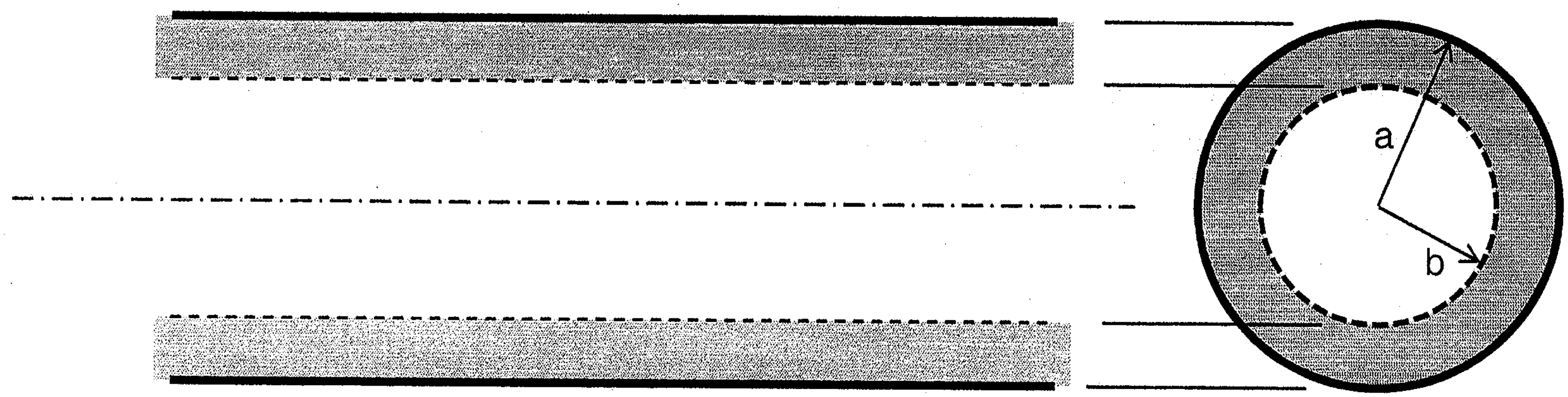
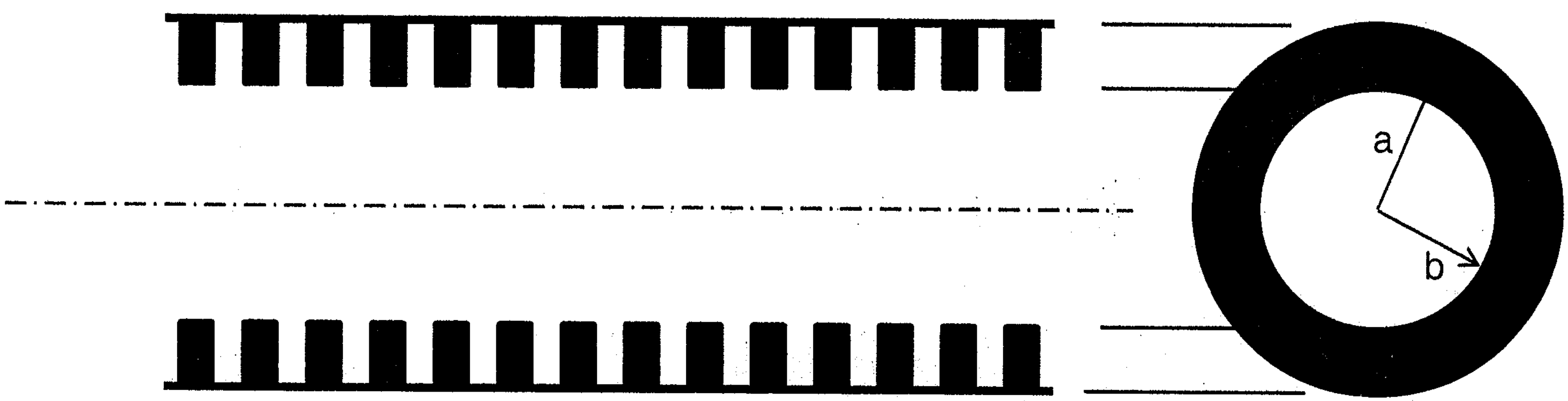
Comparison of HE11 mode in dielectric lined and corrugated waveguides

L3Com CSW, Jan 2009, J M Baird

The hybrid waveguide mode targeted here is the HE11 mode in a dielectric lined circular waveguide.



It is desired to understand the relationship between the loaded HE11 mode and the HE11-like mode in the corrugated type waveguide shown below.



Vector Potential Theory of Circular Waveguide Hybrid Modes

Waveguide vector potential theory begins with setting the electric or magnetic sources to zero. We can then find TE_z ($E_z = 0$)and TM_z ($H_z = 0$) solutions using the z-components of the vector potentials $\mathbf{F} = [0,0,F_z]$ and $\mathbf{A} = [0,0,A_z]$ to get

$$\mathbf{E}^{TE} = \frac{-1}{\varepsilon}\nabla\times\mathbf{F} = \frac{-1}{\varepsilon}\begin{bmatrix} \frac{1}{\rho}\partial_\varphi F_z \\ \partial_\rho F_z \\ 0 \end{bmatrix} \quad ; \quad \mathbf{H}^{TE} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{F} = \frac{1}{j\omega\mu\varepsilon}\begin{bmatrix} \partial_\rho\partial_z F_z \\ \frac{1}{\rho}\partial_\varphi\partial_z F_z \\ \beta_\rho^2 F_z \end{bmatrix}$$

$$\mathbf{H}^{TM} = \frac{-1}{\varepsilon}\nabla\times\mathbf{A} = \frac{-1}{\varepsilon}\begin{bmatrix} \frac{1}{\rho}\partial_\varphi A_z \\ \partial_\rho A_z \\ 0 \end{bmatrix} \quad ; \quad \mathbf{E}^{TM} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{A} = \frac{1}{j\omega\mu\varepsilon}\begin{bmatrix} \partial_\rho\partial_z A_z \\ \frac{1}{\rho}\partial_\varphi\partial_z A_z \\ \beta_\rho^2 A_z \end{bmatrix}$$

The scalar functions F_z and A_z satisfy the Helmholtz wave equation which can be separated into three ordinary differential equations: a Bessel equation in $(\beta_\rho\rho)$, a trig equation in $(m\phi)$ and a second trig equation in $(\beta_z z)$. Thus, F_z and A_z are each made up of the product of three separated functions with a common separation constraint equation:

$$\beta_\rho^2 + \beta_z^2 = k^2 \equiv \left(\frac{\omega}{c}\right)^2 \mu_r\varepsilon_r$$

Hybrid modes are required to match the boundary conditions on dielectric loaded waveguides where the fields are obtained by summing the TE_z and TM_z waveguide fields.

The separated functions for the hybrid solutions are:

$$F_z = F_{z0}\, R(\rho)\,\Phi(\varphi)\,Z(z) \quad \text{and} \quad A_z = A_{z0}\, R(\rho)\,\Phi(\varphi)\,Z(z)$$

For an azimuthal index "m" the general separated functions are as follows where we have used [A's & B's] for the constant coefficients in one case and [C's & D's] in the other.

<u>For the potential function F_z use:</u>

$$R_m \equiv A_\rho J_m(\beta_\rho \rho) + B_\rho Y_m(\beta_\rho \rho)$$
$$\Phi_m \equiv A_\varphi \cos(m\varphi) + B_\varphi \sin(m\varphi)$$
$$Z_m \equiv A_z \exp(-j\beta_z z) + B_z \exp(+j\beta_z z)$$

<u>For the potential function A_z :</u>

$$R_m \equiv C_\rho J_m(\beta_\rho \rho) + D_\rho Y_m(\beta_\rho \rho)$$
$$\Phi_m \equiv C_\varphi \cos(m\varphi) + D_\varphi \sin(m\varphi)$$
$$Z_m \equiv C_z \exp(-j\beta_z z) + D_z \exp(+j\beta_z z)$$

First, TE_z Modes (The primes on $[R,\Phi,Z]$ respectively indicate partials WRT $[\rho,\phi,z]$);

$$\mathbf{E}_m^{TE} = \left(\frac{-F_{0m}}{\varepsilon}\right)\begin{bmatrix} R_m \frac{1}{\rho}\Phi'_m Z_m \\ -R'_m \Phi_m Z_m \\ 0 \end{bmatrix}$$

$$\mathbf{H}_m^{TE} = \left(\frac{F_{0m}}{j\,\omega\mu\varepsilon}\right)\begin{bmatrix} R'_m \Phi_m Z'_m \\ R_m \frac{1}{\rho}\Phi'_m Z'_m \\ \beta_\rho^2 R_m \Phi_m Z_m \end{bmatrix}$$

$$R_m \equiv A_\rho J_m(\beta_\rho \rho) + B_\rho Y_m(\beta_\rho \rho)$$
$$\Phi_m \equiv A_\varphi \cos(m\varphi) + B_\varphi \sin(m\varphi)$$
$$Z_m \equiv A_z \exp(-j\beta_z z) + B_z \exp(+j\beta_z z)$$

Primes on Bessel Functions by convention indicate derivatives WRT the arguments.

$$R'_m \equiv \beta_\rho\left(A_\rho J'_m(\beta_\rho \rho) + B_\rho Y'_m(\beta_\rho \rho)\right)$$
$$\Phi'_m \equiv m\left(-A_\varphi \sin(m\varphi) + B_\varphi \cos(m\varphi)\right)$$
$$Z'_m \equiv -j\beta_z\left(A_z \exp(-j\beta_z z) - B_z \exp(+j\beta_z z)\right)$$

For computational reasons, we re-define the field separation functions :

$$\mathbf{E}_m^{TE} = \left(\frac{-F_{0m}}{\varepsilon}\right)\begin{bmatrix} R_m \frac{1}{\rho}\Phi'_m Z_m \\ -R'_m \Phi_m Z_m \\ 0 \end{bmatrix} = \left(\frac{-F_{0m}}{\varepsilon}\right)\begin{bmatrix} \frac{m}{\rho} R_m \tilde{\Phi}_m Z_m \\ -\beta_\rho \tilde{R}_m \Phi_m Z_m \\ 0 \end{bmatrix} = \left(\frac{F_{0m}}{\varepsilon a}\right)\begin{bmatrix} -\left(\frac{ma}{\rho}\right) R_m \tilde{\Phi}_m Z_m \\ (\beta_\rho a)\tilde{R}_m \Phi_m Z_m \\ 0 \end{bmatrix} \equiv \mathbf{E}_{0m}^{TE}\mathbf{e}_m^{TE}$$

$$\mathbf{H}_m^{TE} = \left(\frac{F_{0m}}{j\omega\mu\varepsilon}\right)\begin{bmatrix} R'_m \Phi_m Z'_m \\ R_m \frac{1}{\rho}\Phi'_m Z'_m \\ \beta_\rho^2 R_m \Phi_m Z_m \end{bmatrix} = \left(\frac{F_{0m}}{\varepsilon}\right)\left(\frac{1}{j\omega\mu}\right)\begin{bmatrix} (-j\beta_z\beta_\rho)\tilde{R}_m \Phi_m \tilde{Z}_m \\ \left(\frac{-j\beta_z m}{\rho}\right) R_m \tilde{\Phi}_m \tilde{Z}_m \\ \beta_\rho^2 R_m \Phi_m Z_m \end{bmatrix} = \left(\frac{F_{0m}}{\varepsilon a}\right)\left(\frac{\beta_z}{\omega\mu}\right)\begin{bmatrix} (-\beta_\rho a)\tilde{R}_m \Phi_m \tilde{Z}_m \\ \left(\frac{-ma}{\rho}\right) R_m \tilde{\Phi}_m \tilde{Z}_m \\ \frac{(\beta_\rho a)^2}{\beta_z a} R_m \Phi_m Z_m \end{bmatrix} \equiv \mathbf{E}_{0m}^{TE} Y_{0z}^{TE} h_m^{TE}$$

$$R_m \equiv A_\rho J_m(\beta_\rho \rho) + B_\rho Y_m(\beta_\rho \rho) \qquad \tilde{R}_m \equiv A_\rho J'_m(\beta_\rho \rho) + B_\rho Y'_m(\beta_\rho \rho)$$

$$\Phi_m \equiv A_\varphi \cos(m\varphi) + B_\varphi \sin(m\varphi) \qquad \tilde{\Phi}_m \equiv -A_\varphi \sin(m\varphi) + B_\varphi \cos(m\varphi)$$

$$Z_m \equiv A_z \exp(-j\beta_z z) + B_z \exp(+j\beta_z z) \qquad \tilde{Z}_m \equiv A_z \exp(-j\beta_z z) - B_z \exp(+j\beta_z z)$$

The Constraint on Separation Constants : $\beta_{\rho,[m,n]}^2 + \beta_{z,[m,n]}^2 = k^2 \equiv \left(\frac{\omega}{c}\right)^2 \varepsilon_r \mu_r$

Indices m & n are azimuthal & radial indices, **Use brackets [m,n] for TEz modes**

TEz Field expansion in terms of **RE-DEFINED** functions:

$$\mathbf{E}_m^{TE} \equiv \mathbf{E}_{0m}^{TE}\mathbf{e}_m^{TE} = \left(\frac{F_{0m}}{\varepsilon a}\right)\begin{bmatrix} -\left(\frac{ma}{\rho}\right) R_m \tilde{\Phi}_m Z_m \\ (\beta_\rho a)\tilde{R}_m \Phi_m Z_m \\ 0 \end{bmatrix}$$

$$= \left(\frac{F_{0m}}{\varepsilon a}\right)\begin{bmatrix} \left(\frac{-ma}{\rho}\right)[A_\rho J_m(\beta_\rho\rho) + B_\rho Y_m(\beta_\rho\rho)][-A_\varphi \sin(m\varphi) + B_\varphi \cos(m\varphi)][A_z \exp(-j\beta_z z) + B_z \exp(+j\beta_z z)] \\ (\beta_\rho a)[A_\rho J'_m(\beta_\rho\rho) + B_\rho Y'_m(\beta_\rho\rho)][A_\varphi \cos(m\varphi) + B_\varphi \sin(m\varphi)][A_z \exp(-j\beta_z z) + B_z \exp(+j\beta_z z)] \\ 0 \end{bmatrix}$$

$$\mathbf{H}_m^{TE} \equiv \mathbf{E}_{0m}^{TE} Y_{0z}^{TE} h_m^{TE} \equiv \left(\frac{F_{0m}}{\varepsilon a}\right)\left(\frac{\beta_z}{\omega\mu}\right)\begin{bmatrix} (-\beta_\rho a)\tilde{R}_m \Phi_m \tilde{Z} \\ \left(\frac{-ma}{\rho}\right) R_m \tilde{\Phi}_m \tilde{Z} \\ \frac{(\beta_\rho a)^2}{\beta_z a} R_m \Phi_m Z \end{bmatrix}$$

$$= \left(\frac{F_{0m}}{\varepsilon a}\right)\left(\frac{\beta_z}{\omega\mu}\right)\begin{bmatrix} (-\beta_\rho a)[A_\rho J'_m(\beta_\rho\rho) + B_\rho Y'_m(\beta_\rho\rho)][A_\varphi \cos(m\varphi) + B_\varphi \sin(m\varphi)][A_z \exp(-j\beta_z z) - B_z \exp(+j\beta_z z)] \\ \left(\frac{-ma}{\rho}\right)[A_\rho J_m(\beta_\rho\rho) + B_\rho Y_m(\beta_\rho\rho)][-A_\varphi \sin(m\varphi) + B_\varphi \cos(m\varphi)][A_z \exp(-j\beta_z z) - B_z \exp(+j\beta_z z)] \\ \left(\frac{(\beta_\rho a)^2}{\beta_z a}\right)[A_\rho J_m(\beta_\rho\rho) + B_\rho Y_m(\beta_\rho\rho)][A_\varphi \cos(m\varphi) + B_\varphi \sin(m\varphi)][A_z \exp(-j\beta_z z) + B_z \exp(+j\beta_z z)] \end{bmatrix}$$

Continuing Re-definition of Field Separation Functions for **TMz Modes** :

$$\mathbf{H}_m = \left(\frac{A_{0m}}{\mu}\right)\begin{bmatrix} R_m \frac{1}{\rho}\Phi'_m Z_m \\ -R'_m \Phi_m Z_m \\ 0 \end{bmatrix} = \left(\frac{A_{0m}}{\mu}\right)\begin{bmatrix} \frac{m}{\rho} R_m \tilde{\Phi}_m Z_m \\ -\beta_\rho \tilde{R}_m \Phi_m Z_m \\ 0 \end{bmatrix} = \left(\frac{A_{0m}}{\mu a}\right)\begin{bmatrix} \left(\frac{ma}{\rho}\right) R_m \tilde{\Phi}_m Z_m \\ -(\beta_\rho a) \tilde{R}_m \Phi_m Z_m \\ 0 \end{bmatrix} \equiv \mathbf{H}_{0m}\mathbf{h}_m$$

$$\mathbf{E}_m = \left(\frac{A_{0m}}{j\omega\mu\varepsilon}\right)\begin{bmatrix} R'_m \Phi_m Z'_m \\ R_m \frac{1}{\rho}\Phi'_m Z'_m \\ \beta_\rho^2 R_m \Phi_m Z_m \end{bmatrix} = \left(\frac{A_{0m}}{\mu}\right)\left(\frac{1}{j\omega\varepsilon}\right)\begin{bmatrix} (-j\beta_z\beta_\rho) \tilde{R}_m \Phi_m \tilde{Z}_m \\ \left(\frac{-j\beta_z m}{\rho}\right) R_m \tilde{\Phi}_m \tilde{Z}_m \\ \beta_\rho^2 R_m \Phi_m Z_m \end{bmatrix} = \left(\frac{A_{0m}}{\mu a}\right)\left(\frac{\beta_z}{\omega\varepsilon}\right)\begin{bmatrix} (-\beta_\rho a) \tilde{R}_m \Phi_m \tilde{Z}_m \\ \left(\frac{-ma}{\rho}\right) R_m \tilde{\Phi}_m \tilde{Z}_m \\ \frac{(\beta_\rho a)^2}{\beta_z a} R_m \Phi_m Z_m \end{bmatrix} \equiv \mathbf{H}_{0m} Z_{0z} \mathbf{e}_m$$

$$R_m \equiv C_\rho J_m(\beta_\rho \rho) + D_\rho Y_m(\beta_\rho \rho) \qquad \tilde{R}_m \equiv C_\rho J'_m(\beta_\rho \rho) + D_\rho Y'_m(\beta_\rho \rho)$$

$$\Phi_m \equiv C_\varphi \cos(m\varphi) + D_\varphi \sin(m\varphi) \qquad \tilde{\Phi}_m \equiv -C_\varphi \sin(m\varphi) + D_\varphi \cos(m\varphi)$$

$$Z_m \equiv C_z \exp(-j\beta_z z) + D_z \exp(+j\beta_z z) \qquad \tilde{Z}_m \equiv C_z \exp(-j\beta_z z) - D_z \exp(+j\beta_z z)$$

Constraint on Separation Constants : $\beta^2_{\rho,(m,n)} + \beta^2_{z,(m,n)} = k^2 \equiv \left(\frac{\omega}{c}\right)^2 \varepsilon_r \mu_r$

Indices m & n are azimuthal & radial indices; **Use Parenthesis (m,n) for TMz modes**

TMz Field expansion in terms of **RE-DEFINED** functions:

$$\mathbf{H}_{0m}\mathbf{h}_m \equiv \left(\frac{A_{0m}}{\mu a}\right)\begin{bmatrix} \left(\frac{ma}{\rho}\right) R_m \tilde{\Phi}_m Z_m \\ -(\beta_\rho a)\tilde{R}_m \Phi_m Z_m \\ 0 \end{bmatrix}$$

$$= \left(\frac{A_{0m}}{\mu a}\right)\begin{bmatrix} \left(\frac{ma}{\rho}\right)[C_\rho J_m(\beta_\rho \rho) + D_\rho Y_m(\beta_\rho \rho)][-C_\varphi \sin(m\varphi) + D_\varphi \cos(m\varphi)][C_z \exp(-j\beta_z z) + D_z \exp(+j\beta_z z)] \\ -(\beta_\rho a)[C_\rho J'_m(\beta_\rho \rho) + D_\rho Y'_m(\beta_\rho \rho)][C_\varphi \cos(m\varphi) + D_\varphi \sin(m\varphi)][C_z \exp(-j\beta_z z) + D_z \exp(+j\beta_z z)] \\ 0 \end{bmatrix}$$

$$\mathbf{H}_{0m} Z_{0z} \mathbf{e}_m \equiv \left(\frac{A_{0m}}{\mu a}\right)\left(\frac{\beta_z}{\omega\varepsilon}\right)\begin{bmatrix} (-\beta_\rho a)\tilde{R}_m \Phi_m \tilde{Z} \\ \left(\frac{-ma}{\rho}\right) R_m \tilde{\Phi}_m \tilde{Z} \\ \frac{(\beta_\rho a)^2}{\beta_z a} R_m \Phi_m Z \end{bmatrix}$$

$$= \left(\frac{A_{0m}}{\mu a}\right)\left(\frac{\beta_z}{\omega\varepsilon}\right)\begin{bmatrix} (-\beta_\rho a)[C_\rho J'_m(\beta_\rho \rho) + D_\rho Y'_m(\beta_\rho \rho)][C_\varphi \cos(m\varphi) + D_\varphi \sin(m\varphi)][C_z \exp(-j\beta_z z) - D_z \exp(+j\beta_z z)] \\ \left(\frac{-ma}{\rho}\right)[C_\rho J_m(\beta_\rho \rho) + D_\rho Y_m(\beta_\rho \rho)][-C_\varphi \sin(m\varphi) + D_\varphi \cos(m\varphi)][C_z \exp(-j\beta_z z) - D_z \exp(+j\beta_z z)] \\ \left(\frac{(\beta_\rho a)^2}{\beta_z a}\right)[C_\rho J_m(\beta_\rho \rho) + D_\rho Y_m(\beta_\rho \rho)][C_\varphi \cos(m\varphi) + D_\varphi \sin(m\varphi)][C_z \exp(-j\beta_z z) + D_z \exp(+j\beta_z z)] \end{bmatrix}$$

Specializing and Summing Fields

for use in dielectric Loaded waveguide analysis.

$$\mathbf{E}_m = \mathbf{E}_m^{TE} + \mathbf{E}_m^{TM} = \left(\frac{F_{0m}}{\varepsilon a}\right)\begin{bmatrix} \left(\frac{-ma}{\rho}\right)[A_\rho J_m(\beta_\rho\rho) + B_\rho Y_m(\beta_\rho\rho)][\sin(m\varphi)][\exp(-j\beta_z z))] \\ (\beta_\rho a)[A_\rho J'_m(\beta_\rho\rho) + B_\rho Y'_m(\beta_\rho\rho)][\cos(m\varphi)][\exp(-j\beta_z z)] \\ 0 \end{bmatrix} +$$

$$\left(\frac{A_{0m}}{\mu a}\right)\left(\frac{\beta_z}{\omega\varepsilon}\right)\begin{bmatrix} (-\beta_\rho a)[C_\rho J'_m(\beta_\rho\rho) + D_\rho Y'_m(\beta_\rho\rho)][\sin(m\varphi)][\exp(-j\beta_z z)] \\ \left(\frac{-ma}{\rho}\right)[C_\rho J_m(\beta_\rho\rho) + D_\rho Y_m(\beta_\rho\rho)][\cos(m\varphi)][\exp(-j\beta_z z)] \\ \left(\frac{(\beta_\rho a)^2}{\beta_z a}\right)[C_\rho J_m(\beta_\rho\rho) + D_\rho Y_m(\beta_\rho\rho)][\sin(m\varphi)][\exp(-j\beta_z z)] \end{bmatrix}$$

$$\mathbf{H}_m = \mathbf{H}_m^{TE} + \mathbf{H}_m^{TM} = \left(\frac{F_{0m}}{\varepsilon a}\right)\left(\frac{\beta_z}{\omega\mu}\right)\begin{bmatrix} (-\beta_\rho a)[A_\rho J'_m(\beta_\rho\rho) + B_\rho Y'_m(\beta_\rho\rho)][\sin(m\varphi)][\exp(-j\beta_z z)] \\ \left(\frac{-ma}{\rho}\right)[A_\rho J_m(\beta_\rho\rho) + B_\rho Y_m(\beta_\rho\rho)][\cos(m\varphi)][\exp(-j\beta_z z)] \\ \left(\frac{(\beta_\rho a)^2}{\beta_z a}\right)[A_\rho J_m(\beta_\rho\rho) + B_\rho Y_m(\beta_\rho\rho)][\sin(m\varphi)][\exp(-j\beta_z z)] \end{bmatrix} +$$

$$\left(\frac{A_{0m}}{\mu a}\right)\begin{bmatrix} \left(\frac{ma}{\rho}\right)[C_\rho J_m(\beta_\rho\rho) + D_\rho Y_m(\beta_\rho\rho)][-\sin(m\varphi)][\exp(-j\beta_z z)] \\ -(\beta_\rho a)[C_\rho J'_m(\beta_\rho\rho) + D_\rho Y'_m(\beta_\rho\rho)][\cos(m\varphi)][\exp(-j\beta_z z)] \\ 0 \end{bmatrix}$$

Zonal Field Forms for Boundary Field Matching via Radial Transition Matrices

The previous specialized fields can be written in the following general form where the G's contain the Bessel functions (which depend on radial position) and any coefficient parameters related to the media region, and the Q_α's contain the $[\phi,z]$ dependencies.

$$\begin{bmatrix} \vec{E}_m \\ \vec{H}_m \end{bmatrix} = \overline{\overline{G}}\vec{Q}_\varphi \equiv \begin{bmatrix} G_{11} & G_{12} & . & G_{14} \\ G_{21} & & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ G_{61} & . & . & G_{66} \end{bmatrix} \begin{bmatrix} A_\varphi \\ B_\varphi \\ C_\varphi \\ D_\varphi \end{bmatrix}$$

This is close to the form required to create radial "transition" matrices which automatically match the $[\phi,z]$ components of the fields at the radial dielectric boundaries. First strip out the field components to be matched:

$$\vec{F}(\rho) = \begin{bmatrix} E_\varphi \\ E_z \\ H_\varphi \\ H_z \end{bmatrix} = \overline{\overline{G}}(\rho)\vec{Q}_\varphi \equiv \begin{bmatrix} G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{51} & G_{52} & G_{53} & G_{54} \\ G_{61} & G_{62} & G_{63} & G_{64} \end{bmatrix} \begin{bmatrix} A_\varphi \\ B_\varphi \\ C_\varphi \\ D_\varphi \end{bmatrix}$$

The transition matrix T is then formed by evaluating this equation at the two boundaries of a radial zone (ρ_1, ρ_2) and eliminating the coefficient vector $\mathbf{Q}_\phi$. Thus,

$$\begin{aligned} \vec{F}(\rho_1) &= \overline{\overline{G}}(\rho_1)\vec{Q}_\varphi \\ \vec{F}(\rho_2) &= \overline{\overline{G}}(\rho_2)\vec{Q}_\varphi \end{aligned} \quad \Rightarrow \quad \vec{F}(\rho_1) = \overline{\overline{G}}(\rho_1)\overline{\overline{G}}^{-1}(\rho_2)\vec{F}(\rho_2) \equiv \overline{\overline{T}}_\rho \vec{F}(\rho_2)$$