DERIVATION OF RADIATING BOUNDARY CONDITIONS:

WAVE EQUATION:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u = 0 \qquad (1)$$

THIS CAN BE WRITTEN IN 2-D:

$$\left.\begin{aligned} &u_{xx} + u_{yy} - \frac{u_{tt}}{c^2} = 0 \\ &L = D_x^2 + D_y^2 - D_t^2 \end{aligned}\right\} \quad Lu = 0 \quad (2)$$

WHERE

$$D_x^2 = \partial^2/\partial x^2 \;, \quad D_y^2 = \partial^2/\partial y^2 \;, \quad D_t^2 = \frac{1}{c^2}\partial^2/\partial t^2$$

$$u_{xx} = \frac{\partial^2 E}{\partial x^2}\;, \text{ etc} \qquad u = E$$

EQUATION (2) CAN BE FACTORED INTO ⊕ ⊖ TRAVELLING WAVES.

$$Lu = L^+ L^- u = 0$$

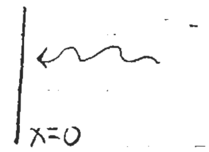
WHERE

$$L^- = D_x - D_t\sqrt{1-S^2}\;, \qquad L^+ = D_x + D_t\sqrt{1-S^2}$$

$$S = \frac{D_y}{D_t}$$

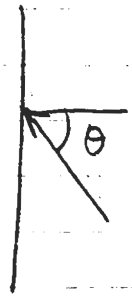
ENQUIST + MAJDA SHOW THAT A $-x$ TRAVELLING WAVE WILL BE ABSORBED IF

$$L^- u\Big|_{x=0} = 0 .$$



APPROXIMATIONS OF L^- ARE MADE... (TAYLOR EXPANSION)

<u>FIRST ORDER:</u>



$$\sqrt{1-s^2} \sim 1$$

$$L^- \sim \frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}$$

(3) $$\boxed{\left(\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}\right) u\Big|_{x=0} = 0}$$

$$\Gamma = \frac{\cos\theta - 1}{\cos\theta + 1}$$

$\sim 17\%$ FOR $45°$

0% FOR NORMAL INCIDEN

<u>SECOND ORDER:</u>

$$\sqrt{1-s^2} \sim 1 - \frac{s^2}{2}$$

$$L^- \sim \frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}\left(1 - \frac{1}{2}\left(\frac{\partial/\partial y}{\frac{1}{c_0}\partial/\partial t}\right)^2\right)$$

MULTIPLY L^- BY $\frac{1}{c_0}\frac{\partial}{\partial t}$

$$\hat{L}^- \cong \frac{1}{c_0^2}\frac{\partial^2}{\partial x\,\partial t} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} + \frac{1}{2}\frac{\partial^2}{\partial y^2}$$

(4) $$\boxed{\left(\frac{1}{C_0^2}\frac{\partial^2}{\partial x \partial t} - \frac{1}{C_0^2}\frac{\partial^2}{\partial t^2} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\right) u \Big|_{x=0} = 0}$$

$$\Gamma = -\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right|^2 \quad \begin{matrix} \sim 3\% & \text{FOR } 45^\circ \\ 0 & \text{FOR } 90^\circ \end{matrix}$$

EQ$^{\underline{n}}$ (12) OF MUR IS 3-D VERSION OF THIS.

* NOTE THAT [1], [2], [3] ARE FOR NORMALIZED MEDIA WHERE $C_0 = 1$. [1], [3] ARE 3-D. [2] IS 2-D.

<u>3rd ORDER:</u>
SEE [3], [4]

HOW ARE THESE PROGRAMMED IN FDTD?
EXAMPLE: <u>1st ORDER</u> 2D (3)

$$\left(\frac{\partial}{\partial x} - \frac{1}{C_0}\frac{\partial}{\partial t}\right) E_z = 0$$

$$\begin{matrix} \bullet & & \bullet & \bullet \\ E_z(0) & & E_z(1/2) & E_z(1) \end{matrix}$$

$$\left[\frac{\dfrac{E_z^{n+1}(1) - E_z^{n+1}(0)}{dx} + \dfrac{E_z^{n}(1) - E_z^{n}(0)}{dx}}{2}\right] - \frac{1}{C_0}\left[\frac{\dfrac{E_z^{n+1}(1) - E_z^{n}(1)}{dt} + \dfrac{E_z^{n+1}(0) - E_z^{n}(0)}{dt}}{2}\right] = 0$$

$$E_z^{n+1}(0)\left[-\frac{1}{dx}-\frac{1}{c_o dt}\right]+E_z^n(0)\left[\frac{-1}{dx}+\frac{1}{c_o dt}\right]$$
$$+E_z^{n+1}(1)\left[\frac{1}{dx}-\frac{1}{c_o dt}\right]+E_z^n(1)\left[\frac{1}{dx}+\frac{1}{c_o dt}\right]=0$$

$$E_z^{n+1}(0)=E_z^n(1)+\frac{c_o dt-dx}{c_o dt+dx}\left[E_z^{n+1}(1)-E_z^n(0)\right]$$

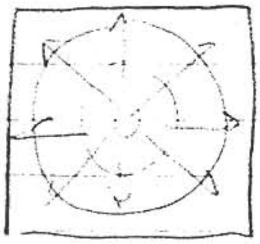
THIS IS MUR EQ^N (15).
* NOTE THAT MUR (16) IS PROGRAMMED IN FDTD 2D PROGRAM: (PAGE 8

$$\left.\begin{aligned}EDL(1,J,T)&=Ez(0,J,T)\\ EDL(2,J,T)&=Ez(1,J,T)\end{aligned}\right\}\text{ LEFT BOUNDARY}$$

$T=1\rightarrow n$
$T=2\rightarrow n-1$

* PRACTICAL APPLICATION:

BC MAY BE USED IN REGION WHERE RADIATION FIELDS EXIST. THIS IS GENERALLY .25 TO 1.0 λ FROM SCATTERER. (5-7 CELLS WITH $dx=\lambda/20$)



BECAUSE WE ASSUME SPHERICAL OUTGOING WAVE, CORNERS ARE WORST FOR REFLECTIONS SINCE THEY ARE ~ 45° ANGLE INCIDENCE.

BC MUST BE REDERIVED FOR EXPANDING GRID.