^{1} e^{1+t}\,dt$.

From Table 6.6.1 for $N = 3$,

$$\frac{1}{2}\int_{-1}^{1} e^{1+t}\,dt \approx \frac{1}{2}\left[\frac{5}{9}e^{1-\sqrt{3/5}} + \frac{8}{9}e + \frac{5}{9}e^{1+\sqrt{3/5}}\right] \approx 3.19444$$

Table 6.6.1

Legendre Polynomials	Zeros	Weights a_j
$P_1(x) = x$	0	2
$P_2(x) = \frac{3}{2}(x^2 - \frac{1}{3})$	$-\frac{1}{\sqrt{3}} = -0.57735$	1.0
	$\frac{1}{\sqrt{3}} = 0.57735$	1.0
$P_3(x) = \frac{5}{2}(x^3 - \frac{3}{5}x)$	$-\sqrt{\frac{3}{5}} = -0.77460$	$\frac{5}{9} = 0.55556$
	0.00000	$\frac{8}{9} = 0.88889$
	$\sqrt{\frac{3}{5}} = 0.77460$	$\frac{5}{9} = 0.55556$
$P_4(x) = \frac{35}{8}(x^4 - \frac{6}{7}x^2 + \frac{3}{35})$	−0.86114	0.34785
	−0.33998	0.65214
	0.33998	0.65214
	0.86114	0.34785
$P_5(x) = \frac{63}{8}(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x)$	−0.90618	0.23692
	−0.53847	0.47863
	0.00000	0.56889
	0.53847	0.47863
	0.90618	0.23692

From Table 6.6.1 for $N = 4$,

$$\frac{1}{2}\int_{-1}^{1} e^{1+t}\,dt \approx \frac{1}{2}[(0.34785)\,e^{1-0.86114} + (0.65214)\,e^{1-0.33998}$$
$$+ (0.65214)\,e^{1+0.33998} + (0.34785)\,e^{1+0.86114}] \approx 3.19450$$

while the exact value of the integral is $(e^2 - 1)/2 = 3.19453$ to five places. ■ ■ ■

EXERCISES

1. Use the two-point and three-point Gaussian integration formulas to approximate:

   (a) $\int_{-1}^{1} e^x\,dx$ (b) $\int_0^{\pi} \sin x\,dx$ (c) $\int_2^3 dx/x$

   Compare each approximated value with the exact value.
2. Find the truncation error formulas for the two-point and three-point Gaussian integration formulas and compare them with the truncation error term for the Simpson rule.
3. One can derive the composite rule for the Gaussian integration formulas by dividing the interval $[a, b]$ into M equal subintervals. Then use the Gaussian integration formula on each subinterval. Approximate $\int_2^3 dx/x$ by dividing [2, 3]