ECE5340 HW 5: Method of Moments

Assignment Objectives:

- Understand the theory behind the Galerkin method and MOM.
- Write a computer program that computes the charge distibution along a thin metal wire using the method of moments (MOM).
- Compare results against previous simulations using FDM.

ASSIGNMENT

This assignment is broken down into two sections. The first section will guide you in the basic MOM theory by taking you step-by-step through the setup. In the second section, you will write a Matlab code that numerically carries out your MOM algorithm.

1. THEORY

1. Write the equation for the 1D triangular basis function, $u_n(x)$, centered at the point x_n and with a width of h. Repeat for the rect function and delta functon.
2. For the Galerkin method, what is the weighting function, $w_m(x)$?
3. For this assignment, we will be solving for the charge distribution along a thin wire from $x = 0$ to $x = L$. The integral equation for voltage potential $V(\mathbf{r})$ in the presence of a charge density $\rho(\mathbf{r})$ is therefore given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\mathbf{r}' \ . \tag{1}$$

Compare this expression with the Fredholm equation of the first kind, which is given as

$$f(x) = \int_a^b K(x, x')\Phi(x')\, dx' \ . \tag{2}$$

Convert Equation (1) into one dimension and clearly indicate which term corresponds to the following from Equation (2):

$x =$ (Position)
$x' =$ (Variable of Integration)
$f(x) =$ (Forcing Function)
$K(x, x') =$ (Kernal)
$\Phi(x') =$ (Unknown Function)

4. $V(x)$ is assumed to be constant along the thin wire. That is to say,

$$V(x) = V_0 \qquad (0 \leq x \leq L) \ .$$

Explain why.

5. Rewrite the unknown function $\Phi(x)$ by approximating it as a linear combination of basis functions. Label your approximate function $\hat{\Phi}(x)$.

6. Suppose that you ran your MOM simulation using $N = 4$ evenly spaced points along the interval $[0, 3]$ m and obtained the following basis coefficients:

$$a_1 = 2$$
$$a_2 = 1$$
$$a_3 = 1$$
$$a_4 = 2$$

Write out the full expression for $\hat{\Phi}(x)$ in terms of the basis functions and graph it along the full domain from $0 \leq x \leq 3$ m.

7. Rewrite Equation (2) using $\hat{\Phi}(x)$ in place of $\Phi(x)$. How many equations and how many unknowns does this give you?

8. Calculate the residual R by subtracting the forcing function from your summed approximation. This represents the error of your approximation. You should be left with an expression of the form

$$R = L\hat{\Phi} - f \ ,$$

where L is some linear operator acting on $\hat{\Phi}$.

9. Our desire is to drive this residual as low as possible, but we cannot do it directly. Therefore, the next best thing is to take the *weighted residual* of R and minimize this over some average sense. This is done by taking the inner product between the residual and a set of N different weight functions $w_m(x)$. We then set each of these values to zero such that

$$\langle w_m, R \rangle = 0 \ .$$

Write out the inner product between w_m and R and set it to zero. Rearrange your result so that it has the following form:

$$\left\langle w_m, L\hat{\Phi} \right\rangle = \langle w_m, f \rangle \ .$$

HINT: Use the following substitution to simplify your results:

$$g_n(x) = \int_0^L \frac{u_n(x')}{|x - x'|} \, dx'$$

10. How many equations and how many unknowns do you have now?

11. Suppose that $u_n(x')$ is a delta function centered at the point x_n. Compute $g_n(x)$. What happens if $x = x_n$? What is going on here? If we instead assume that the wire is a tiny metal cylinder, how does this solve our problem without over-complicating our calculations?

12. We are now going to rewrite the equations from Item (9) in terms of a matrix equation $\mathbf{Ax} = \mathbf{b}$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ \vdots & \vdots & \ddots & \cdots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

Give the sizes of all elements in the matrix equation (rows $\times$ columns). What does the vector $\mathbf{x}$ represent?

13. Specifically write out the elements A_{mn} and b_n in terms of integrals over the basis functions and weighting functions:

$$A_{mn} =$$
$$b_n =$$

14. When calculating the diagonal elements in A (also called the *self-terms*), something bad happens. Explain this problem and how it relates to Item 11.

15. Derive an expression for A_{nn} by assuming a uniform distribution of charge along the outer surface of a metal wire segment with length h and radius a. This will be the self-term you use when filling the moment matrix.

16. After running your simulation, describe how can you might find $V(x, y)$ at some location far from the wire. How would you use this information to calculate $E(x, y)$ at the same location?

17. From your knowledge of V on the wire, and your answer to Item 16, prove that the tangential electric field on the metal wire is zero.

18. Summarize. What are the advantages and disadvantages of the method of moments? What are the limitations of this method when applied to the charge distribution along a thin wire?

2. PROGRAMMING

1. Write a Matlab function called "ww(x)" to compute the integral of the weight function:

$$\mathrm{ww}_m(x) = \int_0^L w_m(x)\, dx$$

Include an argument to specify which basis function to use for $w_m(x)$.

2. Write a second Matlab function that calculates $g_n(x)$. Remember to be careful with the self-terms where $x \approx x_n$. Include an input argument called "opt" to select which basis function to use for $u_n(x)$. Use the convention

$$\text{Delta Function} \rightarrow \text{opt} == 1$$
$$\text{Rect Function} \rightarrow \text{opt} == 2$$
$$\text{Triangle Function} \rightarrow \text{opt} == 3$$

Think about how you might compute $g_n(x)$ for options 2 and 3. You will probably find that computing them analytically is difficult. If so, compute them numerically by using the Trapezoidal rule.

3. Plot $g_n(x)$ on the interval $x \in [0, L]$ under the following parameters:

$$\begin{aligned} L &= 1 \\ x_n &= 0.5 \\ h &= 0.05 \\ u_n(x) &= \text{rect} \end{aligned}$$

4. Write another function called "wmgn(x)" that calculates the integral between the weight function $w_m(x)$ and $g_n(x)$. That is,

$$\text{wmgn}(x_m, x_n) = \int_0^L w_m(x) g_n(x)\, dx \ .$$

   Include two extra input arguments, "opt_w" and "opt_g," that determine which of the three basis functions to use for w_m and g_n. Be sure to write your code to specifically handle each case.

5. Finally, write a program that uses weighted residuals to compute the charge distribution along a thin wire. Use your answers from Part (1) to fill the moment matrix and then solve the system. Generalize your program so that the user can specify the following conditions as input arguments:

   - The number N of sub-divisions to use for computing the charge distribution.
   - The wire radius a and length L.
   - The weight function w_m and basis function u_n to apply when running the simulation.

6. To verify your code, write a script that runs simulations under the following conditions:

   - $L = 1$ m, $a = 1$ mm. Rectangular basis function and dirac delta weighting function.
   - $L = 1$ m, $a = 10$ mm. Rectangular basis function and dirac delta weighting function.
   - $L = 2$ m, $a = 1$ mm. Rectangular basis function and rectangular weighting function.
   - $L = 1$ m, $a = 1$ mm. Triangular basis function and rectangular weighting function.

   Turn in a separate plot of your simulated charge distributions under each case. Remember to clearly label your axes and use correct units! Also make sure that your plot properly represents the basis functions used by the simulation. Experiment with N by yourself to determine an ideal value for each case.

7. **EXTRA CREDIT MAYBE? OR 6340 students only?** Choose one of the above wire conditions and try to simulate it by using your 2D FDM code. After computing the voltages at every point in space, use the Poisson equation $\nabla^2 V = \rho$ to compute the charge distribution along the wire. Compare your result against your simulation using weighted residuals. Discuss any differences you notice between simulation time and the resultant charge distributions.

Helpful Hints

- Be careful not to divide your simulation into too many segments. The closer the segments are to each other, the more our thin-wire approximation fails.
- When numerically computing $g_n(x)$ and wmgn(x), do not use any more than 10-20 sub-intervals with the numerical integration. This will significantly speed up your simulation time.
- The diagonal elements in your moment matrix should dominate the other elements. If not, then there is something wrong.
- The total charge on your wire should be on the order of *pico* Coulombs.
- When using delta basis functions, multiply the result by h to represent a charge distribution over a segment of that size.

3. GRADING BREAKDOWN

- Answers to Part 1: 50 pts.
- Correct results for Part 2: 50 pts.
- Include introduction and summary sections in your report. Organize your source code and properly document each function or script.
- Total: 100 pts.