

ECE 5340/6340

The Finite Difference Frequency Domain Method Applied to Waveguides

The FDFD Method has been used to solve Laplace's and Poisson's Equations:

$$\nabla^2 \phi = 0$$

$$\nabla^2 \phi = \rho_v / \epsilon$$

The 5-point stencil is used to evaluate the Laplacian Operator (∇^2):

$$\nabla^2 \phi = (\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}) / h^2$$

This can also be used to evaluate the Scalar Wave Equation (Helmholtz Equation):

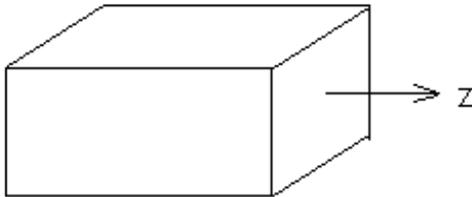
$$\nabla^2 \phi + k^2 \phi = 0$$

Where

$\phi = Ez$ for TM modes

$\phi = Hz$ for TE modes

and all fields vary as $e^{j(\omega t - \beta z)}$



k is the wave number and $k^2 = \omega^2 \mu \epsilon - \beta^2$

The propagation constant is : $\beta = (\omega/c_0) \sqrt{1 - (f_c / f)^2}$ rad/m

Waveguides have several different modes. The mode number indicates the number of half-sine variations of ϕ in the x-y directions. For instance, a TM_{10} mode would have 1 half sine wave of change along the x-direction (Ez is zero on the x-edges and maximum in the center. It is constant in the y-direction). Multiple modes can exist in a waveguide at a given time. The higher the frequency, the more modes can propagate.

A waveguide's mode is determined by two things:

- a) The frequency
- b) The feed system (which must be able to establish a given mode)

The frequency must be ABOVE the cutoff frequency f_c of a given mode in order for the mode to propagate.

The field distribution in the waveguide can often be found as an analytical solution of the Helmholtz equation applying the boundary conditions. But if it cannot be found analytically, FDFD is a good method of solving this.

The steps to apply the FDFD method to the Helmholtz Equation:

- 1) Divide the cross section of the waveguide (x-y plane) into a mesh
- 2) Apply 5-point difference formula

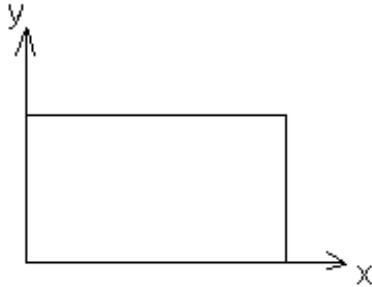
$$\nabla^2 \phi + k^2 \phi$$

$$= (\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}) / h^2 + k^2 \phi_{i,j} = 0$$

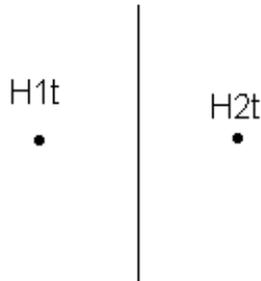
$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - (4 - hk^2) \phi_{i,j} = 0$$

Boundary Conditions

TM = transverse magnetic: E_z is tangential to metal boundaries
 Use Dirichlet BC : $\phi = E_z = 0$ at boundary.



TE = transverse electric: H_z is tangential to metal boundaries
 $\mathbf{n} \times (\mathbf{B}_1 - \mathbf{B}_2) = \mathbf{J}_s = 0$
 $H_{1t} = H_{2t}$



Neumann BC: $\partial\phi / \partial n = 0$

Derive SOR:

$$R_{ij} = [\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - (4 - hk^2)\phi_{i,j}] / (4 - h^2 k^2)$$

$$\phi^{k+1}_{i,j} = \phi^k_{i,j} + \omega R_{ij}$$

But we also don't know k .

So, we GUESS k , and iterate 3 or 4 times, then apply the Raleigh Formula to find k :

$$k = \frac{\int_S \phi \nabla^2 \phi \, dS}{\int_S \phi^2 \, dS}$$

$$k = \frac{\int_S \phi_{i,j} (\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}) / h^2 \, dS}{\int_S \phi_{i,j}^2 \, dS}$$

Find the surface integrals using numerical integration.

Use this new value of k and continue updating ϕ .

Continue until k converges. ϕ will also be converged.

$$R_{ij} = [\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - (4 - hk^2)\phi_{i,j}] / (4 - h^2k^2)$$

$$\phi^{k+1}_{i,j} = \phi^k_{i,j} + \omega R_{ij}$$