

Many numerical simulation problems have several different kinds of materials (and therefore different dielectrics) in them. The microstrip simulation you will be doing has a substrate and a superstrate and/or air above it.

Poisson's Equation assumes uniform dielectric  $\epsilon$ :  $\nabla^2 \phi = -q_v/\epsilon$   
so it can't be used for dielectric interfaces.

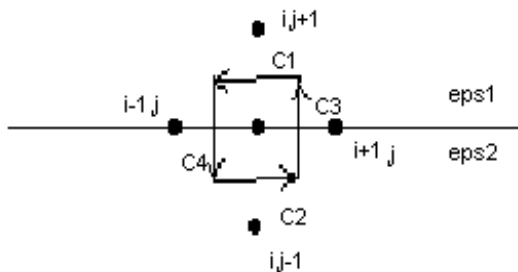
GO back to the basic equation used to derive Poisson's equation (you'll find this in most basic EM books):

$$\nabla \cdot (\epsilon \nabla \phi) = -Q \text{ (total charge)}$$

Apply the Divergence Theorem (also found in your EM book):

$$\int_V \nabla \cdot (\epsilon \nabla \phi) dv = \oint_S \epsilon \nabla \phi \cdot d\vec{S} = -Q = -\int_S q_v dS = -q_v h^2$$

This represents the rate of change of potential across a closed surface. The surface is a 3D box, and the 2D cut through this box is shown below. So what we need is the derivative of potential across this boundary... the derivative defined going out of the box (outward from the box, not outward from the page).



So, let's find the outward going derivative. We will do this over the whole closed surface as shown below. We can assume the ends cancel out, because they are symmetric. (The whole 2D model is symmetric in the 3<sup>rd</sup> dimension in and out of the page.) Now the surface integration becomes a closed contour integration. The contour integral is the sum of the derivative of the potential on C1+C2+C3+C4 shown above.

$$\oint_S = \oint_C + \oint_{\text{ends}}$$

Because front end cancels  
back end



And here it is.... Use Central Difference to define the outward going normal derivative of the potential at every point on the contour shown above. Multiply by the permittivity as shown in the equation. And multiply by dl (which is h). Each term is shown below:

$$\oint_C \epsilon \frac{\partial \phi}{\partial n} dl = \left( \frac{\phi_{i,j+1} - \phi_{i,j}}{h} \right)_{C1} h \epsilon_1 + \left( \frac{\phi_{i,j-1} - \phi_{i,j}}{h} \right)_{C2} h \epsilon_2$$

$$+ \left( \frac{\phi_{i+1,j} - \phi_{i,j}}{h} \right)_{C3} \left( \frac{h}{2} \epsilon_1 + \frac{h}{2} \epsilon_2 \right) + \left( \frac{\phi_{i-1,j} - \phi_{i,j}}{h} \right)_{C4} \left( \frac{h}{2} \epsilon_1 + \frac{h}{2} \epsilon_2 \right) = -q_v h^2$$

Rearranging terms and dividing by  $\epsilon_o$  :

$$\epsilon_{r1} \phi_{i,j+1} + \epsilon_{r2} \phi_{i,j-1} + \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right) \phi_{i-1,j} + \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right) \phi_{i+1,j}$$

$$- 4 \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right) \phi_{i,j} = - \frac{q_v h^2}{\epsilon_o}$$

Notes: In practice, you take each potential and multiply by the effective dielectric at that point (averaging at interfaces). This is not particularly surprising, but we didn't originally ASSUME averaging. We DID assume linear change in  $\phi$  when we applied the central difference formula to obtain the derivatives. That is what gave us the effective averaging in the formula.

Prepare for SOR:

$$\epsilon_{r1} \phi_{i,j+1} + \epsilon_{r2} \phi_{i,j-1} + \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right) \phi_{i-1,j} + \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right) \phi_{i+1,j} - 4 \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right) \phi_{i,j}$$

$$= \frac{-q_v h^2}{\epsilon_o} = b_{ij}$$

Solve for  $\phi_{i,j}$  letting  $\epsilon_{av} = \left( \frac{\epsilon_{r1} + \epsilon_{r2}}{2} \right)$

$$\phi^{(k+1)}_{i,j} = \frac{1}{4\epsilon_{av}} \left[ \epsilon_{r1} \phi_{i,j+1} + \epsilon_{r2} \phi_{i,j-1} + \epsilon_{av} \phi_{i-1,j} + \epsilon_{av} \phi_{i+1,j} + \frac{q_v h^2}{\epsilon_o} \right]$$

We are assuming here that we have updated (k+1) values for all of our potentials. We don't, of course. Some of the potentials will be at (k) and some at (k+1), depending on where you are in your grid. But as the iterations progress, they get closer and closer, and it won't matter in the end.

Apply definition of residual

$$\begin{aligned}
 R_{ij}^{(k)} &= \phi_{i,j}^{(k+1)} - \phi_{i,j}^{(k)} \\
 &= \frac{1}{4\epsilon_{av}} \left[ \epsilon_{r1} \phi_{i,j+1} + \epsilon_{r2} \phi_{i,j-1} + \epsilon_{av} \phi_{i-1,j} + \epsilon_{av} \phi_{i+1,j} + \frac{q_v h^2}{\epsilon_o} \right] - \phi_{i,j}^{(k)} \\
 &= \frac{1}{4\epsilon_{av}} \left[ \epsilon_{r1} \phi_{i,j+1} + \epsilon_{r2} \phi_{i,j-1} + \epsilon_{av} \phi_{i-1,j} + \epsilon_{av} \phi_{i+1,j} - 4\epsilon_{av} \phi_{i,j} + \frac{q_v h^2}{\epsilon_o} \right]
 \end{aligned}$$

Then solve using SOR

$$\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + \omega R_{ij}^{(k)}$$

Perfect dielectrics have  $q_v=0$ , so most of the time there will be no 'b' term in the residual. There will be a 'b' vector ( $Ax=b$ ), caused by boundary conditions.

## Summary:

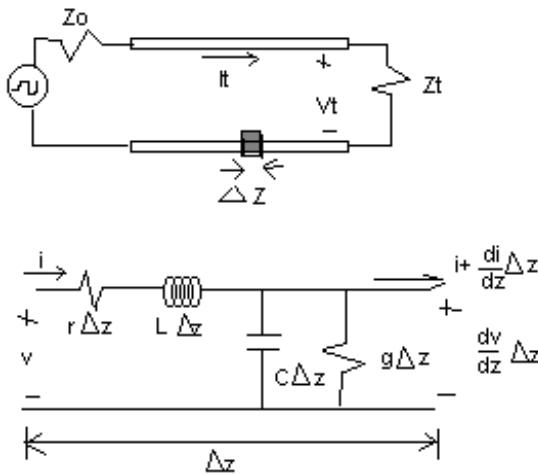
So what do you know about FDFD now?

- How to apply the standard FDFD stencil. Also how to apply it when you have symmetry or multiple dielectrics.
- How to apply boundary conditions. Dirichlet (just set the appropriate points on the grid to the boundary condition) or Neumann (add an extra set of points and write the boundary condition equation for those points, solve for them after the regular stencils are calculated).
- How to solve FDFD using SOR (just write the appropriate  $R_{ij}$  and apply the SOR 'equation').
- From all of this, you can find the potentials (voltages) in the grid. Neat. You can plot them (check out 'pcolor' in Matlab), print them, hang them on your fridge. Now... what ELSE can you do with them?

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## APPLICATION OF FDFD TO TRANSMISSION LINES

We would like to calculate the impedance and velocity of propagation ( $V_p$ ) of a transmission line.



How can we apply transmission line theory with FDFD?

- Can't use Lumped element approach because wavelength is approx. size of the FDFD element.
- Use incremental approach instead.

We know that the characteristic impedance,  $Z_o$ , is a function of the inductance and capacitance of a transmission line:

$$Z_o = \sqrt{\frac{L}{C}}$$

The inductance just depends on the metallic shape/structure of the transmission line. It doesn't depend on the dielectric material. The capacitance depends on both the shape and dielectric. We have voltage from FDFD. We can't calculate inductance from that, but we can calculate capacitance:

$$C = q/V$$

So, let's find a way to calculate  $Z_o$  without needing to calculate inductance. We know we need two equations (We have two variables,  $L$  and  $C$ , so surely any transformation we will do will also end up with two variables). We can get these two things from doing two different capacitance calculations. The first one, for  $C$ , comes from our original simulation, with all of the dielectrics in place. The second one, for  $C_o$ , comes from the original shape, but with all of the dielectrics changed to air. Also recall that  $V_{po} = 1/(\sqrt{L_o C_o})$  And that inductance doesn't change with dielectric (so  $L = L_o$ ).

Then:

$$Z_o = \sqrt{\frac{LCo}{CCo}} = \frac{1}{V_{po} \sqrt{CCo}}$$

$$V_p = \frac{1}{\sqrt{LC}} = V_{po} \sqrt{\frac{Co}{C}}$$

$$V_{po} = \frac{1}{\sqrt{L_o C_o}} = 2.996 \times 10^8 \text{ m/s}$$

For TL in free space  $V_p = V_{po}$ .

$Co$  = Capacitance/unit length of TL without dielectric

$C$  = Capacitance/unit length of TL with dielectric

TO CALCULATE CAPACITANCE from numerical simulation:

$$C = q / V$$

$$q = \int_S \epsilon \cdot \bar{E} \cdot d\bar{S} = \int_S \epsilon (-\nabla \phi) \cdot d\bar{S} = \int_S \epsilon \left( \frac{-d\phi}{dn} \right) dS$$

$$\oint_S = \oint_C + \oint_{\cancel{AB}}$$

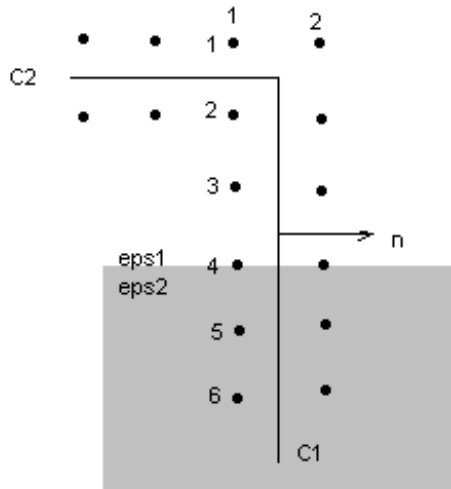
Because front end cancels  
back end



Use Numerical differentiation and integration:

$$dV/dn = (V_{n+1} + V_n) / dn$$

trapezoidal integration =  $h [ f(a)/2 + f(b)/2 + \text{sum}(f) ]$



$$dV/dn = (V_{j,2} - V_{j,1})/h_x$$

$$\oint_{C1} \epsilon \frac{\partial \phi}{\partial n} \cdot dl_1 = h_y \left\{ \begin{aligned} &\epsilon_1 \left[ \frac{\phi_{21} - \phi_{11}}{h_x} + \frac{\phi_{22} - \phi_{12}}{h_x} + \frac{\phi_{23} - \phi_{13}}{h_x} \right] \\ &+ \left( \frac{\epsilon_1 + \epsilon_2}{2} \right) \left[ \frac{\phi_{24} - \phi_{14}}{h_x} \right] + \epsilon_2 \left[ \frac{\phi_{25} - \phi_{15}}{h_x} + \frac{\phi_{26} - \phi_{16}}{h_x} \right] \end{aligned} \right\}$$

METHOD TO FIND characteristic impedance  $Z_o$ :

- 1) Simulate Microstripline (complete, with two dielectrics)
  - a) Find potential ( $\phi$ ) distribution
  - b) Integrate 4 line integrals around the top conductor to find charge on top conductor. (May use symmetry.)
  - c) Find  $Q_1$  and  $C_1$
- 2) Simulation Microstripline (air ONLY, no dielectrics)
  - a) Find potential ( $\phi$ ) distribution
  - b) Integrate 4 line integrals around the top conductor to find charge on top conductor. (May use symmetry.)
  - a) Find  $Q_o$  and  $C_o$
- 3) Calculate  $V_p$  and  $Z_o$ :
 
$$V_p = V_o \sqrt{(C_o/C)}$$

$$Z_o = 1 / (V_{po} \sqrt{(C C_o)})$$