Stability, Convergence, and Robustness of Adaptive Systems

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Abstract

The thesis addresses three issues of prime importance to adaptive systems: the stability under ideal conditions, the convergence of the adaptive parameters, and the robustness to modeling errors and to measurement noise. New results are presented, as well as simplified and unified proofs of existing results.

First, some identification algorithms are reviewed, and their stability and parameter convergence properties are established. Then, a new, input error, direct adaptive control scheme is presented. It is an alternate scheme to the output error scheme of Narendra, Lin, and Valavani, which does not require a strictly positive real condition on the reference model, or overparametrization when the high-frequency gain is unknown. Useful lemmas are presented and unified stability proofs are derived for the input and output error schemes, as well as for an indirect adaptive control scheme. The results show that all three schemes have similar stability and convergence properties. However, the input error and the indirect schemes have the advantage of leading to a linear error equation, and of allowing for a useful separation of identification and control.

The parameter convergence of the adaptive schemes is further analyzed using averaging techniques, assuming that the reference input possesses some stationarity properties, and that the adaptation gain is sufficiently small. It is shown that the nonautonomous adaptive systems can be approximated by autonomous systems, thereby considerably simplifying the analysis. In particular, estimates of the rates of exponential convergence of the parameters are obtained for the linear identification scheme, as well for the nonlinear adaptive control scheme. The approach is particularly useful, as it leads to a frequency domain analysis, and has a vast potential of interesting extensions.

The Rohrs examples of instability in the presence of unmodeled dynamics are reviewed. A connection between exponential convergence and robustness is established in a general framework. The result is applied to a model reference adaptive control scheme, and stresses the importance of the persistency of excitation condition for robustness.

Robustness margins of the adaptive control scheme are also obtained. The mechanisms of instability observed in the Rohrs examples are explained, and methods to improve robustness are briefly investigated.

To my parents

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List of Abbreviations

BIBO	Bounded-Input Bounded-Output
BIBS	Bounded-Input Bounded-State
LHP	Left-Half Plane
LTI	Linear Time Invariant
RHP	Right-Half Plane
SISO	Single-Output Single-Output
SPR	Strictly Positive Real
PE	Persistently Exciting
UCO	Uniformly Completely Observable

Introduction

Motivation - Objectives

This thesis studies stability, convergence, and robustness properties of identification and adaptive control systems, generally referred to as adaptive systems. Identification methods are of considerable importance to several areas, especially control, communications, and signal processing. The performance of any control system for example depends crucially on the accuracy of the model used to design it. Adaptive control, a direct aggregation of identification and control, has current and potential applications to a large number of systems with parametric uncertainty, and/or time-varying dynamics. Among these, we find flight and space vehicles, robotic manipulators, chemical processes, and many others. Therefore, our main motivation in studying adaptive systems is a large number of current, and potential applications.

With this motivation comes the need for better understanding of the dynamical behavior of adaptive systems. Although such systems have been studied at least since the 1960's, the field still lacks methods of analysis comparable to the classical methods for nonadaptive linear time invariant systems. This has limited practical applications, especially in adaptive control, despite a significant research effort. Many of the existing results concern either algorithms, structures, or specific applications, and much still needs to be understood concerning the dynamic behavior of adaptive systems, and their robustness to uncertainties. This is another motivation for this work.

Our goal is to study the dynamic properties of adaptive systems: their stability and convergence under ideal conditions, and their ability to maintain stability in the presence of noise and modeling errors. We do not intend to find the optimal algorithm, given a specific problem, but to develop techniques to analyze and compare various algorithms. As much as possible, we wish to derive new results on stability, convergence, and robustness that are sufficiently general to be applied to a large class of algorithms. Adaptive systems are essentially time-varying, and usually non linear systems. This accounts for much of the difficulty encountered in analyzing them, and causes the need to develop appropriate methods of analysis. We do not want to restrict our attention to simplifications based on either eliminating these characteristics by considering constant or periodic inputs, or by linearizing the adaptive system around some nominal trajectory.

The number of existing identification and adaptive control schemes is considerable, due to the variety of possible choices during their derivations. For simplicity, we will limit the plant under consideration to be single-input, single-output, linear time invariant, continuous time and deterministic. The identification schemes are parametric, and recursive, that is with parameters that are updated as time progresses. Their application to adaptive control is therefore immediate. The adaptive control schemes considered are model reference adaptive control schemes.

Finally, our objective in this thesis is to present a reasonably self-contained treatment of stability, convergence, and robustness issues in adaptive systems. We present results in a unified framework, sometimes simplifying proofs of existing results. Our purpose there is to make this work accessible to a wider audience, and clarify the connections between various adaptive schemes, and between different topics. For example, we will show the connections between apparently very different direct and indirect adaptive control schemes, and between input error and output error adaptive control schemes. We will also show connections between robustness and convergence, and between convergence results obtained by exact methods and by averaging.

Review of Literature

We do not intend to review here the considerable literature in identification and adaptive control, but to show the evolution of the research connected to the topics of the thesis.

Model reference adaptive control techniques appear to have been first proposed for the control of aircraft and spacecraft in the work of Whitaker (1959), and Osburn, Whitaker, and Kezer (1959). Their purpose was to design a self-adapting control system such that, over the whole flight envelope, the controlled aircraft would behave in a satisfactory way, as described by a reference model. Adaptation algorithms were based on an analysis of the sensitivity of the output error to adjustable parameters, followed by a steepest descent search. The resulting update law was called the MIT rule, and was the topic of much research such as in Donalson and Leondes (1963a & b), Horrocks (1964), Dymock et al (1965), and White (1966). These papers already showed the difficulties encountered by the authors in dealing with the dynamics of these nonlinear time-varying systems, and their attempts to reduce their complexity and analyze them with conventional LTI techniques.

The lack of stability proofs, and instabilities observed on examples induced the redesign of the model reference adaptive control system by Parks (1966). This design was supported by a stability proof based on Lyapunov techniques. It also marked the beginning of a more rigorous approach, accounting for the nonlinearity and time variation of the adaptive system. The scheme was further extended by Monopoli (1974), Narendra and Valavani (1978), and Landau (1979). Stability proofs for the general case appeared simultaneously in Narendra, Lin, and Valavani (1980), Morse (1980), and in the discrete-time literature, in Goodwin, Ramadge, and Caines (1980). In addition to Lyapunov analysis, these papers introduced the use of functional analysis techniques (such as studied in Desoer and Vidyasagar (1975)) to establish stability of the adaptive systems.

The stability and convergence of identifiers was independently addressed as early as in Lion (1967), and proofs of exponential convergence were derived by Sondhi and Mitra (1976), Anderson (1977), Kreisselmeier (1977), Morgan and Narendra (1977a & b). These results were then extended to the adaptive control case by Boyd and Sastry (1983) and (1984).

The robustness issue appeared with a controversial paper by Rohrs et al (1982) and (1985). The example led to further discussion by Astrom (1983), Astrom (1984), Chen and Cook (1984), Reidle, Cyr, and Kokotovic (1984), and Rohrs (1985). Anderson (1985) showed the existence of unstable bursting phenomena in adaptive control systems, even without unmodeled dynamics. Besides the controversy related to the discussion, a significant research effort was started, that led to robustness analyses, and to methods of improvement of robustness in work by Kreisselmeier and Narendra (1982), Peterson and Narendra (1982), Anderson and Johnstone (1983), Bodson and Sastry (1984), Kosut and Johnson (1984), Sastry (1984), Ortega, Praly, and Landau (1985), Kreisselmeier (1986), Kreisselmeier and Anderson (1986), Narendra and Annaswamy (1986), and others. A significant step was the introduction of averaging methods to analyze instabilities of adaptive systems in the work of Astrom (1984) and (1985), Riedle and Kokotovic (1985), Kokotovic, Riedle, and Praly (1985), Riedle and Kokotovic (1986), Mareels et al (1986), the book by Anderson et al (1986), and Fu and Sastry (1986). Averaging methods were also introduced for the analysis of convergence of adaptive systems in Fu, Bodson, and Sastry (1985), and Bodson et al (1986).

Contributions of the Thesis

The topics of stability, convergence, and robustness are addressed successively for identification and control algorithms. Along these lines, the thesis brings the following contributions

1) In chapter 3, we present a new continuous time, input error adaptive control algorithm. Since the connections of this scheme to known schemes, especially in the discrete time literature, are strong, the main interest is in unifying known results, and explaining some discrepancies between continuous time and discrete time results. We also present stability proofs for direct adaptive control schemes, and for an indirect scheme. Thereby, we show that their stability properties are essentially identical. Although the stability proofs rely strongly on known results, some new proofs are provided for intermediary lemmas, and the presentation of the stability proofs is original and unified for the various schemes. In particular, the stability proof for the indirect adaptive control scheme, without persistency of excitation conditions is original.

2) A significant contribution of our research is the development of averaging methods for adaptive systems, and the derivation of results justifying the use of these methods to determine convergence rates of adaptive systems. We review in chapter 4 results obtained with other coworkers, and published in Fu, Bodson, and Sastry (1985) and Bodson et al (1986). This research is original in providing convergence rates estimates, using a frequency domain analysis in the linear as well as in the nonlinear cases.

3) The connection between exponential convergence and robustness is established in a general result in chapter 5. This result is then used to establish robustness margins of a specific adaptive control system. Although the result is more conceptual than practical, it

gives useful insight into mechanisms of instabilities found by Rohrs et al (1982). It also shows the strong connection between the exponential convergence of the nominal system and the robustness of the actual system.

4) Besides the original contributions of the thesis, we concentrate on presenting a reasonably self-contained analysis of the three main topics of the thesis. Therefore, some known schemes are reviewed, and some known results are presented in a unified framework. Sometimes, original or reviewed proofs are given, such as in the study of the convergence of identifiers for example. We hope these results will be useful to the reader unfamiliar with the literature in that area.

Overview of the Thesis

Chapter 1 introduces the notation followed throughout the thesis, and presents basic definitions and results to be used in the sequel.

Chapter 2 reviews a basic identification scheme for SISO LTI plants, with several identification algorithms. General properties of the identification algorithms are established, and the stability of the identifier is proved under general conditions. Conditions for exponential parameter convergence are also derived, with an analysis of convergence rates and factors influencing them. Finally, similar properties are established for strictly positive real error equations arising in other identification and adaptive control schemes.

Chapter 3 presents three model reference adaptive control schemes, among which is an original input error scheme. The connections between them and their respective advantages are discussed. The stability of the adaptive control systems is proved, together with the convergence of the output error to zero. Exponential parameter convergence is also deduced for the adaptive control algorithms, under condiditons similar to the identification schemes.

Chapter 4 introduces averaging techniques for the approximation of adaptive systems by autonomous (i.e. time invariant) systems. Several useful results are established, together with a general framework serving as a basis for further developments. The methods are applied to study parameter convergence properties of identification and adaptive control schemes. In particular, estimates of the exponential convergence rates are obtained, together with their dependence on the frequency content of the reference input.

Chapter 5 reviews the Rohrs examples of instability in adaptive control systems, and studies the mechanisms of instability. The relationship between exponential convergence and robustness is analyzed, and guaranteed robustness margins are obtained. More refined methods to guarantee robustness are required however, and the chapter concludes with a review of some proposed methods to improve robustness of adaptive systems.

Finally, we present some general conclusions resulting from this work, and suggestions for future research.

Chapter 1 Preliminaries

This chapter introduces the notation used in this work, as well as some basic definitions and results. The notation used in the adaptive systems literature varies widely. We elected to use a notation close to that of Narendra and Valavani (1978), and Narendra, Lin and Valavani (1980), since many connections exist between this work, and their results. We will refer to texts such as Desoer and Vidyasagar (1975), Vidyasagar (1978) for standard results, and this chapter will concentrate on the definitions used most often, and on non-standard results.

1.1 Notation

Lower case letters are used to denote scalars or vectors. Upper case letters are used to denote matrices, operators, or sets. When $u(t)$ is a function of time, $\hat{u}(s)$ denotes its Laplace transform. Without ambiguity, we will drop the arguments, and simply write u and $\hat{u}$. Rational transfer functions of linear time invariant (LTI) systems will be denoted using upper case letters, e.g. $\hat{H}(s)$ or $\hat{H}$. Polynomials in s will be denoted using lower case letters, for example $\hat{n}(s)$, or simply $\hat{n}$. Thus, we may have $\hat{H} = \hat{n} / \hat{d}$, where $\hat{H}$ is both the ratios of polynomials in s, and an operator in the Laplace transform domain. Sometimes, the time domain and the Laplace transform domain will be mixed, and parentheses will determine the sense to be made of an expression. For example, $\hat{H}(u)$ or $\hat{H}\,\hat{u}$ is the output of the LTI system $\hat{H}$ with input u. $\hat{H}(u)v$ is $\hat{H}(u)$ multiplied by v in the time domain, while $\hat{H}(uv)$ is $\hat{H}$ operating on the product $u(t)v(t)$.

1.2 L_p Spaces, Norms

We denote by $|x|$ the absolute value of x if x is a scalar, and the euclidean norm of x if x is a vector. The notation $\|\ \|$ will be used to denote the induced norm of an operator, in particular the induced matrix norm

$$\|A\| = \sup_{|x|=1} |A\,x| \tag{1.2.1}$$

and for functions of time, the notation is used for the L_p norm

$$\|u\|_p = \left(\int_0^\infty |u(\tau)|^p \, d\tau\right)^{1/p} \tag{1.2.2}$$

When p is omitted, $\|u\|$ denotes the L_2 norm. Truncated functions are defined as

$$\begin{aligned} f_s(t) &= f(t) \qquad t \leqslant s \\ &= 0 \qquad\quad\; t > s \end{aligned} \tag{1.2.3}$$

and the extended L_p spaces are defined by

$$L_{pe} = \{ f \mid \text{ for all } s < \infty, f_s \in L_p \} \tag{1.2.4}$$

For example, e^t does not belong to L_∞, but $e^t \in L_{\infty e}$. When $u \in L_{\infty e}$, we have

$$\|u_t\|_\infty := \sup_{\tau \leqslant t} |u(\tau)| \tag{1.2.5}$$

Note that $f \in L_2$ does not imply that $f \to 0$ as $t \to \infty$. This is not even guaranteed if f is bounded. However, note the following results.

Lemma 1.2.1 Barbalat Lemma

If $f(t)$ is a uniformly continuous function, such that $\lim_{t \to \infty} \int_0^t f(\tau) \, d\tau$ exists and is finite

Then $f(t) \to 0$ as $t \to \infty$

Proof of Lemma 1.2.1 cf Popov (1973) p. 211.

Corollary 1.2.2

If $f, \dot{f} \in L_\infty$, and $f \in L_2$

Then $f(t) \to 0$ as $t \to \infty$

Proof of Corollary 1.2.2

Direct from lemma 1.2.1, since $f, \dot{f}$ bounded implies that f is uniformly continuous. □

1.3 Positive Definite Matrices

Positive definite matrices are frequently found in work on adaptive systems. We summarize here several facts that will be useful. We consider real matrices. Recall that a scalar u, or a function of time $u(t)$, is said to be *positive* if $u \geq 0$, or $u(t) \geq 0$ for all t. It is *strictly positive* if $u > 0$, or, for some $\alpha > 0$, $u(t) \geq \alpha$ for all t. A square matrix $A \in R^{n \times n}$ is *positive semidefinite* if $x^T A x \geq 0$ for all x. It is *positive definite* if, for some $\alpha > 0$, $x^T A x \geq \alpha x^T x = \alpha |x|^2$ for all x. Equivalently, we can require $x^T A x \geq \alpha$ for all x such that $|x| = 1$. The matrix A is negative semidefinite if $-A$ is positive semidefinite and we write $A \geq B$ if $A - B \geq 0$. Note that a matrix can be neither positive semidefinite, nor negative semidefinite, so that this only establishes a partial order of the matrices.

The eigenvalues of a positive semidefinite matrix lie in the closed right-half plane (RHP), while those of a positive definite matrix lie in the open RHP. If $A \geq 0$ and $A = A^T$, then A is *symmetric positive semidefinite*. In particular, if $A \geq 0$, then $A + A^T$ is symmetric positive semidefinite. The eigenvalues of a symmetric positive semidefinite matrix are all real and positive. Such matrix also has n orthogonal eigenvectors, so that we can decompose A as

$$A = U^T \Lambda U \tag{1.3.1}$$

where U is the matrix of eigenvectors satisfying $U^T U = I$ (i.e. U is a *unitary* matrix), and Λ is a diagonal matrix composed of the eigenvalues of A. The square root matrix $\Lambda^{1/2}$ is a diagonal matrix composed of the square roots of the eigenvalues of A, and

$$A^{1/2} = U^T \Lambda^{1/2} U \tag{1.3.2}$$

is the square root matrix of A, with $A = A^{1/2} . A^{1/2}$ and $(A^{1/2})^T = A^{1/2}$.

If $A \geq 0$ and $B \geq 0$, then $A + B \geq 0$ but it is not true in general that $A.B \geq 0$. However, if A, B are symmetric positive semidefinite matrices, then AB - although not necessarily symmetric, or positive semidefinite - has all eigenvalues real positive.

Another property of symmetric positive semidefinite matrices, following from (1.3.1), is

$$\lambda_{\min}(A)|x|^2 \leq x^T A x \leq \lambda_{\max}(A)|x|^2 \tag{1.3.3}$$

This simply follows from the fact that $x^T A x = x^T U^T \Lambda U x = z^T \Lambda z$ and

$|z|^2 = z^T z = |x|^2$. We also have that

$$\|A\| = \lambda_{max}(A) \tag{1.3.4}$$

and, when A is positive definite

$$\|A^{-1}\| = 1/\lambda_{min}(A) \tag{1.3.5}$$

1.4 Stability of Dynamic Systems

This section is concerned with differential equations of the form

$$\dot{x} = f(t,x) \qquad x(t_0) = x_0 \tag{1.4.1}$$

where $x \in \mathbf{R}^n$, $t \geq 0$.

The system defined by (1.4.1) is said to be *autonomous*, or *time-invariant*, if f does not depend on t, and *non autonomous*, or *time-varying*, otherwise. It is said to be *linear* if $f(t,x) = A(t)x$ for some $A(.) : \mathbf{R}_+ \to \mathbf{R}^{n \times n}$, and *nonlinear* otherwise.

We will always assume that $f(t,x)$ is *piecewise continuous* with respect to t. By this, we mean that there are only a finite number of discontinuity points in any compact set.

We define by B_h the closed ball of radius h centered at 0 in $\mathbf{R}^n$.

Properties will be said to be true

- *locally* if true for all x_0 in some ball B_h
- *globally* if true for all $x_0 \in \mathbf{R}^n$
- *in any closed ball* if true for all $x_0 \in B_h$, with h arbitrary
- *uniformly* if true for all $t_0 \geq 0$.

By default, properties will be true locally.

Lipschitz Condition and Consequences

The function f is said to be *Lipschitz* in x if, for some $h > 0$, there exists $l \geq 0$ such that

$$|f(t,x_1) - f(t,x_2)| \leq l\,|x_1 - x_2| \tag{1.4.2}$$

for all x_1, $x_2 \in B_h$, $t \geq 0$. The constant l is called the *Lipschitz constant*. This defines locally Lipschitz functions. Globally Lipschitz functions satisfy (1.4.2) for all x_1, $x_2 \in \mathbf{R}^n$, while functions that are Lipschitz in any closed ball satisfy (1.4.2) for all x_1, $x_2 \in B_h$, with l possibly depending on h. The Lipschitz property is by default assumed to be satisfied uniformly, i.e. l does not depend on t.

If f is Lipschitz in x, then it is continuous in x. On the other hand, if f has continuous and bounded partial derivatives in x, then it is Lipschitz. We denote

$$D_2 f := \left[\frac{\partial f_i}{\partial x_j} \right] \tag{1.4.3}$$

If $\| D_2 f \| \leqslant l$, then f is Lipschitz with constant l.

From the theory of ordinary differential equations (cf. Coddington and Levinson (1955)), it is known that f locally bounded, and f locally Lipschitz in x imply the existence and uniqueness of the solutions of (1.4.1) on some time interval (for as long as $x \in B_h$).

By definition, an *equilibrium point* x satisfies $f(t, x) = 0$ for all $t \geqslant 0$. We will often assume that, by change of coordinates, the equilibrium point is transformed to be $x = 0$. The following proposition gives bounds on the solutions of (1.4.1) when f is Lipschitz in x.

Proposition 1.4.1

If $x = 0$ is an equilibrium point of (1.4.1), f is Lipschitz in x with constant l, and is piecewise continuous with respect to t

Then the solution $x(t)$ of (1.4.1) satisfies

$$|x_0| e^{l(t-t_0)} \geqslant |x(t)| \geqslant |x_0| e^{-l(t-t_0)} \tag{1.4.4}$$

as long as $x(t)$ remains in B_h.

Proof of Proposition 1.4.1

Note that $|x|^2 = x^T x$ implies that

$$\left| \frac{d}{dt} |x|^2 \right| = 2|x| \left| \frac{d}{dt} |x| \right|$$

$$= 2 \left| x^T \frac{d}{dt} x \right| \leqslant 2|x| \left| \frac{d}{dt} x \right| \tag{1.4.5}$$

so that

$$\left| \frac{d}{dt} |x| \right| \leqslant \left| \frac{d}{dt} x \right| \tag{1.4.6}$$

Since f is Lipschitz

$$-l\,|x| \leqslant \frac{d}{dt}|x| \leqslant l\,|x| \tag{1.4.7}$$

and there exists a positive function $s(t)$ such that

$$\frac{d}{dt}|x| = -l\,|x| + s \tag{1.4.8}$$

Solving (1.4.8)

$$|x(t)| = |x_0| e^{-l(t-t_0)} + \int_0^t e^{-l(t-\tau)} s(\tau)\, d\tau$$

$$\geqslant |x_0| e^{-l(t-t_0)} \tag{1.4.9}$$

The other inequality follows similarly. □

Proposition 1.4.1 implies that solutions starting inside B_h will remain inside B_h for at least a finite time interval. Or, conversely, given a time interval, the solutions will remain in B_h provided that the initial conditions are sufficiently small. Also, f globally Lipschitz implies that $x \in L_{\infty e}$. Proposition 1.4.1 also says that x cannot tend to zero faster than exponentially.

The following lemma is an important result generalizing the well-known Bellman-Gronwall lemma (Bellman (1943)). The proof is similar to the proof of proposition 1.4.1, and is left to the appendix.

Lemma 1.4.2 Bellman-Gronwall Lemma

Let $x(.), a(.), u(.)$: $\mathbf{R}_+ \to \mathbf{R}_+$. Let $T \geqslant 0$.

If

$$x(t) \leqslant \int_0^t a(\tau) x(\tau) d\tau + u(t) \qquad \text{for all } t \in [0,T] \tag{1.4.10}$$

Then

$$x(t) \leq \int_0^t a(\tau)u(\tau)e^{\int_\tau^t a(\sigma)d\sigma} d\tau + u(t) \qquad \text{for all } t \in [0,T] \qquad (1.4.11)$$

When $u(.)$ is differentiable

$$x(t) \leq u(0)e^{\int_0^t a(\sigma)d\sigma} + \int_0^t \dot{u}(\tau)e^{\int_\tau^t a(\sigma)d\sigma} d\tau \qquad \text{for all } t \in [0,T] \qquad (1.4.12)$$

Proof of Lemma 1.4.2 in appendix

Stability Definitions

Definition Stability in the sense of Lyapunov

$x = 0$ is called a *stable* equilibrium point of (1.4.1), if for all $\epsilon > 0$, there exists $\delta > 0$ such that $x_0 \in B_\delta$ implies that the solution $x(t) \in B_\epsilon$ for all $t \geq t_0, t_0 \geq 0$.

Definition Asymptotic Stability

$x = 0$ is called an *asymptotically stable* equilibrium point of (1.4.1), if it is stable, and for all $x_0 \in B_h$, $t_0 \geq 0$, the solution $x(t) \to 0$ as $t \to \infty$ (i.e. $x = 0$ is *attractive*).

Definition Exponential Stability, Rate of Convergence

$x = 0$ is called an *exponentially stable* equilibrium point of (1.4.1) if there exist $m, \alpha > 0$ such that the solution $x(t)$ satisfies

$$|x(t)| \leq m\, e^{-\alpha(t - t_0)}|x_0| \qquad (1.4.13)$$

for all $x_0 \in B_h$, $t \geq t_0 \geq 0$. The constant α is called the *rate of convergence*.

Global exponential stability means that (1.4.13) is satisfied for any $x_0 \in \mathbf{R}^n$. Exponential stability in any closed ball is similar except that m and α may be a function of h. Exponential stability is assumed to be uniform with respect to t_0. It can be shown that uniform asymptotic stability is equivalent to exponential stability for linear systems (Vidyasagar (1978), p. 170), but it is not true in general.

Exponential Stability Theorems

We will pay special attention to exponential stability for two reasons. When considering the convergence of adaptive algorithms, exponential stability means convergence, and the rate of convergence is a useful measure of how fast estimates converge to their nominal values. In chapter 5, we will also observe that exponentially stable systems possess at least some tolerance to perturbations, and are therefore desirable in engineering applications.

The following theorem will be useful in proving several results, and relates exponential stability to the existence of a specific Lyapunov function.

Theorem 1.4.3 Converse Theorem of Lyapunov

Consider the system (1.4.1). Assume that f has continuous and bounded first partial derivatives in x, and is piecewise continuous in t for all $x \in B_h$, $t \geq 0$. Then, the following statements are equivalent

(a) $x = 0$ is an *exponentially stable* equilibrium point of (1.4.1)

(b) there exists a function $v(t, x)$, and some strictly positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that, for all $x \in B_h$, $t \geq 0$

$$\alpha_1 |x|^2 \leq v(t, x) \leq \alpha_2 |x|^2 \tag{1.4.14}$$

$$\left. \frac{dv(t, x)}{dt} \right|_{(1.4.1)} \leq -\alpha_3 |x|^2 \tag{1.4.15}$$

$$\left| \frac{\partial v(t, x)}{\partial x} \right| \leq \alpha_4 |x| \tag{1.4.16}$$

Comments

The derivative in (1.4.15) is a derivative taken along the trajectories of (1.4.1), that is

$$\left. \frac{dv(t, x)}{dt} \right|_{(1.4.1)} = \frac{\partial v(t, x)}{\partial t} + \frac{\partial v(t, x)}{\partial x} f(t, x) \tag{1.4.17}$$

This means that we consider x to be a function of t to calculate the derivative along the trajectories of (1.4.1) passing through x at t. It does *not* require of x to be the solution $x(t)$ of (1.4.1) starting at $x(t_0)$

Theorem 1.4.3 can be found in Krasovskii (1963) p. 60, and Hahn (1967) p. 273. It is known as one of the converse theorems. The proof of the theorem is constructive: it provides an explicit Lyapunov function $v(t,x)$. This is a rather unusual circumstance, and makes the theorem particularly valuable. In the proof, we derive explicit values of the constants involved in (1.4.14)-(1.4.16)

Proof of Theorem 1.4.3

(a) implies (b)

(i) Denote by $p(\tau, x, t)$ the solution of (1.4.1) starting at $x(t)$, t, and define

$$v(t,x) = \int_t^{t+T} |p(\tau, x, t)|^2 \, d\tau \tag{1.4.18}$$

where $T > 0$ will be defined in (ii). From the exponential stability and the Lipschitz condition

$$m\,|x|\,e^{-\alpha(\tau - t)} \geqslant |p(\tau, x, t)| \geqslant |x|\,e^{-l(\tau - t)} \tag{1.4.19}$$

and inequality (1.4.14) follows with

$$\alpha_1 := \left(1 - e^{-2lT}\right) / \, 2l \qquad \alpha_2 := m^2 \left(1 - e^{-2\alpha T}\right) / \, 2\alpha \tag{1.4.20}$$

(ii) Differentiating (1.4.18) with respect to t, we obtain

$$\frac{dv(t,x)}{dt} = |p(t+T, x, t)|^2 - |p(t, x, t)|^2 + \int_t^{t+T} \frac{d}{dt} \left(|p(\tau, x, t)|^2\right) d\tau \tag{1.4.21}$$

Note that d/dt is a derivative with respect to the *initial* time t, and is taken along the trajectories of (1.4.1). By definition of the solution p

$$p(\tau, x(t + \Delta t), t + \Delta t) = p(\tau, x(t), t) \tag{1.4.22}$$

for all Δt, so that the term in the integral is identically zero over $[t, t+T]$. The second term in the right-hand side of (1.4.21) is simply $|x|^2$, while the first is related to $|x|^2$ by the assumption of exponential stability. It follows that

$$\frac{dv(t,x)}{dt} \leqslant -\left(1 - m^2 e^{-2\alpha T}\right) |x|^2 \tag{1.4.23}$$

Inequality (1.4.15) follows, provided that $T > (1/\alpha) \ln m$, and

$$\alpha_3 := 1 - m^2 e^{-2\alpha T} \tag{1.4.24}$$

(iii) Differentiating (1.4.18) with respect to x_i, we have

$$\frac{\partial v(t,x)}{\partial x_i} = 2 \int_t^{t+T} \sum_{j=1}^{n} p_j(\tau,x,t) \frac{\partial p_j(\tau,x,t)}{\partial x_i} \, d\tau \tag{1.4.25}$$

Under the assumptions, the partial derivative of the solution with respect to the initial conditions satisfies

$$\frac{d}{d\tau}\left[\frac{\partial p_j(\tau,x,t)}{\partial x_i}\right] = \frac{\partial}{\partial x_i}\left[\frac{d}{d\tau}\, p_j(\tau,x,t)\right] = \frac{\partial}{\partial x_i}\left[f_j(\tau, p(\tau,x,t))\right]$$

$$= \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k}\bigg|_{\tau, p(\tau,x,t)} \bullet \frac{\partial p_k(\tau,x,t)}{\partial x_i} \tag{1.4.26}$$

(except possibly at points of discontinuity of $f(\tau,x)$). Denote

$$Q_{ij}(\tau,x,t) := \partial p_i(\tau,x,t) / \partial x_j \qquad A_{ij}(x,t) := \partial f_i(t,x) / \partial x_j \tag{1.4.27}$$

so that (1.4.26) becomes

$$\frac{d}{d\tau}\, Q(\tau,x,t) = A\,(p\,(\tau,x,t),\tau\,).Q(\tau,x,t) \tag{1.4.28}$$

Eqn (1.4.28) defines $Q(\tau,x,t)$, when integrated from $\tau = t$ to $\tau = t + T$, with initial conditions $Q(t,x,t) = I$. Thus, $Q(\tau,x,t)$ is the *transition matrix* associated with the time varying matrix $A(p\,(\tau,x,t),\tau)$. By assumption, $\|A(.,.)\| \leqslant k$ for some k, so that

$$\|Q(\tau,x,t)\| \leqslant e^{k(\tau - t)} \tag{1.4.29}$$

and, using the exponential stability again, (1.4.26) becomes

$$\left|\frac{\partial v(t,x)}{\partial x}\right| \leqslant 2 \int_t^{t+T} m\,|x|\,e^{(k-\alpha)(\tau-t)} d\tau \tag{1.4.30}$$

which is (1.4.16) if we define

$$\alpha_4 := 2m\,(e^{(k-\alpha)T} - 1\,) \,/\, (k - \alpha) \tag{1.4.31}$$

Note that the function $v(t,x)$ is only really defined for $x \in B_{h'}$ with $h' = h / m$, if we wish to guarantee that $p\,(\tau,x,t) \in B_h$ for all $\tau \geqslant t$. This is a technicality which will have no consequence.

(b) implies (a)

This direction is straightforward, using only (1.4.14)-(1.4.15), and we find

$$m := \left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} \qquad \alpha := \frac{1}{2}\frac{\alpha_3}{\alpha_2} \tag{1.4.32}$$

□

Comments

The Lyapunov function $v(t,x)$ can be interpreted as an average of the squared norm of the state along the solutions of (1.4.1). This approach is actually the basis of exact proofs of exponential convergence presented in sections 2.5-2.6 for identification algorithms. On the other hand, the approximate proofs presented in chapter 4 rely on methods for averaging *the differential system* itself. Then the norm squared of the state itself becomes a Lyapunov function, from which the exponential convergence can be deduced.

Theorem 1.4.3 is mostly useful to establish the existence of the Lyapunov function corresponding to exponentially stable systems. To establish exponential stability from a Lyapunov function, the following theorem will be more appropriate. Again, the derivative is to be taken along the trajectories of (1.4.1).

Theorem 1.4.4 Exponential Stability Theorem

If There exists a function $v(t,x)$, and strictly positive constants α_1, α_2, α_3, and δ, such that for all $x \in B_h$, $t \geqslant 0$

$$\alpha_1 |x|^2 \leqslant v(t,x) \leqslant \alpha_2 |x|^2 \tag{1.4.33}$$

$$\frac{d}{dt} v(t, x(t))\Big|_{(1.4.1)} \leqslant 0 \tag{1.4.34}$$

$$\int_t^{t+\delta} \frac{d}{d\tau} v(\tau, x(\tau))\Big|_{(1.4.1)} d\tau \leqslant -\alpha_3 |x(t)|^2 \tag{1.4.35}$$

Then $x(t)$ converges exponentially to 0.

Proof of Theorem 1.4.4

From (1.4.35)

$$v(t, x(t)) - v(t+\delta, x(t+\delta)) \geqslant (\alpha_3/\alpha_2)\, v(t, x(t)) \qquad (1.4.36)$$

for all $t \geqslant 0$, so that

$$v(t+\delta, x(t+\delta)) \leqslant (1-\alpha_3/\alpha_2)\, v(t, x(t)) \qquad \text{for all } t \geqslant 0 \qquad (1.4.37)$$

From (1.4.34)

$$v(t_1, x(t_1)) \leqslant v(t, x(t)) \qquad \text{for all } t_1 \in [t, t+\delta] \qquad (1.4.38)$$

Choose for t the sequence $t_0, t_0+\delta, t_0+2\delta, \ldots$ so that $v(t, x(t))$ is bounded by a staircase $v(t_0, x(t_0)), v(t_0+\delta, x(t_0+\delta)), \ldots$ where the steps are related in geometric progression through (1.4.36). It follows that

$$v(t, x(t)) \leqslant m_v\, e^{-\alpha_v (t-t_0)}\, v(t_0, x(t_0)) \qquad \text{for all } t \geqslant t_0 \geqslant 0 \qquad (1.4.39)$$

where

$$m_v = \frac{1}{(1-\alpha_3/\alpha_2)} \qquad \alpha_v = \frac{1}{\delta} \ln\left(\frac{1}{(1-\alpha_3/\alpha_2)}\right) \qquad (1.4.40)$$

Similarly

$$|x(t)| \leqslant m\, e^{-\alpha (t-t_0)} |x(t_0)| \qquad (1.4.41)$$

where

$$m = \left(\frac{\alpha_2}{\alpha_1} \frac{1}{1-\alpha_3/\alpha_2}\right)^{1/2} \qquad \alpha = \frac{1}{2\delta} \ln\left(\frac{1}{1-\alpha_3/\alpha_2}\right) \qquad (1.4.42)$$

□

Chapter 2 Identification

2.1 Identification Problem

In this chapter, we review some identification methods for single-input single-output (SISO) linear time invariant (LTI) systems. We concentrate our attention on *recursive* identification methods, where the estimates of the parameters are updated in real-time, thus leading naturally to adaptive control schemes in the following chapter.

Note that a polynomial in s is called *monic* if the coefficient of the highest power in s is 1, and *Hurwitz* if its roots lie in the open left-half plane. Rational transfer functions are called *stable* if their denominator polynomial is Hurwitz, and *minimum phase* if their numerator polynomial is Hurwitz. The *relative degree* of a transfer function is by definition the difference between the degrees of the denominator and numerator polynomials. A rational transfer function is called *proper* if its relative degree is at least 0, and *strictly proper* if its relative degree is at least 1.

We consider the identification problem of SISO LTI systems, given the following assumptions.

Assumptions

(A1) Plant Assumptions

the plant is a SISO LTI system, described by a transfer function

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{2.1.1}$$

where $\hat{r}(s)$ and $\hat{y}_p(s)$ are the Laplace transforms of the input and output of the plant respectively, $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are monic, coprime polynomials of degrees n and $m \leqslant n-1$ respectively (m is unknown).

(A2) Reference Input Assumptions

the input $r(.)$ is piecewise continuous, and bounded on $\mathbf{R}_+$.

The objective of the identifier is to obtain estimates of k_p and of the coefficients of the polynomials $\hat{n}_p(s)$ and $\hat{d}_p(s)$ from measurements of the input $r(t)$ and output $y_p(t)$ only. Note that we do not assume that $\hat{P}$ is stable.

2.2 Identifier Structure

The identifier structure presented in this section is similar to that of Kreisselmeier (1977). The transfer function $\hat{P}(s)$ can be explicitly written as

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{\alpha_n s^{n-1} + \cdots + \alpha_1}{s^n + \beta_n s^{n-1} + \cdots + \beta_1} \tag{2.2.1}$$

where the $2n$ coefficients $\alpha_1 \cdots \alpha_n$, and $\beta_1 \cdots \beta_n$ are unknown. This expression is a *parametrization* of the unknown plant, i.e. a model in which only a finite number of parameters are to be determined. For identification purposes, it is convenient to find an expresssion which depends linearly on the unknown parameters. For example, the expression

$$s^n \hat{y}_p(s) = (\alpha_n s^{n-1} + \cdots + \alpha_1)\hat{r}(s) - (\beta_n s^{n-1} + \cdots + \beta_1)\hat{y}_p(s) \tag{2.2.2}$$

is linear in the parameters α_i and β_i. However, it would require explicit differentiations to be implemented. To avoid this problem, we introduce a monic n th order polynomial denoted $\hat{\lambda}(s) = s^n + \lambda_n s^{n-1} + \cdots + \lambda_1$. This polynomial is assumed to be Hurwitz, but is otherwise arbitrary. Then, using (2.1.1)

$$\hat{\lambda}(s)\hat{y}_p(s) = k_p \hat{n}_p(s)\hat{r}(s) + (\hat{\lambda}(s) - \hat{d}_p(s))\hat{y}_p(s) \tag{2.2.3}$$

or, with (2.2.1)

$$\hat{y}_p(s) = \frac{\alpha_n s^{n-1} + \cdots \alpha_1}{\hat{\lambda}(s)} \hat{r}(s) + \frac{(\lambda_n - \beta_n) s^{n-1} + \cdots (\lambda_1 - \beta_1)}{\hat{\lambda}(s)} \hat{y}_p(s) \tag{2.2.4}$$

This expression is a new parametrization of the plant. Let

$$\begin{aligned} \hat{a}^*(s) &= \alpha_n s^{n-1} + \cdots \alpha_1 = k_p \hat{n}_p(s) \\ \hat{b}^*(s) &= (\lambda_n - \beta_n) s^{n-1} + \cdots (\lambda_1 - \beta_1) = \hat{\lambda}(s) - \hat{d}_p(s) \end{aligned} \tag{2.2.5}$$

so that the new representation of the plant can be written

$$\hat{y}_p(s) = \frac{\hat{a}^*(s)}{\hat{\lambda}(s)}\,\hat{r}(s) + \frac{\hat{b}^*(s)}{\hat{\lambda}(s)}\,\hat{y}_p(s) \tag{2.2.6}$$

The transfer function from $r \to y_p$ is given by

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \frac{\hat{a}^*(s)}{\hat{\lambda}(s) - \hat{b}^*(s)} \tag{2.2.7}$$

and it is easy to verify that this transfer function is $\hat{P}(s)$ when $\hat{a}^*(s)$ and $\hat{b}^*(s)$ are given by (2.2.5). Further, this choice is unique when $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are coprime: indeed, suppose that there existed $\hat{a}^*(s) + \delta\hat{a}(s)$, $\hat{b}^*(s) + \delta\hat{b}(s)$, such that the transfer function is still $k_p\,\hat{n}_p(s)/\,\hat{d}_p(s)$. The following equation would then have to be satisfied

$$\frac{\delta\hat{a}(s)}{\delta\hat{b}(s)} = -k_p\,\frac{\hat{n}_p(s)}{\hat{d}_p(s)} = -\hat{P}(s) \tag{2.2.8}$$

However, equation (2.2.8) has no solution since the degree of $\hat{d}_p$ is n, and $\hat{n}_p$, $\hat{d}_p$ are coprime, while the degree of $\delta\hat{b}$ is at most $n-1$.

State-Space Realization

A state-space realization of the above representation can be found by choosing $\Lambda \in \mathbf{R}^{n\times n}$, $b_\lambda \in \mathbf{R}^n$ in controllable canonical form such that $\det(sI - \Lambda) = \hat{\lambda}(s)$, and

$$(sI-\Lambda)^{-1}b_\lambda = \frac{1}{\hat{\lambda}(s)}\begin{pmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1}\end{pmatrix} \tag{2.2.9}$$

In analogy with (2.2.5), define

$$a^{*T} := (\alpha_1, \cdots \alpha_n) \qquad b^{*T} := (\lambda_1 - \beta_1, \cdots \lambda_n - \beta_n) \tag{2.2.10}$$

and the vectors $w_p^{(1)}(t)$, $w_p^{(2)}(t) \in \mathbf{R}^n$

$$\begin{aligned} \dot{w}_p^{(1)} &= \Lambda w_p^{(1)} + b_\lambda r \\ \dot{w}_p^{(2)} &= \Lambda w_p^{(2)} + b_\lambda y_p \end{aligned} \tag{2.2.11}$$

with initial conditions $w_p^{(1)}(0)$, $w_p^{(2)}(0)$. In Laplace transforms

$$\begin{aligned} \hat{w}_p^{(1)}(s) &= (sI-\Lambda)^{-1}b_\lambda\,\hat{r}(s) \\ \hat{w}_p^{(2)}(s) &= (sI-\Lambda)^{-1}b_\lambda\,\hat{y}_p(s) \end{aligned} \tag{2.2.12}$$

With this notation, the description of the plant (2.2.6) becomes

$$\hat{y}_p(s) = a^{*T}\, \hat{w}_p^{(1)}(s) + b^{*T}\, \hat{w}_p^{(2)}(s) \tag{2.2.13}$$

and, since the plant parameters are constant, the same expression is valid in the time domain

$$y_p(t) = a^{*T}\, w_p^{(1)}(t) + b^{*T}\, w_p^{(2)}(t) := \theta^{*T}\, w_p(t) \tag{2.2.14}$$

where

$$\theta^{*T} := (a^{*T}, b^{*T}) \in \mathbf{R}^{2n} \qquad w_p(t)^T := (w_p^{(1)T}(t), w_p^{(2)T}(t)) \in \mathbf{R}^{2n} \tag{2.2.15}$$

Eqns (2.2.10)-(2.2.14) define a realization of the new parametrization. The vector w_p is the *generalized state* of the plant, and has dimension $2n$. Therefore, the realization of $\hat{P}(s)$ is not minimal, but the unobservable modes are those of $\hat{\lambda}(s)$, and are all stable.

The vector θ^* is a vector of unknown parameters related linearly to the original plant parameters α_i, β_i by (2.2.10)-(2.2.15). Knowledge of a set of parameters is equivalent to the knowledge of the other, and each corresponds to one of the (equivalent) parametrizations. In the last form however, the plant output depends linearly on the unknown parameters, so that standard identification algorithms can be used. This plant parametrization is represented in figure 2.1.

Identifier Structure

The purpose of the identifier is to produce a recursive estimate $\theta(t)$ of the *nominal* parameter θ^*. Since r and y_p are available, we define the *observer*

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda r$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p \tag{2.2.16}$$

to reconstruct the states of the plant. The initial conditions in (2.2.16) are arbitrary. We also define the identifier signals

$$\theta^T(t) := (a^T(t), b^T(t)) \in \mathbf{R}^{2n} \qquad w^T(t) := (w^{(1)T}(t), w^{(2)T}(t)) \in \mathbf{R}^{2n} \tag{2.2.17}$$

By (2.2.11), (2.2.16), the *observer error* $w(t) - w_p(t)$ decays exponentially to zero, *even when the plant is unstable*. We note therefore that the generalized state of the plant $w_p(t)$ is such that it can be reconstructed from available signals, without knowledge of the plant parameters.

The plant output can be written

$$y_p(t) = \theta^{*T} w(t) \qquad (\epsilon) \tag{2.2.18}$$

where the notation (ϵ) is to remind one of the presence of an additive exponentially decaying term

$$\epsilon(t) = \theta^{*T} (w_p(t) - w(t)) \tag{2.2.19}$$

due to the initial conditions in the observer. We will first neglect the presence of the (ϵ) term, but later show that it does not affect the properties of the identifier.

In analogy with the expression of the plant output, the output of the identifier is defined to be

$$y_i(t) = \theta^T(t) w(t) \quad \in \mathbf{R} \tag{2.2.20}$$

We also define the *parameter error*

$$\phi(t) := \theta(t) - \theta^* \quad \in \mathbf{R}^{2n} \tag{2.2.21}$$

and the *identifier error*

$$e_1(t) := y_i(t) - y_p(t) = \phi^T(t) w(t) \qquad (\epsilon) \tag{2.2.22}$$

These signals will be used by the identification algorithm, and are represented in figure 2.2.

2.3 Linear Error Equation and Identification Algorithms

Many identification algorithms (cf. Eykhoff (1974), Ljung and Soderstrom (1983)) rely on a linear expression of of the form obtained above, that is

$$y_p(t) = \theta^{*T} w(t) \tag{2.3.1}$$

where $y_p(t)$, $w(t)$ are known signals, and θ^* is unknown. The vector $w(t)$ is usually called the *regressor* vector. With the expression of $y_p(t)$ is associated the standard *linear error equation*

$$e_1(t) = \phi^T(t) w(t) \tag{2.3.2}$$

We arbitrarily separated the identifier into an *identifier structure* and an *identification algorithm*. The identifier structure constructs the regressor w and other signals, related by the identifier error equation. The identification algorithm is defined by a differential equation, called the *update law*, of the form

$$\dot{\theta} = \dot{\phi} = F(y_p, e_1, \theta, w) \tag{2.3.3}$$

where F is a causal operator explicitly independent of θ^*, which defines the evolution of the identifier parameter θ.

2.3.1 Gradient Algorithms

The update law

$$\dot{\theta} = -g\, e_1 w \qquad g > 0 \tag{2.3.4}$$

defines the standard *gradient algorithm*. The right-hand side is proportional to the gradient of the output error squared, viewed as a function of ϕ, that is

$$\frac{\partial}{\partial \phi} \left(e_1^2(\phi) \right) = 2\, e_1 w \tag{2.3.5}$$

This update law can thus be seen as a *steepest descent* method. The parameter g is a *fixed*, strictly positive gain called the *adaptation gain*, and allows us to vary the rate of adaptation of the parameters. The initial condition $\theta(0)$ is arbitrary, but can be chosen to take any a priori knowledge of the plant parameters into account.

An alternative to this algorithm is the *normalized gradient algorithm*

$$\dot{\theta} = -g\, \frac{e_1 w}{1 + \gamma w^T w} \qquad g, \gamma > 0 \tag{2.3.6}$$

where g and γ are constants. This update law is equivalent to the previous update law, with w replaced by $w / \sqrt{1 + \gamma w^T w}$ in (2.3.2) and (2.3.4). The new regressor is thus a normalized form of w. The right-hand side of the differential equation (2.3.6) is globally Lipschitz in ϕ (using (2.3.2)), even when w is unbounded.

When the nominal parameter θ^* is known *a priori* to lie in a set $\Theta \in \mathbf{R}^{2n}$ (which we will assume to be closed, convex, and delimited by a smooth boundary), it is useful to modify the update law to take this information into account. For example, the *normalized gradient algorithm with projection* is defined by

$$\dot{\theta} = -g \frac{e_1 w}{1+\gamma w^T w} \qquad \theta \in \mathrm{int}(\Theta)$$

$$= \mathrm{Pr}\left(-g \frac{e_1 w}{1+\gamma w^T w}\right) \qquad \theta \in \partial\Theta \tag{2.3.7}$$

where intΘ and $\partial\Theta$ denote the interior and boundary of Θ, and Pr(z) denotes the projection of the vector z onto the hyperplane tangent to $\partial\Theta$ at θ.

The gradient algorithms, as well as the least-squares algorithms, can be used to identify the plant parameters with the identifier structure described in section 2.2. Using the normalized gradient algorithm for example, the practical implementation is as follows.

Identifier - Practical Implementation

Assumptions

(A1)-(A2)

Data

n

Input

$r(t), y_p(t) \in \mathbf{R}$

Output

$\theta(t), y_i(t) \in \mathbf{R}$

Internal Signals

$w(t) \in \mathbf{R}^{2n}$ $(w^{(1)}(t), w^{(2)}(t) \in \mathbf{R}^n)$

$\theta(t) \in \mathbf{R}^{2n}$ $(a(t), b(t) \in \mathbf{R}^n)$

$y_i(t), e_1(t) \in \mathbf{R}$

Initial conditions are arbitrary

Design Parameters

Choose

- $\Lambda \in \mathbf{R}^{n \times n}$, $b_\lambda \in \mathbf{R}^n$ in controllable canonical form such that

det$(sI - \Lambda) = \hat{\lambda}(s)$ is Hurwitz

- $g, \gamma > 0$

Identifier Structure

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda r$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\theta^T = (a^T, b^T) \quad \text{estimates of } (\alpha_1, \cdots \alpha_n, \lambda_1 - \beta_1, \cdots \lambda_n - \beta_n)$$

$$w^T = (w^{(1)^T}, w^{(2)^T})$$

$$y_i = \theta^T w$$

$$e_1 = y_i - y_p$$

Normalized Gradient Algorithm

$$\dot{\theta} = -g \frac{e_1 w}{1 + \gamma w^T w}$$

□

2.3.2 Least-Squares Algorithms

Least-squares algorithms can be derived by several methods. An interesting approach is to connect the parameter identification problem to the *state estimation* problem of a linear time varying system. The parameter θ^* can be considered to be the unknown state of the system

$$\dot{\theta}^*(t) = 0 \tag{2.3.8}$$

with output

$$y_p(t) = w^T(t)\,\theta^*(t) \tag{2.3.9}$$

Assuming that the right-hand sides of (2.3.8)-(2.3.9) are perturbed by zero-mean white gaussian noises of spectral intensities $Q \in \mathbf{R}^{2n \times 2n}$ and $r \in \mathbf{R}$ respectively, the least-squares estimator is the well-known Kalman filter (Kalman and Bucy (1961))

$$\dot{\theta} = -\frac{1}{r} P\, w\, e_1 = -g\; P\, w\, e_1$$

$$\dot{P} = Q - \frac{1}{r} P\, w\, w^T P = Q - g\; P\, w\, w^T P \qquad Q, g > 0 \tag{2.3.10}$$

Q and g are fixed design parameters of the algorithm. The update law for θ is very similar to the gradient update law, with the presence of the so-called *correlation* term $w\,e_1$. The matrix P is called the *covariance matrix*, and acts in the θ update law as a time-varying, *directional* adaptation gain. The covariance update law in (2.3.10) is called the *covariance propagation equation*. The initial conditions are arbitrary, except that $P(0) > 0$. $P(0)$ is usually chosen to reflect the confidence in the initial estimate $\theta(0)$.

In the identification literature, the least-squares algorithm referred to is usually the algorithm with $Q=0$, since the parameter θ^* is assumed to be constant. The covariance propagation equation is then replaced by

$$\dot{P} = -g\,P\,w\,w^T\,P \qquad \text{i.e.} \quad (\dot{P^{-1}}) = g\,w\,w^T \qquad g > 0 \tag{2.3.11}$$

where g is a constant.

The new expression for P^{-1} shows that $\frac{d}{dt}P^{-1} \geqslant 0$, so that P^{-1} may grow without bound. Then P will become arbitrarily small in some directions, and the adaptation of the parameters in those directions becomes very slow. This so-called *covariance wind-up* problem, can be prevented using the *least-squares with forgetting factor* algorithm, defined by

$$\dot{P} = -g\,(-\lambda\,P + P\,w\,w^T\,P\,)$$

$$\text{i.e.} \quad (\dot{P^{-1}}) = g\,(-\lambda\,P^{-1} + w\,w^T\,) \qquad \lambda, g > 0 \tag{2.3.12}$$

Another possible remedy is the *covariance resetting*, where P is reset to a predetermined positive definite value, whenever $\lambda_{\min}(P)$ falls under some threshold.

The *normalized least-squares* algorithm is defined (cf Goodwin and Mayne (1985)) by

$$\dot{\theta} = -g\,\frac{P\,w\,e_1}{1+\gamma w^T\,P\,w} \qquad g, \gamma > 0$$

$$\dot{P} = -g\,\frac{P\,w\,w^T\,P}{1+\gamma w^T\,P\,w} \quad \text{i.e.} \quad (\dot{P^{-1}}) = g\,\frac{w\,w^T}{1+\gamma w^T\,(P^{-1})^{-1}w} \tag{2.3.13}$$

Again g, γ are fixed parameters, and $P(0) > 0$. The same modifications can also be made to avoid covariance windup.

The least-squares algorithms are somewhat more complicated to implement, but are found in practice to have faster convergence properties.

2.4 Properties of the Identification Algorithms - Identifier Stability

In this section, we establish properties of the gradient algorithm

$$\dot{\phi} = \dot{\theta} = -g\, e_1 w \qquad g > 0 \tag{2.4.1}$$

and the normalized gradient algorithm

$$\dot{\phi} = \dot{\theta} = -g \frac{e_1 w}{1 + \gamma w^T w} \qquad g, \gamma > 0 \tag{2.4.2}$$

assuming the linear error equation

$$e_1 = \phi^T w \tag{2.4.3}$$

Theorems 2.4.1-2.4.4 establish general properties of the gradient algorithms, and concern solutions of the differential equations (2.4.1)-(2.4.2), with e_1 defined by (2.4.3). The properties do not require that the vector w originates from the identifier described in section 2.2, but only require that w be a piecewise continuous function of time, to guarantee the existence of the solutions. The theorems are also valid for vectors w of any dimension, not necessarily even.

Theorem 2.4.1 Linear Error Equation with Gradient Algorithm

Consider the linear error equation (2.4.3), together with the gradient algorithm (2.4.1). Let $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous.

Then (a) $e_1 \in L_2$

(b) $\phi \in L_\infty$

Proof of Theorem 2.4.1

The differential equation describing ϕ is $\dot{\phi} = -g\, w\, w^T \phi$. Let $v = \phi^T \phi$ so that $\dot{v} = -2g\, (\phi^T w)^2 = -2g\, e_1^2 \leqslant 0$. Hence, $0 \leqslant v(t) \leqslant v(0)$ for all $t \geqslant 0$, so that $v, \phi \in L_\infty$

Since v is a positive, monotonically decreasing function, the limit $v(\infty)$ is well-defined, and $-1/\,2g \int_0^\infty \dot{v}\, dt = \int_0^\infty e_1^2\, dt < \infty$, i.e. $e_1 \in L_2$. □

Theorem 2.4.2 Linear Error Equation with Normalized Gradient Algorithm

Consider the linear error equation (2.4.3), together with the normalized gradient algorithm (2.4.2). Let $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous.

Then (a) $\dfrac{e_1}{\sqrt{1+\gamma w^T w}} \in L_2 \cap L_\infty$

(b) $\phi \in L_\infty$, $\dot{\phi} \in L_2 \cap L_\infty$

(c) $\beta = \dfrac{\phi^T w}{1 + \| w_t \|_\infty} \in L_2 \cap L_\infty$

Proof of Theorem 2.4.2

Let $v = \phi^T \phi$, so that $\dot{v} = -2g\, e_1^2 / 1 + \gamma w^T w \leqslant 0$. Hence, $0 \leqslant v(t) \leqslant v(0)$ for all $t \geqslant 0$, so that $v, \phi, e_1 / \sqrt{1 + \gamma w^T w}\, , \beta \in L_\infty$. Using the fact that $x / 1+x \leqslant 1$ for all $x \geqslant 0$, we get that $|\dot{\phi}| \leqslant (g / \gamma) |\phi|$, and $\dot{\phi} \in L_\infty$.

Since v is a positive, monotonically decreasing function, the limit $v(\infty)$ is well-defined, and $-\int_0^\infty \dot{v}\, dt < \infty$ implies that $e_1 / \sqrt{1 + \gamma w^T w} \in L_2$. Note that $\beta = (e_1 / \sqrt{1 + \gamma w^T w}\,) (\sqrt{1 + \gamma w^T w} / 1 + \| w_t \|_\infty)$, where the first term is in L_2, and the second in L_∞, so that $\beta \in L_2$. Since $|\dot{\phi}|^2 \leqslant (g^2 / \gamma)(e_1^2 / 1 + \gamma w^T w)$, $\dot{\phi} \in L_2$. □

Effect of Initial Conditions and Projection

In the derivation of the linear error equation in section 2.2, we found exponentially decaying terms, such that (2.4.3) is replaced by

$$e_1(t) = \phi^T(t)\, w(t) + \epsilon(t) \tag{2.4.4}$$

where $\epsilon(t)$ is an exponentially decaying term due to the initial conditions in the observer. It may also be useful, or necessary, to replace the gradient algorithms by the algorithms with projection. The following theorem asserts that these modifications do not affect the previous results.

Theorem 2.4.3 Effect of initial conditions and projection

If the linear error equation (2.4.3) is replaced by (2.4.4), and/or the gradient algorithms are replaced by the gradient algorithms with projection,

Then the conclusions of theorems 2.4.1-2.4.2 are valid.

Proof of Theorem 2.4.3

(a) Effect of initial conditions

Modify the Lyapunov function to $v = \phi^T \phi + \frac{g}{2} \int_t^\infty \epsilon^2(\tau)\, d\tau$. Note that the additional term is bounded, and tends to zero as t tends to infinity. Consider first the gradient algorithm (2.4.1), so that

$$\dot{v} = -2g\, (\phi^T w)^2 - 2g\, (\phi^T w)\epsilon - \frac{g}{2}\, \epsilon^2$$

$$= -2g\, (\phi^T w + \frac{\epsilon}{2})^2 \quad \leqslant 0 \tag{2.4.5}$$

The proof can be completed as in theorem 2.4.1, noting that $\epsilon \in L_2 \cap L_\infty$, and similarly for theorem 2.4.2.

(b) Effect of projection

Denote by z the right-hand side of the update law (2.4.1) or (2.4.2). When $\theta \in \partial\Theta$, z is replaced by $\Pr(z)$ in the update law. Note that it is sufficient to prove that the derivative of the Lyapunov function on the boundary is less than or equal to its value with the original differential equation. Therefore, denote by z_0 the component of z perpendicular to the tangent plane at θ, so that $z = \Pr(z) + z_0$. Since $\theta^* \in \Theta$ and Θ is convex, $(\theta - \theta^*) . z_0 = \phi^T z_0 \geqslant 0$. Using the Lyapunov function $v = \phi^T \phi$, we find that, for the original differential equation $\dot{v} = 2\phi^T z$. For the differential equation with projection, $\dot{v}_{Pr} = 2\phi^T \Pr(z) = \dot{v} - 2\phi^T\, z_0$ so that $\dot{v}_{Pr} \leqslant \dot{v}$, i.e. the projection can only improve the convergence of the algorithm. The proof can again be completed as before. □

Least-Squares Algorithms

We now turn to the *normalized LS algorithm with covariance resetting*, defined by the following update law

$$\dot{\phi} = \dot{\theta} = -g \frac{P\, w\, e_1}{1 + \gamma w^T P\, w} \qquad g, \gamma > 0 \tag{2.4.6}$$

and a *discontinuous* covariance propagation

$$\dot{P} = -g \frac{P\, w\, w^T P}{1 + \gamma w^T P\, w} \quad \text{i.e.} \quad (\dot{P^{-1}}) = g \frac{w\, w^T}{1 + \gamma w^T (P^{-1})^{-1} w}$$

$$P(0) = P(t_r^+) = k_0 I > 0$$

$$\text{where } t_r = \{ t \mid \lambda_{\min}(P(t)) \leqslant k_1 < k_0 \} \tag{2.4.7}$$

This update law has similar properties as the normalized gradient update law, as stated in the following theorem.

Theorem 2.4.4 Linear Error Equation with Normalized LS Algorithm and Covariance Resetting

Consider the linear error equation (2.4.3), together with the normalized LS algorithm with covariance resetting (2.4.6)-(2.4.7).

Let $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous.

Then (a) $\dfrac{e_1}{\sqrt{1+\gamma w^T P w}} \in L_2 \cap L_\infty$

(b) $\phi \in L_\infty \; \dot{\phi} \in L_2 \cap L_\infty$

(c) $\beta = \dfrac{\phi^T w}{1 + \| w_t \|_\infty} \in L_2 \cap L_\infty$

Proof of Theorem 2.4.4

The covariance matrix P is a discontinuous function of time. Between discontinuities, the evolution is described by the differential equation in (2.4.7). We note that $d / dt\ P^{-1} \geqslant 0$, so that $P^{-1}(t_1) - P^{-1}(t_2) \geqslant 0$ for all $t_1 \geqslant t_2 \geqslant 0$ between covariance resettings. At the resettings, $P^{-1}(t_r^+) = k_0^{-1} I$, so that $P^{-1}(t) \geqslant P^{-1}(t_0) = k_0^{-1} I$, for all $t \geqslant 0$.

On the other hand, due to the resetting, $P(t) \geq k_1 I$ for all $t \geq 0$, so that

$$k_0 I \geq P(t) \geq k_1 I \qquad k_1^{-1} I \geq P^{-1}(t) \geq k_0^{-1} I \tag{2.4.8}$$

where we used results of section 1.3.

Note that the interval between resettings is bounded below, since

$$\|(\dot{P^{-1}})\| \leq g \frac{|w|^2}{1+\gamma \lambda_{min}(P)|w|^2}$$

$$\leq \frac{g}{\gamma} \|P^{-1}\| \tag{2.4.9}$$

where we used the fact that $x/\ 1+x \leq 1$ for all $x \geq 0$. Thus, the differential equation governing P^{-1} is globally Lipschitz. It also follows that $\{t_r\}$ is a set of measure zero.

Let now $v = \phi^T P^{-1}\phi$, so that $\dot{v} = -g\, e_1^2 /\ 1+\gamma w^T P w \leq 0$ between resettings. At the points of discontinuity of P, $v(t_r^+)-v(t_r) = \phi^T(P^{-1}(t_r^+)-P^{-1}(t_r))\phi \leq 0$. It follows that $0 \leq v(t) \leq v(0)$ for all $t \geq 0$, and, from the bounds on P, we deduce that $\phi, \dot{\phi}, \beta \in L_\infty$. Also $-\int_0^\infty \dot{v}\, dt < \infty$, so that $e_1 / \sqrt{1+\gamma w^T P w} \in L_2$. Note that

$$\frac{\phi^T w}{1+\|w_t\|_\infty} = \frac{\phi^T w}{\sqrt{1+\gamma w^T P w}} \frac{\sqrt{1+\gamma w^T P w}}{1+\|w_t\|_\infty} \tag{2.4.10}$$

$$\dot{\phi} = -g \frac{e_1}{\sqrt{1+\gamma w^T P w}} \frac{P w}{\sqrt{1+\gamma w^T P w}} \tag{2.4.11}$$

where the first terms in the right-hand sides of (2.4.10)-(2.4.11) are in L_2, and the last terms are bounded. The conclusions follow from this observation. □

Comments

a) Theorems 2.4.1-2.4.4 state general properties of differential equations arising from the identification algorithms described in section 2.3. The theorems can be directly applied to the identifier with the structure described in section 2.2, and the results interpreted in terms of the parameter error ϕ, and the identifier error e_1.

b) The conclusions of theorems 2.4.1-2.4.4 may appear somewhat weak, since none of the errors involved actually converge to zero. The reader should note however that the conclusions are valid under very general conditions regarding the input signal w. In particular, no assumption is made on the boundedness, or on the differentiability of w.

c) The conclusions of theorem 2.4.2 can be interpreted in the following way. The function $\beta(t)$ is defined by

$$\beta(t) = \frac{\phi^T(t)\,w(t)}{1+\|w_t\|_\infty} = \frac{e_1(t)}{1+\|w_t\|_\infty} \tag{2.4.12}$$

so that

$$|e_1(t)| = |\phi^T(t)\,w(t)| = \beta(t)\|w_t\|_\infty + \beta(t) \tag{2.4.13}$$

The purpose of the identification algorithms is to reduce the parameter error ϕ to zero, or at least the error e_1. In (2.4.12), β can be interpreted as a *relative error*, i.e. e_1 normalized by $\|w_t\|_\infty$. In (2.4.13), β can be interpreted as the *gain* from w to $\phi^T w$. From theorem 2.4.2, this gain is guaranteed to become small as $t \to \infty$ in an L_2 sense.

Stability of the Identifier

We are not guaranteed the convergence of the *parameter error* ϕ to zero. Since only one output y_p is measured to determine a vector of unknown parameters, some additional condition on the signal w (see section 2.5) must be satisfied in order to guarantee parameter convergence. In fact, we are not even guaranteed the convergence of the identifier error e_1 to zero. This can be obtained under the following additional assumption

(A3) Bounded Output Assumption

Assume that the plant is either stable, or located in a control loop such that r and y_p are bounded.

Theorem 2.4.5 Stability of the Identifier

Consider the identification problem, with (A1)-(A3), the identifier structure of section 2.2, and the gradient algorithms (2.4.1), (2.4.2), or the normalized LS algorithm with covariance resetting (2.4.6)-(2.4.7).

Then The output error $e_1 \in L_2 \cap L_\infty$, $e_1 \to 0$ as $t \to \infty$, and $\phi, \dot{\phi} \in L_\infty$.

Proof of Theorem 2.4.5

Since r and y_p are bounded, it follows from (2.2.16)-(2.2.17), and the stability of Λ, that w and $\dot{w}$ are bounded. By theorems 2.4.1 - 2.4.4, ϕ and $\dot{\phi}$ are bounded so that e_1 and $\dot{e}_1$ are bounded. Also $e_1 \in L_2$, and by corollary 1.2.2, $e_1, \dot{e}_1 \in L_\infty$ and $e_1 \in L_2$ implies that $e_1 \to 0$ as $t \to \infty$. □

Regular Signals

Theorem 2.4.5 relies on the boundedness of $w, \dot{w}$, guaranteed by (A3). It is of interest to relax this condition, and to replace it by a weaker condition. We will present such a result using a *regularity* condition on the regressor w. This condition guarantees a certain degree of smoothness of the signal w. In discrete time, such a condition is not necessary, because it is automatically verified. The definition presented here corresponds to a definition in Narendra, Lin, and Valavani (1980).

Definition Regular Signals

Let $z : \mathbf{R}_+ \to \mathbf{R}^n$, such that $z, \dot{z} \in L_{\infty e}$.

z is called *regular* if, for some $k_1, k_2 \geq 0$

$$|\dot{z}(t)| \leq k_1 \| z_t \|_\infty + k_2 \qquad \text{for all } t \geq 0 \tag{2.4.14}$$

The class of regular signals includes bounded signals with bounded derivatives, but also unbounded signals (e.g. e^t). It typically excludes signals with "increasing frequency" such as $\sin(e^t)$. We will also derive some properties of regular signals in chapter 3. Note that it will be sufficient for (2.4.14) to hold everywhere except on a set of measure zero. Therefore, piecewise differentiable signals can also be considered.

This definition allows us to state the following theorem, extending the properties derived in theorems 2.4.2-2.4.4 to the case when w is regular.

Theorem 2.4.6

Let $\phi, w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be such that $w, \dot{w} \in L_{\infty e}$, and $\phi, \dot{\phi} \in L_\infty$.

If (*a*) w is regular

(*b*) $\beta = \dfrac{\phi^T w}{1 + \| w_t \|_\infty} \in L_2$

Then $\beta, \dot{\beta} \in L_\infty$, and $\beta \to 0$ as $t \to \infty$.

Proof of Theorem 2.4.6

Clearly, $\beta \in L_\infty$, and since $\beta, \dot{\beta} \in L_\infty$ and $\beta \in L_2$ implies that $\beta \to 0$ as $t \to \infty$ (corollary 1.2.2), we are left to show that $\dot{\beta} \in L_\infty$.

We have that

$$|\dot{\beta}| \leq \left| \dot{\phi}^T \frac{w}{1 + \|w_t\|_\infty} \right| + \left| \phi^T \frac{\dot{w}}{1 + \|w_t\|_\infty} \right| + \left| \frac{\phi^T w}{1 + \|w_t\|_\infty} \frac{(d \ / \ dt \ \|w_t\|_\infty)}{1 + \|w_t\|_\infty} \right| \quad (2.4.15)$$

The first and second terms are bounded, since $\phi, \dot{\phi} \in L_\infty$, and w is regular. On the other hand

$$\left| \frac{d}{dt} \|w_t\|_\infty \right| = \left| \frac{d}{dt} \sup_{\tau \leq t} |w(\tau)| \right|$$

$$\leq \left| \frac{d}{dt} |w(t)| \right| \leq \left| \frac{d}{dt} w(t) \right| \quad (2.4.16)$$

The regularity assumption then implies that the last term in (2.4.15) is bounded, and hence $\dot{\beta} \in L_\infty$. □

Stability of the Identifier with Unstable Plant

Theorem 2.4.6 shows that when w is possibly unbounded, but nevertheless satisfies the regularity condition, the relative error $e_1 / \ 1 + \|w_t\|_\infty$ or gain from $w \to \phi^T w$ tends to zero as $t \to \infty$.

The conclusions of theorem 2.4.6 are useful in proving stability in adaptive control, where the boundedness of the regressor w is not guaranteed a priori. In the identification problem, we are now allowed to consider the case of an unstable plant with bounded input, i.e. to relax assumption (A3).

Theorem 2.4.7 Stability of the Identifier - Unstable Plant

Consider the identification problem with (A1)-(A2), the identifier structure of section 2.2, and the gradient algorithms (2.4.1), (2.4.2), or the normalized LS with covariance resetting (2.4.6)-(2.4.7).

Then $\beta = \dfrac{\phi^T w}{1 + \| w_t \|_\infty} \in L_2 \cap L_\infty$, $\beta \to 0$ as $t \to \infty$, and $\phi, \dot{\phi} \in L_\infty$.

Proof of Theorem 2.4.7

It suffices to show that w is regular, to apply theorem 2.4.4 followed by theorem 2.4.6. Combining (2.2.16) - (2.2.18), it follows that

$$\dot{w}(t) = \begin{bmatrix} \Lambda & 0 \\ b_\lambda a^{*T} & \Lambda + b_\lambda b^{*T} \end{bmatrix} w(t) + \begin{bmatrix} b_\lambda \\ 0 \end{bmatrix} r(t) \qquad (2.4.17)$$

Since r is bounded by (A2), (2.4.17) shows that w is regular. □

2.5 Persistent Excitation and Exponential Parameter Convergence

In the previous section, we derived results on the stability of the identifiers, and on the convergence of the output error $e_1 = \theta^T w - \theta^{*T} w = \phi^T w$ to zero. We are now concerned with the convergence of the parameter θ to its nominal value θ^*, i.e. the convergence of the parameter error ϕ to zero.

The convergence of the identification algorithms is related to the asymptotic stability of the differential equation

$$\dot{\phi}(t) = -g\, w(t) w^T(t) \phi(t) \qquad g > 0 \qquad (2.5.1)$$

which is of the form

$$\dot{\phi}(t) = -A(t) \phi(t) \qquad (2.5.2)$$

where $A(t) \in \mathbf{R}^{2n \times 2n}$ is a positive semidefinite matrix for all t. Using the Lyapunov function $v = \phi^T \phi$, $\dot{v} = -\phi^T (A + A^T) \phi$. When $A(t)$ is uniformly positive definite, with $\lambda_{\min}(A + A^T) \geqslant 2\alpha$, then $\dot{v} \leqslant -2\alpha v$, which implies that system (2.5.2) is exponentially stable with rate α. For the original differential equation (2.5.1), such is never the case, however, since at any instant the matrix $w(t) w^T(t)$ is of rank 1. In fact, any vector ϕ perpendicular to w lies in the null space of $w\, w^T$, and results in $\dot{\phi} = 0$. However, since w varies with time, we can expect ϕ to still converge to 0 if w completely spans $\mathbf{R}^n$ as t varies. This leads naturally to the following definition

Definition Persistency of Excitation (PE)

A vector $z : \mathbf{R}_+ \to \mathbf{R}^n$ is *persistently exciting* if there exist $\alpha_1, \alpha_2, \delta > 0$ such that

$$\alpha_2 I \geq \int_{t_0}^{t_0+\delta} z(\tau) z^T(\tau) d\tau \geq \alpha_1 I \qquad \text{for all } t_0 \geq 0 \qquad (2.5.3)$$

Although the matrix $z(\tau) z^T(\tau)$ is singular for all τ, the PE condition requires that z rotates sufficiently in space that the integral of the matrix $z(\tau) z^T(\tau)$ is uniformly positive definite over any interval of some length δ.

The condition has another interpretation, by re-expressing the PE condition in scalar form

$$\alpha_2 \geq \int_{t_0}^{t_0+\delta} (z^T(\tau) x)^2 d\tau \geq \alpha_1 \qquad \text{for all } t_0 \geq 0, |x| = 1 \qquad (2.5.4)$$

which appears as a condition on the energy of z in all directions.

With this, we establish the following convergence theorem. For consistency, the dimension of w is assumed to be $2n$, but it is in fact arbitrary.

Theorem 2.5.1 PE and Exponential Stability

Let $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous.

If w is PE

Then (2.5.1) is globally exponentially stable

The proof of theorem 2.5.1 can be found in various places in the literature (Sondhi and Mitra (1976), Morgan and Narendra (1977a&b), Anderson (1977), Kreisselmeier (1977)). The proof by Anderson has the advantage of leading to interesting interpretations, while those by Sondhi and Mitra, Kreisselmeier give estimates of the convergence rates. We will present here a combined proof. Before proving the theorem, it is suitable to recall a few definitions and results.

Uniform Complete Observability - Definition and Results

Consider a linear time-varying system [A, C] defined by

$$\dot{x}(t) = A(t)x(t)$$
$$y(t) = C(t)x(t) \tag{2.5.5}$$

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n \times n}$, $C \in \mathbf{R}^{m \times n}$, and $y \in \mathbf{R}^m$.

Definition Uniform Complete Observability (UCO)

The system [A,C] is *uniformly completely observable* if there exist positive constants β_1, β_2, δ, and a positive function $\beta_3 \in L_{\infty e}$, such that, for all $t, t_0 \geqslant 0$

$$\beta_2 I \geqslant N(t_0, t_0+\delta) \geqslant \beta_1 I \tag{2.5.6}$$

$$\|\Phi(t, t_0)\| \leqslant \beta_3(|t - t_0|) \tag{2.5.7}$$

where $\Phi(t, t_0)$ is the transition matrix associated with $A(t)$, and $N(t_0, t_0+\delta)$ is the *observability grammian*

$$N(t_0, t_0+\delta) = \int_{t_0}^{t_0+\delta} \Phi^T(t, t_0)\, C^T(t) C(t) \Phi(t, t_0)\, dt \tag{2.5.8}$$

Note that condition (2.5.6) can be rewritten as

$$\beta_2 |x(t_0)|^2 \geqslant \int_{t_0}^{t_0+\delta} |C(t)x(t)|^2\, dt \geqslant \beta_1 |x(t_0)|^2 \quad \text{for all } x(t_0) \in \mathbf{R}^n, t_0 \geqslant 0 \tag{2.5.9}$$

where $x(t)$ is the solution of (2.5.5) starting at $x(t_0)$.

Similarly, condition (2.5.7) can be written

$$|x(t)| \leqslant \beta_3(|t - t_0|)\, |x(t_0)| \quad \text{for all } x(t_0) \in \mathbf{R}^n, t, t_0 \geqslant 0 \tag{2.5.10}$$

The following lemma is a result by Anderson and Moore (1969), stating that the UCO of the system [A,C] is equivalent to the UCO of the system with output injection [A+KC,C]. The proof is given in the appendix. It is an alternate proof to the original proof, and relates the eigenvalues of the associated observability grammians, thereby leading to estimates of the convergence rates in the proof of theorem 2.5.1 given afterwards.

Lemma 2.5.2 Uniform Complete Observability under Output Injection

Assume that, for all $\delta > 0$, there exists $k_\delta \geqslant 0$ such that, for all $t_0 \geqslant 0$

$$\int_{t_0}^{t_0+\delta} \| K(\tau) \|^2 d\tau \leqslant k_\delta \tag{2.5.11}$$

Then The system [A,C] is uniformly completely observable *if and only if* the system [A+KC,C] is uniformly completely observable.

Moreover, if the system [A,C] satisfies inequalities (2.5.6) and (2.5.7) with β_1, β_2, δ, and $\beta_3(.)$, then the system [A+KC,C] satisfies these inequalities with identical δ, and

$$\beta_1' = \beta_1 / \left(1 + \sqrt{k_\delta \beta_2} \right)^2 \tag{2.5.12}$$

$$\beta_2' = \beta_2 \exp (k_\delta \beta_2) \tag{2.5.13}$$

$$\beta_3'(|t - t_0|) = \beta_3(|t - t_0|) + \sup_{\tau \in [0, t - t_0]} \beta_3(|\tau|) \left(k_{t - t_0} \left(1 + \left| \frac{t - t_0}{\delta} \right| \right) \beta_2' \right)^{1/2} \tag{2.5.14}$$

Proof of Lemma 2.5.2 in appendix.

With these preliminaries, we are now ready to return to the proof of theorem 2.5.1. The idea of the proof of exponential stability is to note that the PE condition is a UCO condition on the system

$$\dot{\theta}^*(t) = 0$$

$$y(t) = w^T(t)\,\theta^*(t) \tag{2.5.15}$$

which is the system described earlier in the context of the least-squares identification algorithms (cf (2.3.8)-(2.3.9)). We recall that the identification problem is equivalent to the state estimation problem for the system described by (2.5.15). We now find that the persistency of excitation condition, which turns out to be an *identifiability condition*, is equivalent to a uniform complete observability condition on system (2.5.15).

Proof of Theorem 2.5.1

Let $v = \phi^T \phi$, so that $\dot{v} = -2g\,(w^T \phi)^2 \leqslant 0$ along the trajectories of (2.5.1). For all $t_0 \geqslant 0$

$$\int_{t_0}^{t_0+\delta} \dot{v}\, d\tau = -2g \int_{t_0}^{t_0+\delta} (w^T(\tau)\phi(\tau))^2\, d\tau \qquad (2.5.16)$$

By the PE assumption, the system $[0,\ w^T(t)]$ is UCO. Under output injection with $K(t) = -g\ w(t)$, the system becomes $[-g\ w(t)\, w^T(t),\ w^T(t)]$, with

$$k_\delta = \int_{t_0}^{t_0+\delta} |g\ w(\tau)|^2\, d\tau = g^2\, \mathrm{tr}\,(\int_{t_0}^{t_0+\delta} w(\tau) w^T(\tau)\, d\tau) \leqslant 2n\ g^2 \beta_2 \qquad (2.5.17)$$

where $2n$ is the dimension of w. By lemma 2.5.2, the system with output injection is UCO. Therefore, for all $t_0 \geqslant 0$

$$\int_{t_0}^{t_0+\delta} \dot{v}\, d\tau \leqslant \frac{-2g\ \beta_1}{(1+\sqrt{2n}\ g\ \beta_2)^2}\ |\phi(t_0)|^2 \qquad (2.5.18)$$

Exponential convergence then follows from theorem 1.4.4.

The constants α and m are related to the PE constants α_1, α_2, δ (equal here to β_1, β_2, δ), and the adaptation gain g through

$$\alpha = \frac{1}{2\delta} \ln \left(\frac{1}{1 - \dfrac{2g\ \alpha_1}{(1+\sqrt{2n}\ g\ \alpha_2)^2}} \right) \qquad m = \left(\frac{1}{1 - \dfrac{2g\ \alpha_1}{(1+\sqrt{2n}\ g\ \alpha_2)^2}} \right)^{1/2} \qquad (2.5.19)$$

□

Exponential Convergence of the Identifier

Theorem 2.5.1 can be applied to the identification problem as follows.

Theorem 2.5.3 Exponential Convergence of the Identifier

Consider the identification problem with assumptions (A1)-(A3), the identifier structure of section 2.2, and the gradient algorithms (2.4.1) or (2.4.2), or the normalized LS algorithm with covariance resetting (2.4.6)-(2.4.7).

If w is PE

Then the identifier parameter θ converges to the nominal parameter θ^* exponentially fast.

Proof of Theorem 2.5.3

This theorem follows directly from theorem 2.5.1. Note that when w is bounded, w PE is equivalent to $w / \sqrt{1+\gamma w^T w}$ PE, so that the exponential convergence is guaranteed for both gradient update laws. The bounds on P obtained in the proof of theorem 2.4.4 allow to extend the proof of exponential convergence to the LS algorithm. □

Exponential Convergence Rates

Estimates of the convergence rates can be found from the results in the proof of theorem 2.5.1. For the standard gradient algorithm (2.4.1) for example, the convergence rate is as given in (2.5.19). The influence of some design parameters can be studied with this relationship. The constants α_1, α_2, δ depend in a complex manner on the input signal r and on the plant being identified. However, if r is multiplied by 2, then α_1, α_2 are multiplied by 4. In the limiting case when the adaptation gain g or the reference input r are made small, the rate of convergence $\alpha \to g\ \alpha_1 / \delta$. In this case, the convergence rate is proportional to the adaptation gain g, and to the lower bound in the PE condition. Through the PE condition, it is also proportional to the square of the amplitude of the reference input r. This result will be found again in chapter 5, using averaging techniques.

When the adaptation gain and reference input get sufficiently large, this approximation is not valid anymore, and (2.5.19) shows that above some level, the convergence rate estimate saturates, and even decreases (cf. Sondhi and Mitra (1976)).

It is also possible to show that the presence of the exponentially decaying terms due to initial conditions in the observer do not affect the exponential stability of the system. The rate of convergence will however be as found previously only if the rate of dacay of the transients is faster than the rate of convergence of the algorithm (cf. Kreisselmeier (1977)).

2.6 Strictly Positive Real Error Equation and Identification Algorithms

In previous sections, we derived properties of identification algorithms for the linear error equation

$$e_1(t) = \phi^T(t) w(t) \tag{2.6.1}$$

A more general error equation encountered in identification and adaptive control problems is the *strictly positive real* (SPR) error equation

$$e_1(t) = \hat{M}(\phi^T(t)w(t)) \tag{2.6.2}$$

where $\hat{M}$ is a stable, strictly positive real transfer function. For uniformity with previous discussions, we assume that $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$, but the dimension of w is in fact completely arbitrary.

Definition Strictly Positive Real Function (SPR)

A rational function $\hat{M}(s)$ of the complex variable $s = \sigma + j\omega$ is *positive real* (PR), if $\hat{M}(\sigma) \in \mathbf{R}$, $\mathrm{Re}(\hat{M}(s)) > 0$ for $\sigma > 0$, and $\mathrm{Re}\,\hat{M}(j\omega) \geqslant 0$, for all $\omega \geqslant 0$. It is *strictly positive real* (SPR) if, for some $\epsilon > 0$, $\hat{M}(s-\epsilon)$ is PR.

SPR *transfer functions* form a rather restricted class. In particular, an SPR transfer function must be minimum phase, and its phase may never exceed 90°. An important lemma concerning SPR transfer functions is the Kalman-Yacubovitch-Popov lemma, of which a form is given below.

Lemma 2.6.1 Minimal Realization of an SPR Transfer Function

Let $[A, b, c^T]$ be a minimal realization of a strictly proper, stable, rational transfer function $\hat{M}(s)$. Then, the following statements are equivalent

(a) $\hat{M}(s)$ is *SPR*

(b) there exist symmetric positive definite matrices P, Q, such that

$$PA + A^T P = -Q$$

$$Pb = c \tag{2.6.3}$$

Proof of Lemma 2.6.1 cf. Anderson and Vongpanitlerd (1973).

SPR Error Equation with Gradient Algorithm

A remarkable fact about SPR transfer functions is that the gradient update law

$$\dot{\phi}(t) = \dot{\theta}(t) = -g\, e_1(t)\, w(t) \qquad g > 0 \tag{2.6.4}$$

has properties similar to the SPR error equation (2.6.2), as with the linear error equation

(2.6.1).

Using lemma 2.6.1, a state-space realization of $\hat{M}(s)$ with state e_m can be obtained so that

$$\dot{e}_m(t) = A\, e_m(t) + b\,\phi^T(t) w(t)$$

$$e_1(t) = c^T(t)\, e_m(t)$$

$$\dot{\phi}(t) = -g\, c^T e_m(t)\, w(t) \qquad g > 0 \tag{2.6.5}$$

Theorem 2.6.2 SPR Error Equation with Gradient Algorithm

Let $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous. Consider the SPR error equation (2.6.2) with $\hat{M}(s)$ SPR, together with the gradient update law (2.6.4). Equivalently, consider the state-space realization (2.6.5) where $[A, b, c^T]$ satisfy the conditions of lemma 2.6.1.

Then

(a) $e_m, e_1 \in L_2$

(b) $e_m, e_1, \phi \in L_\infty$

Proof of Theorem 2.6.2

Let P, Q be as in lemma 2.6.1, and $v = g\, e_m^T P\, e_m + \phi^T \phi$. Along the trajectories of (2.6.5)

$$\dot{v} = g\, e_m^T P\, A\, e_m + g\, e_m^T P\, b\phi^T w + g\, e_m^T A^T P\, e_m + g\, \phi^T w\, b^T P\, e_m - 2g\, c^T e_m\, \phi^T w$$

$$= -g\, e_m^T Q\, e_m \leqslant 0 \tag{2.6.6}$$

where we used (2.6.3). The conclusions follow as in theorem 2.4.1, since P and Q are positive definite. □

Modified SPR Error Equation

The normalized gradient update law presented for the linear error equation is not usually applied to the SPR error equation. Instead, a *modified SPR error equation* is considered

$$e_1(t) = \hat{M}(\phi^T(t)\, w(t) - \gamma\, w^T(t)\, w(t)\, e_1(t)) \qquad \gamma > 0 \tag{2.6.7}$$

where γ is a constant. The same gradient algorithm is applied to this error equation, so that in state-space form

$$\dot{e}_m(t) = A\, e_m(t) + b(\phi^T(t) w(t) - \gamma w^T(t)\, w(t)\, c^T e_m(t))$$

$$e_1(t) = c^T e_m(t)$$

$$\dot{\phi}(t) = -g\; c^T e_m(t)\, w(t) \qquad\qquad g, \gamma > 0 \tag{2.6.8}$$

Theorem 2.6.3 Modified SPR Error Equation with Gradient Algorithm

Let $w : \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous. Consider the modified SPR error equation (2.6.7) with $\hat{M}(s)$ SPR, together with the gradient update law (2.6.4). Equivalently, consider the state-space realization (2.6.8), where $[A, b, c^T]$ satisfy the conditions of lemma 2.6.1.

Then

(a) $\quad e_m, e_1, \dot{\phi} \in L_2$

(b) $\quad e_m, e_1, \phi \in L_\infty$

Proof of Theorem 2.6.3

Let P, Q be as in lemma 2.6.1, and $v = g\, e_m^T P\, e_m + \phi^T \phi$. Along the trajectories of (2.6.8)

$$\dot{v} = -g\; e_m^T Q\, e_m - 2\, g\; \gamma (e_1 w)^T . (e_1 w) \leqslant 0 \tag{2.6.9}$$

Again, it follows that e_m, e_1, ϕ are bounded, and $e_m, e_1 \in L_2$. Moreover, it also follows now that $e_1 w \in L_2$, so that $\dot{\phi} \in L_2$. □

Exponential Convergence of the Gradient Algorithms with SPR Error Equations

As stated in the following theorem, the gradient algorithm is also exponentially convergent with the SPR error equations, under the PE condition.

Theorem 2.6.4 Exponential Convergence of the Gradient Algorithms with SPR Error Equations

Let $w: \mathbf{R}_+ \to \mathbf{R}^{2n}$. Let $[A, b, c^T]$ satisfy the conditions of lemma 2.6.1.

If w is PE, and $w, \dot{w} \in L_\infty$

Then (2.6.5), (2.6.8) are globally exponentially stable.

The proof given hereafter is similar to the proof by Anderson (1977) (with some differences however). The main condition for exponential convergence is the PE condition, as required previously. The additional boundedness requirement on $\dot{w}$ guarantees that PE is not lost through the transfer function $\hat{M}$ (cf. lemma 2.6.6 hereafter). It is sufficient that the boundedness conditions hold almost everywhere, so that piecewise differentiable signals may be considered.

Auxiliary Lemmas on PE Signals

The following auxiliary lemmas will be useful in proving the theorem. Note that the sum of two PE signals is not necessarily PE. On the other hand, an L_2 signal is necessarily *not* PE. Lemma 2.6.5 asserts that PE is not altered by the addition of a signal belonging to L_2. In particular, this implies that terms due to initial conditions do not affect PE. Again, we assume the dimension of the vectors to be $2n$, for uniformity, but the dimension is in fact arbitrary.

Lemma 2.6.5 PE and L_2 Signals

Let $w, e: \mathbf{R}_+ \to \mathbf{R}^{2n}$ be piecewise continuous.

If w is PE

$e \in L_2$

Then $w + e$ is PE.

Proof of Lemma 2.6.5 in appendix.

Lemma 2.6.6 shows that PE is not lost if the signal is filtered by a stable, minimum phase transfer function, provided that the signal is sufficiently smooth.

Lemma 2.6.6 PE Through LTI Systems

Let $w : \mathbf{R}_+ \rightarrow \mathbf{R}^{2n}$.

If w is PE, and w , $\dot{w} \in L_\infty$

$\hat{H}$ is a stable, minimum phase, rational transfer function

Then $\hat{H}(w)$ is PE.

Proof of Lemma 2.6.6 in appendix.

We now prove theorem 2.6.4.

Proof of Theorem 2.6.4

As previously, let $v = g\, e_m^T P\, e_m + \phi^T \phi$, so that for both SPR error equations

$$\int_{t_0}^{t_0+\delta} \dot{v}\ d\tau \leqslant -g \int_{t_0}^{t_0+\delta} e_m^T Q\, e_m\, d\tau \leqslant -g\, \frac{\lambda_{\min}(Q)}{|c|^2} \int_{t_0}^{t_0+\delta} e_1^2\ d\tau \leqslant 0 \qquad (2.6.10)$$

By theorem 1.4.4, exponential convergence will be guaranteed if, for some $\alpha_3 > 0$

$$\int_{t_0}^{t_0+\delta} e_1^2(\tau) d\tau \geqslant \alpha_3(|e_m(t_0)|^2 + |\phi(t_0)|^2) \qquad (2.6.11)$$

for all t_0, $e_m(t_0)$, $\phi(t_0)$.

Derivation of (2.6.11)

This condition can be interpreted as a UCO condition on the system

$$\dot{e}_m = A\ e_m + b\,\phi^T w$$

$$\dot{\phi} = -g\ c^T e_m\ w$$

$$e_1 = c^T e_m \qquad (2.6.12)$$

An additional term $-b\,\gamma\, w^T w\ c^T e_m$ is added in the differential equation governing e_m in the case of the modified SPR error equation. Using lemma 2.5.2 about UCO under output injection, we find that inequality (2.6.11) will be satisfied if the following system

$$\dot{e}_m = A\ e_m + b\,\phi^T w$$

$$\dot{\phi} = 0$$

$$e_1 = c^T e_m \qquad (2.6.13)$$

is UCO. For this, we let

$$K = \begin{pmatrix} 0 \\ gw \end{pmatrix} \qquad \text{or} \qquad K = \begin{pmatrix} b\gamma w^T w \\ gw \end{pmatrix} \tag{2.6.14}$$

for the basic SPR, or modified SPR error equations respectively. The condition on K in lemma 2.5.2 is satisfied, since w is bounded.

We are thus left to show that system (2.6.13) is UCO, i.e. that

$$e_1(t) = c^T e^{A(t-t_0)} e_m(t_0) + \int_{t_0}^{t} c^T e^{A(t-\tau)} b\, w^T(\tau)\, d\tau\ \phi(t_0)$$

$$:= x_1(t) + x_2(t) \tag{2.6.15}$$

satisfies, for some $\beta_1, \beta_2, \delta > 0$

$$\beta_2(\,|e_m(t_0)|^2 + |\phi(t_0)|^2) \geqslant \int_{t_0}^{t_0+\delta} e_1^2(\tau)d\tau \geqslant \beta_1(\,|e_m(t_0)|^2 + |\phi(t_0)|^2) \tag{2.6.16}$$

for all t_0, $e_m(t_0)$, $\phi(t_0)$.

Derivation of (2.6.16)

By assumption, w is PE, and w, $\dot{w} \in L_\infty$. Therefore, using lemma 2.6.6, we have that, for all $t_0 \geqslant 0$, the signal

$$w_f(t) = \int_{t_0}^{t} c^T e^{A(t-\tau)} b\, w(\tau)\, d\tau \tag{2.6.17}$$

is PE. This means that, for some $\alpha_1, \alpha_2, \sigma > 0$

$$\alpha_2\, |\phi(t_0)|^2 \geqslant \int_{t_1}^{t_1+\sigma} x_2^2(\tau)\, d\tau \geqslant \alpha_1\, |\phi(t_0)|^2 \tag{2.6.18}$$

for all $t_1 \geqslant t_0 \geqslant 0$, and $\phi(t_0)$.

On the other hand, since A is stable, there exist $\gamma_1, \gamma_2 > 0$, such that

$$\int_{t_0+m\sigma}^{\infty} x_1^2(\tau)\, d\tau \leqslant \gamma_1\, |e_m(t_0)|^2 e^{-\gamma_2 m \sigma} \tag{2.6.19}$$

for all t_0, $e_m(t_0)$, and an arbitrary integer $m > 0$ to be defined later. Since $[A, c^T]$ is observable, there exists $\gamma_3(m\,\sigma) > 0$, with $\gamma_3(m\,\sigma)$ increasing with $m\,\sigma$, such that

$$\int_{t_0}^{t_0+m\sigma} x_1^2(\tau)\,d\tau \geqslant \gamma_3(m\sigma)\,|e_m(t_0)|^2 \tag{2.6.20}$$

for all t_0, $e_m(t_0)$, and $m>0$.

Let $n>0$ be another integer to be defined, and let $\delta=(m+n)\sigma$. Using the triangle inequality

$$\int_{t_0}^{t_0+\delta} e_1^2(\tau)d\tau \geqslant \int_{t_0}^{t_0+m\sigma} x_1^2(\tau)d\tau - \int_{t_0}^{t_0+m\sigma} x_2^2(\tau)\,d\tau + \int_{t_0+m\sigma}^{t_0+\delta} x_2^2(\tau)d\tau - \int_{t_0+m\sigma}^{t_0+\delta} x_1^2(\tau)d\tau$$

$$\geqslant \gamma_3(m\sigma)|e_m(t_0)|^2 - m\,\alpha_2|\phi(t_0)|^2$$

$$+\,n\,\alpha_1|\phi(t_0)|^2 - \gamma_1 e^{-\gamma_2 m\sigma}|e_m(t_0)|^2 \tag{2.6.21}$$

Let m be large enough to get

$$\gamma_3(m\sigma) - \gamma_1 e^{-\gamma_2 m\sigma} \geqslant \gamma_3(m\sigma)/\,2 \tag{2.6.22}$$

and n sufficiently large to obtain

$$n\,\alpha_1 - m\,\alpha_2 \geqslant \alpha_1 \tag{2.6.23}$$

Further, define

$$\beta_1 = \min\,(\alpha_1, \gamma_3(m\sigma)/\,2) \tag{2.6.24}$$

The lower inequality in (2.6.16) follows from (2.6.21), with β_1 as defined, while the upper inequality is easily found to be valid with

$$\beta_2 = \max(\gamma_1, (m+n)\alpha_2) \tag{2.6.25}$$

□

Comments

a) Although the proof of theorem 2.6.4 is somewhat long and tedious, it has some interesting features. First, it relies on the same basic idea as the proof of exponential convergence for the linear error equation (cf. theorem 2.5.1). It interprets the condition for exponential convergence as a uniform complete observability condition. Then, it uses lemma 2.5.2 concerning UCO under output injection to transform the UCO condition to a UCO condition on a similar system, but *where the vector ϕ is constant* (cf. (2.6.13)). The UCO condition leads then to a PE condition on a vector w_f, which is a filtered version of

w, through the LTI system $\hat{M}(s)$.

b) The steps of the proof can be followed to obtain guaranteed rates of exponential convergence. Although such rates would be useful to the designer, the expression one obtains is quite complex, and examination of the proof leaves little hope that the estimate would be tight. A more successful approach is found in chapter 4, using averaging techniques.

2.7 Conclusions

In this chapter, we derived a simple identification scheme for SISO LTI plants. The scheme involved a generic linear error equation, relating the identifier error, the regressor, and the parameter error. Several gradient and least-squares algorithms were reviewed, and common properties were established, that are valid under general conditions. It was shown that for any of these algorithms, and provided that the regressor was a bounded function of time, the identifier error converged to zero as t approached infinity. The parameter error was also guaranteed to remain bounded. When the regressor was not bounded, but satisfied a regularity condition, then it was shown that a normalized error still converged to zero.

The exponential convergence of the parameter error to its nominal value followed from a persistency of excitation condition on the regressor. Guaranteed rates of exponential convergence were also obtained, and showed the influence of various design parameters. In particular, the reference input was found to be a dominant factor influencing the parameter convergence.

The stability and convergence properties were further extended to strictly positive real error equations. Although more complex to analyze, the SPR error equation was found to have similar stability and convergence properties. In particular, PE appeared as a fundamental condition to guarantee exponential parameter convergence.

Most results derived in this chapter are known, but scattered in the literature. We presented here these results in a unified framework, and established the basis for subsequent developments.

Chapter 3 Adaptive Control

3.1 Model Reference Adaptive Control Problem

The motivation for adaptive control arises from applications where plant parameters are unknown, or vary with time to a sufficient degree that robust control is not satisfactory. Initial interest appears to have been concentrated on applications to advanced aerospace vehicles, which experience substantial changes in dynamical behavior as altitude and velocity are varied. Current and potential applications span a large class of problems, including process control, robotics, and others.

Model reference adaptive control consists in designing an adaptive controller such that the behavior of the controlled plant remains close to the behavior of a desirable model, despite uncertainties or variations in the plant parameters. More formally, a *reference* model $\hat{M}$ is given, with input $r(t)$ and output $y_m(t)$. The unknown *plant* $\hat{P}$ has input $u(t)$ and output $y_p(t)$. The control objective is to design $u(t)$ such that $y_p(t)$ asymptotically tracks $y_m(t)$, with all generated signals remaining bounded. In this chapter, we consider the problem of attaining this objective under the following assumptions.

Assumptions

(A1) Plant Assumptions

the plant is a SISO LTI system, described by a transfer function

$$\frac{\hat{y}_p(s)}{\hat{u}(s)} = \hat{P}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{3.1.1}$$

where $\hat{n}_p(s)$, $\hat{d}_p(s)$ are monic, coprime polynomials of degree m and n respectively. The plant is strictly proper, and minimum phase. The sign of the so-called *high-frequency gain* k_p is known, and without loss of generality, we will assume $k_p > 0$.

(A2) Reference Model Assumptions

The reference model is described by

$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \hat{M}(s) = k_m \frac{\hat{n}_m(s)}{\hat{d}_m(s)} \tag{3.1.2}$$

where $\hat{n}_m(s)$, $\hat{d}_m(s)$ are monic, coprime polynomials of degree m and n respectively (i.e. the same degrees as the corresponding plant polynomials). The reference model is stable, minimum phase, and $k_m > 0$.

(A3) Reference Input Assumptions

The reference input $r(.)$ is piecewise continuous, and bounded on $\mathbb{R}_+$.

Note that $\hat{P}(s)$ is assumed to be minimum phase, but is *not* assumed to be stable.

3.2 Controller Structure

To achieve the control objective, we consider the controller structure shown in figure 3.1. By inspection of the figure, we see that

$$u = c_0 r + \frac{\hat{c}(s)}{\hat{\lambda}(s)}(u) + \frac{\hat{d}(s)}{\hat{\lambda}(s)}(y_p) \tag{3.2.1}$$

where c_0 is a scalar, $\hat{c}(s)$, $\hat{d}(s)$, and $\hat{\lambda}(s)$ are polynomials of degrees $n-2$, $n-1$, and $n-1$ respectively. From (3.2.1)

$$u = \frac{\hat{\lambda}}{\hat{\lambda}-\hat{c}}\left(c_0 r + \frac{\hat{d}}{\hat{\lambda}}(y_p)\right) \tag{3.2.2}$$

which is shown in figure 3.2. Since

$$y_p = k_p \frac{\hat{n}_p}{\hat{d}_p}(u) \tag{3.2.3}$$

the transfer function from r to y_p is

$$\frac{\hat{y}_p}{\hat{r}} = \frac{c_0 k_p \hat{\lambda} \hat{n}_p}{(\hat{\lambda} - \hat{c})\hat{d}_p - k_p \hat{n}_p \hat{d}} \tag{3.2.4}$$

Note that the derivation of (3.2.4) relies on the cancellation of polynomials $\hat{\lambda}(s)$. Physically, this would correspond to the exact cancellation of modes of $\hat{c}(s)/\hat{\lambda}(s)$ and $\hat{d}(s)/\hat{\lambda}(s)$. For numerical considerations, we will therefore require that $\hat{\lambda}(s)$ is a Hurwitz polynomial.

The following proposition indicates that the controller structure is adequate to achieve the control objective, i.e. that it is possible to make the transfer function from r to y_p equal to $\hat{M}(s)$. For this, it is clear from (3.2.4) that $\hat{\lambda}(s)$ must contain the zeros of $\hat{n}_m(s)$, so that we write

$$\hat{\lambda}(s) = \hat{\lambda}_0(s)\,\hat{n}_m(s) \tag{3.2.5}$$

where $\hat{\lambda}_0(s)$ is an arbitrary minimum phase polynomial of degree $n-m-1$.

Proposition 3.2.1 Matching Equality

There exist unique c_0^*, $\hat{c}^*(s)$, $\hat{d}^*(s)$ such that the transfer function from $r \to y_p$ is $\hat{M}(s)$.

Proof of Proposition 3.2.1

Existence

The transfer function from r to y_p is $\hat{M}$ if and only if the following *matching equality* is satisfied

$$(\hat{\lambda} - \hat{c}^*)\,\hat{d}_p - k_p\,\hat{n}_p\,\hat{d}^* = c_0^*\,\frac{k_p}{k_m}\,\hat{\lambda}_0\,\hat{n}_p\,\hat{d}_m \tag{3.2.6}$$

The solution can be found by inspection. Divide $\hat{\lambda}_0\,\hat{d}_m$ by $\hat{d}_p$, let $\hat{q}$ be the quotient (of degree $n-m-1$), and $-k_p\,\hat{d}^*$ the remainder (of degree $n-1$). Thus $\hat{d}^*$ is given by

$$\hat{d}^* = \frac{1}{k_p}\,(\hat{q}\,\hat{d}_p - \hat{\lambda}_0\,\hat{d}_m) \tag{3.2.7}$$

Let $\hat{c}^*$ (of degree $n-2$), c_0^* be given by

$$\hat{c}^* = \hat{\lambda} - \hat{q}\,\hat{n}_p \tag{3.2.8}$$

$$c_0^* = \frac{k_m}{k_p} \tag{3.2.9}$$

Eqns (3.2.7)-(3.2.9) define a solution to (3.2.6), as can easily be seen by substituting c_0^*, c^*, d^* in (3.2.6).

Uniqueness

Assume that there exist $c_0 = c_0^* + \delta c_0$, $\hat{c} = \hat{c}^* + \delta\hat{c}$, $\hat{d} = \hat{d}^* + \delta\hat{d}$ satisfying (3.2.6). The following equality must then be satisfied

$$\delta\hat{c}\,\hat{d}_p + k_p\,\hat{n}_p\,\delta\hat{d} = -\delta c_0 \frac{k_p}{k_m}\hat{\lambda}_0 \hat{n}_p\,\hat{d}_m \tag{3.2.10}$$

Recall that $\hat{d}_p$, $\hat{n}_p$, $\hat{\lambda}_0$, $\hat{d}_m$ have degrees n, m, $n-m-1$, and n respectively, with $m \leqslant n-1$, and $\delta\hat{c}$, $\delta\hat{d}$ have degrees at most $n-2$, and $n-1$. Consequently, the right-hand side is a polynomial of degree $2n-1$, and the left-hand side is a polynomial of degree at most $2n-2$. No solution exists unless $\delta c_0 = 0$, so that c_0^* is unique. Let then $\delta c_0 = 0$, so that (3.2.10) becomes

$$\frac{\delta\hat{c}}{\delta\hat{d}} = -k_p \frac{\hat{n}_p}{\hat{d}_p} = -\hat{P} \tag{3.2.11}$$

This equation has no solution since $\hat{n}_p$, $\hat{d}_p$ are coprime, so that $\hat{c}^*$ and $\hat{d}^*$ are also unique. □

Comments

a) The coprimeness of $\hat{n}_p$, $\hat{d}_p$ is only necessary to guarantee a *unique* solution. If this assumption is not satisfied, a solution can still be found using (3.2.7)-(3.2.9). Equation (3.2.11) characterizes the set of solutions in this case.

b) Using (3.2.2), the controller structure can be expressed as in figure 3.2, with a forward block $\hat{\lambda}/\hat{\lambda}-\hat{c}$, and a feedback block $\hat{d}/\hat{\lambda}$. When matching with the model occurs, (3.2.7)-(3.2.8) show that the compensator becomes

$$\frac{\hat{\lambda}}{\hat{\lambda}-\hat{c}^*} = \frac{\hat{\lambda}_0 \hat{n}_m}{\hat{q}\,\hat{n}_p} \tag{3.2.12}$$

and

$$\frac{\hat{d}^*}{\hat{\lambda}} = \frac{1}{k_p}\,\frac{\hat{q}\,\hat{d}_p - \hat{\lambda}_0 \hat{d}_m}{\hat{\lambda}_0 \hat{n}_m} \tag{3.2.13}$$

Thus the forward block actually cancels the zeros of $\hat{P}$, and replaces them by the zeros of $\hat{M}$.

c) The transfer function from r to y_p is of order n, while the plant and controller have $3n-2$ states. It can be checked (see section 3.5) that the $2n-2$ extra modes are unobservable, and that they are those of $\hat{\lambda}$, $\hat{\lambda}_0$, and $\hat{n}_p$. The modes corresponding to $\hat{\lambda}, \hat{\lambda}_0$ are stable by choice, and those of $\hat{n}_p$ are stable by assumption (A1).

d) The structure of the controller is not unique. In particular, it is equivalent to the familiar structure found, e.g. in Callier and Desoer (1982) p. 164, and represented in figure 3.3. The polynomials found in this case are related to the previous ones through

$$\hat{n}_\pi = c_0 \hat{\lambda} \qquad \hat{d}_c = \hat{\lambda} - \hat{c} \qquad \hat{n}_f = -\hat{d} \tag{3.2.14}$$

The motivation in using the previous controller structure is to obtain an expression that is *linear* in the unknown parameters. These parameters are the coefficients of the polynomials $\hat{c}$, $\hat{d}$, and the gain c_0. The expression in (3.2.1) shows that the control signal is the sum of the parameters multiplied by known or reconstructible signals.

State-Space Representation

To make this more precise, we consider a state-space representation of the controller. Choose $\Lambda \in \mathbf{R}^{n-1 \times n-1}$, and $b_\lambda \in \mathbf{R}^{n-1}$, such that (Λ, b_λ) is in controllable canonical form, and $\det(sI - \Lambda) = \hat{\lambda}(s)$. It follows that

$$(sI - \Lambda)^{-1} b_\lambda = \frac{1}{\hat{\lambda}(s)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-2} \end{bmatrix} \tag{3.2.15}$$

Let $c \in \mathbf{R}^{n-1}$ be the vector of coefficients of the polynomial $\hat{c}(s)$, so that

$$\frac{\hat{c}(s)}{\hat{\lambda}(s)} = c^T (sI - \Lambda)^{-1} b_\lambda \tag{3.2.16}$$

Consequently, this transfer function can be realized by

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\frac{\hat{c}}{\hat{\lambda}} (u) = c^T w^{(1)} \tag{3.2.17}$$

where the state $w^{(1)} \in \mathbf{R}^{n-1}$, and the initial condition $w^{(1)}(0)$ is arbitrary. Similarly, there exist $d_0 \in \mathbf{R}$, and $d \in \mathbf{R}^{n-1}$ such that

$$\frac{\hat{d}(s)}{\hat{\lambda}(s)} = d_0 + d^T (sI - \Lambda)^{-1} b_\lambda \tag{3.2.18}$$

and

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\frac{\hat{d}}{\hat{\lambda}}(y_p) = d_0 y_p + d^T w^{(2)} \tag{3.2.19}$$

where the state $w^{(2)} \in \mathbf{R}^{n-1}$, and the initial condition $w^{(2)}(0)$ is arbitrary. The controller can be represented as in figure 3.4, with

$$\begin{aligned} u &= c_0 r + c^T w^{(1)} + d_0 y_p + d^T w^{(2)} \\ &:= \theta^T w \end{aligned} \tag{3.2.20}$$

where

$$\theta^T := (c_0, \bar{\theta}^T) := (c_0, c^T, d_0, d^T) \in \mathbf{R}^{2n} \tag{3.2.21}$$

is the vector of *controller parameters*, and

$$w^T := (r, \bar{w}^T) := (r, w^{(1)T}, y_p, w^{(2)T}) \in \mathbf{R}^{2n} \tag{3.2.22}$$

is a vector of signals that can be obtained without knowledge of the plant parameters. Note the definitions of $\bar{\theta}$ and $\bar{w}$ which correspond to the vectors θ and w with their first components removed.

In analogy to the previous definitions, we let

$$\theta^{*T} := (c_0^*, \bar{\theta}^{*T}) := (c_0^*, c^{*T}, d_0^*, d^{*T}) \in \mathbf{R}^{2n} \tag{3.2.23}$$

be the vector of *nominal* controller parameters that achieves a matching of the transfer function $r \to y_p$ to the model transfer function $\hat{M}$. We also define the *parameter errors*

$$\phi := \theta - \theta^* \in \mathbf{R}^{2n} \qquad \bar{\phi} := \bar{\theta} - \bar{\theta}^* \in \mathbf{R}^{2n-1} \tag{3.2.24}$$

The linear dependence of u on the parameters is clear in (3.2.20). In the sequel, we will consider *adaptive* control algorithms, and the parameter θ will be a function of time. Similarly, $\hat{c}(s)$, $\hat{d}(s)$ will be polynomials in s whose coefficients vary with time. Eqns (3.2.17) and (3.2.19) give a meaning to (3.2.1) in that case.

3.3 Adaptive Control Schemes - Identifier Structure

In section 3.2, we showed how a controller can be designed to achieve tracking of the reference output y_m by the plant output y_p, when the plant transfer function is known. We now consider the case when the plant is unknown, and the control parameters are updated recursively using an identifier. Several approaches are possible. In an indirect adaptive control scheme, the plant parameters (i.e. k_p, and the coefficients of $\hat{n}_p(s)$, $\hat{d}_p(s)$) are identified using a recursive identification scheme, such as the one described in chapter 2. The estimates are then used to compute the control parameters through (3.2.7)-(3.2.9).

In a direct adaptive control scheme, an identification scheme is designed that *directly* identifies the controller parameters c_0, c, d_0, d. A typical procedure is to derive an identifier error signal which depends linearly on the parameter error ϕ. The output error $e_0(t) = y_p(t) - y_m(t)$ is the basis for output error adaptive control schemes such as those of Narendra and Valavani (1978), Narendra, Lin and Valavani (1980), and Morse (1980). An output error direct adaptive control scheme, and an indirect adaptive control scheme will be described in sections 3.3.2 and 3.3.3, but we will first turn to an input error direct adaptive control scheme in section 3.3.1.

Note that we made the distinction between controller and identifier, even in the case of direct adaptive control. The *controller* is by definition the system that determines the value of the control input, using some controller parameters as in a nonadaptive context. The *identifier* obtains estimates of these parameters - directly or indirectly.

As in chapter 2, we also make the distinction, within the identifier, between the *identifier structure* and the *identification algorithm*. The identifier structure constructs signals which are related by some error equation, and are to be used by the identification algorithm. The identification algorithm defines the evolution of the identifier parameters, from which the controller parameters depend. Given an identifier structure with linear error equation for example, several identification algorithms exist from which we can choose (cf. section 2.3).

Although we make the distinction between controller and identifier, we will see that, for efficiency, some internal signals will be shared by both systems.

3.3.1 Input Error Direct Adaptive Control

Define

$$r_p = \hat{M}^{-1}(y_p) = \hat{M}^{-1}\hat{P}(u) \tag{3.3.1}$$

and let the *input error* e_i be defined by

$$e_i := r_p - r$$
$$= \hat{M}^{-1}(y_p - y_m) = \hat{M}^{-1}(e_0) \tag{3.3.2}$$

where $e_0 = y_p - y_m$ is the *output error*.

By definition, an input error adaptive control scheme is a scheme based on this error, or an approximation of it.

Preliminaries

Since the relative degree of $\hat{M}$ is at least 1, its inverse is not proper. $\hat{M}^{-1}(.)$ is well-defined, provided that the argument is sufficiently smooth. However, in the frequency domain, the gain of the operator $\hat{M}^{-1}$ is arbitrarily large at high frequencies. Therefore, due to the presence of measurement noise, the use of $\hat{M}^{-1}$ is not desirable in practice. Although we will use $\hat{M}^{-1}(.)$ in the analysis, we will consider it not implementable, so that r_p and e_i are not available. Instead, we will construct an approximate inverse of $\hat{M}$ as follows.

Since $\hat{M}$ is minimum phase with relative degree $n-m$, for any stable, minimum phase transfer function $\hat{L}^{-1}$ of relative degree $n-m$, the transfer function $\hat{M}\,\hat{L}$ has a proper and stable inverse. For example, we can let $\hat{L}$ be a minimum phase polynomial of degree $n-m$. The signal $\hat{L}^{-1}(r_p)$ is available since

$$\hat{L}^{-1}(r_p) = (\hat{M}\,\hat{L})^{-1}(y_p) \tag{3.3.3}$$

where $(\hat{M}\,\hat{L})^{-1}$ is a proper, stable transfer function.

Approximate Input Error

To obtain the identification scheme of chapter 2, it was useful to derive an expression in which a known signal depended linearly on the unknown parameter θ^*. We now derive such an identity based on the matching equality (3.2.6).

First transform (3.2.6), by dividing both sides by $\hat{\lambda}\,\hat{d}_p\,\hat{L}$ so that it becomes, using (3.2.5)

$$\frac{\hat{\lambda}-\hat{c}^*}{\hat{\lambda}}\hat{L}^{-1} - k_p\frac{\hat{n}_p}{\hat{d}_p}\frac{\hat{d}^*}{\hat{\lambda}}\hat{L}^{-1} = c_0^*\,k_p\frac{\hat{n}_p}{\hat{d}_p}\cdot\frac{\hat{d}_m}{k_m\hat{n}_m}\cdot\hat{L}^{-1} \tag{3.3.4}$$

and, with the definition of $\hat{P}, \hat{M}$

$$\left[\hat{L}^{-1}\frac{\hat{d}^*}{\hat{\lambda}} + c_0^*(\hat{M}\hat{L})^{-1}\right]\hat{P} = \hat{L}^{-1} - \hat{L}^{-1}\frac{\hat{c}^*}{\hat{\lambda}} \qquad (3.3.5)$$

(3.3.5) is an equality of two polynomial ratios, but also an equality of two LTI operators. The right-hand side is a stable transfer function, while the left-hand side is possibly unstable (since $\hat{P}$ is not assumed to be stable).

To transform (3.3.5) into an equality in the time domain, care must be taken of the effect of the initial conditions related to the unstable modes of $\hat{P}$. These will be unobservable or uncontrollable, depending on the realization of the transfer function. If the left-hand side is realized by $\hat{P}$ followed by $L^{-1}\frac{\hat{d}^*}{\hat{\lambda}} + c_0^*(\hat{M}\hat{L})^{-1}$, the unstable modes of $\hat{P}$ will be controllable, and therefore unobservable.

The operator equality (3.3.5) can be transformed to a signal equality by applying both operators to u, so that

$$\hat{L}^{-1}\frac{\hat{d}^*}{\hat{\lambda}}(y_p) + c_0^*(\hat{M}\hat{L})^{-1}(y_p) = \hat{L}^{-1}(u) - \hat{L}^{-1}\frac{\hat{c}^*}{\hat{\lambda}}(u) \qquad (\epsilon) \qquad (3.3.6)$$

where (ϵ) reminds us of the presence of exponentially *decaying* terms due to initial conditions. These are decaying because the transfer functions are stable, and the unstable modes are unobservable. Therefore, (3.3.6) is valid for arbitrary initial conditions in the realizations of $\hat{L}^{-1}, \hat{\lambda}$, and $(\hat{M}\hat{L})^{-1}$.

Now, recall that $\overline{w} \in \mathbb{R}^{2n-1}$ is given by

$$\overline{w} = \begin{bmatrix} (sI-\Lambda)^{-1}b_\lambda(u) \\ y_p \\ (sI-\Lambda)^{-1}b_\lambda(y_p) \end{bmatrix} \qquad (3.3.7)$$

and, since $\overline{\theta}^*$ is constant, $\overline{\theta}^{*T}\hat{L}^{-1}(\overline{w})$ is given by

$$\begin{aligned}\overline{\theta}^{*T}\hat{L}^{-1}(\overline{w}) &= \hat{L}^{-1}(\overline{\theta}^{*T}\overline{w}) \\ &= \hat{L}^{-1}\left[\frac{\hat{c}^*}{\hat{\lambda}}(u) + \frac{\hat{d}^*}{\hat{\lambda}}(y_p)\right] \\ &= \hat{L}^{-1}(u) - c_0^*(\hat{M}\hat{L})^{-1}(y_p) \qquad (\epsilon) \qquad (3.3.8)\end{aligned}$$

where we used (3.3.6). Define now

$$v^T := \left[\hat{L}^{-1}(r_p), \hat{L}^{-1}(\overline{w}^T)\right] = \left[(\hat{M}\hat{L})^{-1}(y_p), \hat{L}^{-1}(\overline{w}^T)\right] \in \mathbb{R}^{2n} \qquad (3.3.9)$$

so that (3.3.8) can be written

$$\hat{L}^{-1}(u) = \theta^{*T} v \qquad (\epsilon) \qquad (3.3.10)$$

where θ^* is defined in (3.2.23). (3.3.10) is essential to subsequent derivations, so that we summarize the result in the following proposition.

Proposition 3.3.1 Fundamental Identity

Let $\hat{P}$, $\hat{M}$ satisfy assumptions (A1)-(A2). Let $\hat{L}^{-1}$ be any stable, minimum phase transfer function of relative degree $r = n - m$. Let v and $\bar{w}$ be as defined by (3.3.9), and (3.3.7), with arbitrary initial conditions in the realizations of the transfer functions. Let θ^* be defined by (3.2.7)-(3.2.9), with (3.2.23).

Then for all piecewise continuous $u \in L_{\infty e}$, (3.3.10) is satisfied.

Input Error Identifier Structure

Equation (3.3.10) is of the form studied in chapter 2 for recursive identification. Both the signal $\hat{L}^{-1}(u)$ and v are available from measurements, and the expression is linear in the unknown parameter θ^*.

Therefore, we define the *approximate input error* to be

$$e_2 := \theta^T v - \hat{L}^{-1}(u) \qquad (3.3.11)$$

so that, using (3.3.10)

$$e_2 = \phi^T v \qquad (\epsilon) \qquad (3.3.12)$$

which is of the form of the *linear error equation* studied in chapter 2. Although we considered the input error e_i not to be available, because it would require the realization of a nonproper transfer function, the approximate input error e_2, and the signal v are available, given these considerations.

We also observed in chapter 2 that standard properties of the identification algorithms are not affected by the (ϵ) term. For simplicity, we will omit this term in subsequent derivations.

Relationship Between the Approximate Input Error and the Input Error

The approximate input error e_2 in (3.3.11) can be related to the input error e_i in (3.3.2) by assuming that $u = \theta^T w = c_0 r + \overline{\theta}^T \overline{w}$, and that the controller parameter θ is constant. Then

$$
\begin{aligned}
e_2 &= c_0 \hat{L}^{-1}(r_p) + \overline{\theta}^T \hat{L}^{-1}(\overline{w}) - \hat{L}^{-1}(c_0 r) - \hat{L}^{-1}(\overline{\theta}^T \overline{w}) \\
&= c_0 \hat{L}^{-1}(r_p) + \overline{\theta}^T \hat{L}^{-1}(\overline{w}) - c_0 \hat{L}^{-1}(r) - \overline{\theta}^T \hat{L}^{-1}(\overline{w}) \\
&= c_0 \hat{L}^{-1}(r_p - r) = c_0 \hat{L}^{-1}(e_i) \qquad (3.3.13)
\end{aligned}
$$

Equation (3.3.13) shows the relationship between the approximate input error e_2, and the input error e_i. It should be remembered that it is only valid under the conditions that $u = \theta^T w$, and that θ is constant, but this is *not* necessary for previous derivations.

Assumptions

The algorithm relies on assumptions (A1)-(A3), and the following additional assumption.

(A4) Bound on the High-Frequency Gain

Assume that an upper bound on k_p is known, i.e. that $k_p \leqslant k_{\max}$ for some $k_{\max}$.

The structure of the controller and identifier is shown in figure 3.5, while the complete algorithm is summarized hereafter. The need for assumption (A4), and for the projection of c_0 will be discussed later, in connection with alternate schemes. It will be more obvious from the proof of stability of the algorithm in section 3.7.

Input Error Direct Adaptive Control Algorithm - Practical Implementation

Assumptions

(A1)-(A4)

Data

$n, m, \hat{M}$ (i.e. $k_m, \hat{n}_m, \hat{d}_m$), $k_{\max}$

Input

$r(t), y_p(t) \in \mathbb{R}$

Output

$u(t) \in \mathbf{R}$

Internal Signals

$w(t) \in \mathbf{R}^{2n}$ $(w^{(1)}(t), w^{(2)}(t) \in \mathbf{R}^{n-1})$

$\theta(t) \in \mathbf{R}^{2n}$ $(c_0(t), d_0(t) \in \mathbf{R}, c(t), d(t) \in \mathbf{R}^{n-1})$

$v(t) \in \mathbf{R}^{2n}, e_2(t) \in \mathbf{R}$

Initial conditions are arbitrary, except $c_0(0) \geqslant c_{\min} = k_m / k_{\max} > 0$

Design Parameters

Choose

- $\Lambda \in R^{n-1 \times n-1}$, $b_\lambda \in R^{n-1}$ in controllable canonical form, such that $\det(sI - \Lambda)$ is Hurwitz, and contains the zeros of $\hat{n}_m(s)$
- $\hat{L}^{-1}$ stable, minimum phase transfer function of relative degree $n-m$
- $g, \gamma > 0$

Controller Structure

$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$

$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$

$\theta^T = (c_0, c^T, d_0, d^T)$

$w^T = (r, w^{(1)}, y_p, w^{(2)})$

$u = \theta^T w$

Identifier Structure

$v^T = ((\hat{M}\ \hat{L})^{-1}(y_p), \hat{L}^{-1}(w^{(1)^T}), \hat{L}^{-1}(y_p), \hat{L}^{-1}(w^{(2)^T}))$

$e_2 = \theta^T v - \hat{L}^{-1}(u)$

Normalized Gradient Algorithm with Projection

$$\dot{\theta} = -g \frac{e_2 v}{1 + \gamma v^T v} \qquad \text{if } c_0 = c_{\min} \text{ and } \dot{c}_0 < 0, \text{ then let } \dot{c}_0 = 0.$$

□

Adaptive Observer

The signal generators for $w^{(1)}$ and $w^{(2)}$ ((3.2.17) and (3.2.19)) are almost identical to those used in chapter 2 for identification of the plant parameters (their dimension is now $n-1$ instead of n previously). They are shared by the controller and the identifier. The signal generators for $w^{(1)}$ and $w^{(2)}$ form a *generalized observer*, reconstructing the states of the plant in a specific parametrization. This parametrization has the characteristic of allowing the reconstruction of the states without knowledge of the parameters. The states are used for the state feedback of the controller to the input in what is called a *certainty equivalence* manner, meaning that the parameters used for feedback are the current estimates multiplying the states *as if* they were the true parameters. The identifier with the generalized observer is sometimes called an *adaptive observer* since it provides at the same time estimates of the states and of the parameters.

Separation of Identification and Control

Although we derived a direct adaptive control scheme, the identifier and the controller can distinguished. The gains c_0, c, d_0, d serving to generate u are associated with the controller, while those used to compute e_2 are associated with the identifier. In fact, it is not necessary that these be identical for the identifier error to be as defined in (3.3.12). This is because (3.3.12) was derived using the fundamental identity (3.3.10), which is valid no matter how u is actually computed. In other words, the identifier can be used off-line, without actually updating the controller parameters if necessary. This is also useful for example in case of saturation of the input (cf Goodwin and Mayne (1985)). If the actual input to the LTI plant is different from the computed input $u=\theta^T w$ (due to actuator saturation for example), the identifier will still have consistent input signals, provided that the signal u entering the identifier is the actual input entering the LTI plant.

3.3.2 Output Error Direct Adaptive Control

An output error scheme is based on the output error $e_0 = y_p - y_m$. Note that by applying $\hat{M}\,\hat{L}$ to both sides of (3.3.10), we find

$$\hat{M}(u) = c_0^* \, y_p + \hat{M}(\bar{\theta}^{*T}\bar{w}) \tag{3.3.14}$$

As before, the control input u is set equal to $u = \theta^T w$, but now, this equality is used to derive the identifier error equation

$$e_0 = y_p - y_m = \frac{1}{c_0^*} \hat{M} (u - \bar{\theta}^{*T} \bar{w}) - \hat{M} (r)$$

$$= \frac{1}{c_0^*} \hat{M} ((c_0 - c_0^*) r + (\bar{\theta}^T - \bar{\theta}^{*T}) \bar{w})$$

$$= \frac{1}{c_0^*} \hat{M} (\phi^T w) \qquad (3.3.15)$$

which has the form of the *basic* SPR error equation of chapter 2. The gradient identification algorithms of section 2.6 can therefore be used, provided that $\hat{M}$ is SPR. However, since this requires $\hat{M}$ to have relative degree at most 1, this scheme does not work for plants with relative degree greater than 1.

The approach can however be saved by modifying the scheme, as for example in Narendra, Lin, and Valavani (1980). We now review their scheme for the case when the high-frequency gain k_p is known, and we let $c_0 = c_0^*$.

The controller structure of the output error scheme is identical to the controller structure of the input error scheme, while the identifier structure is different. It relies on the identifier error

$$e_1 = \frac{1}{c_0^*} \hat{M} \hat{L} (\bar{\phi}^T \bar{v} - \gamma \bar{v}^T \bar{v} e_1) \qquad (3.3.16)$$

which is now of the form of the *modified* SPR error equation of chapter 2. As previously, $\bar{v}$ is identical to v, but with the first component removed. Practically, (3.3.16) is not implemented as such. Instead, we use (3.3.10) to obtain

$$e_1 = \frac{1}{c_0^*} \hat{M} \hat{L} (\bar{\theta}^T \bar{v} - \hat{L}^{-1}(u) + c_0^* (\hat{M}\hat{L})^{-1}(y_p) - \gamma \bar{v}^T \bar{v} e_1)$$

$$= y_p - \frac{1}{c_0^*} \hat{M} (u) + \frac{1}{c_0^*} \hat{M} \hat{L} (\bar{\theta}^T \hat{L}^{-1}(\bar{w}) - \gamma \bar{v}^T \bar{v} e_1) \qquad (3.3.17)$$

As before, the control signal is set equal to $u = \theta^T w = c_0^* r + \bar{\theta}^T \bar{w}$, and the equality is used to derive the error equation for the identifier

$$e_1 = y_p - \hat{M} (r) - \frac{1}{c_0^*} \hat{M} (\bar{\theta}^T \bar{w}) + \frac{1}{c_0^*} \hat{M} \hat{L} (\bar{\theta}^T \hat{L}^{-1}(\bar{w}) - \gamma \bar{v}^T \bar{v} e_1)$$

$$= y_p - y_m - \frac{1}{c_0^*} \hat{M} \hat{L} \left((\hat{L}^{-1} \overline{\theta}^T - \overline{\theta}^T \hat{L}^{-1})(\overline{w}) + \gamma \overline{v}^T \overline{v} e_1 \right) \tag{3.3.18}$$

Again, the identifier error involves the output error $e_0 = y_p - y_m$. The additional term, which appeared somewhat mysteriously starting with the work of Monopoli (1974), is denoted

$$y_a = \frac{1}{c_0^*} \hat{M} \hat{L} \left((\hat{L}^{-1} \overline{\theta}^T - \overline{\theta}^T \hat{L}^{-1})(\overline{w}) + \gamma \overline{v}^T \overline{v} e_1 \right) \tag{3.3.19}$$

and the resulting error $e_1 = y_p - y_m - y_a$ is called the *augmented error*, in contrast with the original output error $e_0 = y_p - y_m$.

The error (3.3.16) is of the form of the modified SPR error equation of chapter 2 provided that $\hat{M} \hat{L}$ is a strictly positive real transfer function. If this condition is satisfied, the properties of the identifier will follow, and are the basis of the stability proof of section 3.7.

Assumptions

The algorithm relies on assumptions (A1)-(A3), and the following assumption.

(A5) High-Frequency Gain and SPR Assumptions

Assume that k_p is known, and that there exists $\hat{L}^{-1}$, a stable, minimum phase transfer function of relative degree $n - m - 1$, such that $\hat{M} \hat{L}$ is SPR

The practical implementation of the algorithm is summarized hereafter.

Output Error Direct Adaptive Control Algorithm - Practical Implementation

Assumptions

(A1)-(A3), (A5)

Data

$n, m, \hat{M}$ (i.e. $k_m, \hat{n}_m, \hat{d}_m$), k_p

Input

$r(t), y_p(t) \in \mathbf{R}$

Output

$u(t) \in \mathbf{R}$

Internal Signals

$\bar{w}(t) \in \mathbf{R}^{2n-1}$ $(w^{(1)}(t), w^{(2)}(t) \in \mathbf{R}^{n-1})$

$\bar{\theta}(t) \in \mathbf{R}^{2n-1}$ $(c(t), d(t) \in \mathbf{R}^{n-1}, d_0(t) \in \mathbf{R})$

$\bar{v}(t) \in \mathbf{R}^{2n-1}$

$e_1(t), y_a(t), y_m(t) \in \mathbf{R}$

Initial conditions are arbitrary

Design Parameters

Choose

- $\Lambda \in R^{n-1 \times n-1}$, $b_\lambda \in R^{n-1}$, in controllable canonical form, such that $\det(sI-\Lambda)$ is Hurwitz, and contains the zeros of $\hat{n}_m(s)$
- $\hat{L}^{-1}$ stable, minimum phase transfer function of relative degree $n-m-1$, such that $\hat{M}\,\hat{L}$ is SPR
- $g, \gamma > 0$

Controller Structure

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\bar{\theta}^T = (c^T, d_0, d^T)$$

$$\bar{w}^T = (w^{(1)^T}, y_p, w^{(2)^T})$$

$$c_0^* = k_m / k_p > 0$$

$$u = c_0^* r + \bar{\theta}^T \bar{w}$$

Identifier Structure

$$\bar{v}^T = \hat{L}^{-1}(\bar{w})$$

$$y_m = \hat{M}(r)$$

$$y_a = \frac{1}{c_0^*} \hat{M}\,\hat{L}\,(\hat{L}^{-1}(\bar{\theta}^T \bar{w}) - \bar{\theta}^T \hat{L}^{-1}(\bar{w}) - \gamma \bar{v}^T \bar{v}\, e_1)$$

$$e_1 = y_p - y_m - y_a$$

Gradient Algorithm

$$\dot{\bar{\theta}} = -g\, e_1 \bar{v}$$

□

Differences between input and output error

Traditionally, the starting point in the derivation of model reference adaptive control schemes has been the output error $e_0 = y_p - y_m$. Using the error between the plant and the reference model to update controller parameters is intuitive. However, stability considerations suggest that SPR conditions must be satisfied by the model, and that an augmented error should be used when the relative degree of the plant is greater than 1. The derivation of the input error scheme shows that model reference adaptive control can in fact be achieved *without* formally involving the output error, and without SPR conditions on the reference model.

Important differences should be noted between the input and output error schemes. The first is that the derivation of the equation error (3.3.18) from (3.3.16) relies on the input signal u being equal to the computed value $u = \theta^T w$, at all times. If the input saturates, updates of the identifier will be erroneous. When the input error scheme is used, this problem can be avoided, provided that the actual input entering the LTI plant is available and used in the identifier. This is because (3.3.12) is based on (3.3.10), and does not assume any particular value of u. If needed, the parameters used for identification and control can also be separated, and the identifier can be used "off-line".

A second difference appears between the input and output error schemes when the high-frequency gain k_p is unknown, and the relative degree of the plant is greater than 1. The error e_1 derived in (3.3.16) is not implementable if c_0^* is unknown. Although an SPR error equation can still be obtained in the unknown high-frequency gain case, the solution proposed by Morse (1980) (and also Narendra, Lin, and Valavani (1980)) requires an overparametrization of the identifier which excludes the possibility of asymptotic stability even when PE conditions are satisfied (cf. Boyd and Sastry (1984), Anderson, Dasgupta, and Tsoi (1985)). In view of the recent examples due to Rohrs, and the connections between exponential convergence and robustness (see chapter 5), this appears to be a major drawback of the algorithm.

Another advantage of the input error scheme is to lead to a linear error equation for which other identification algorithms, such as the least-squares algorithm, are available. These algorithms may be an advantageous alternative to the gradient algorithm.

Output Error Direct Adaptive Control - The Relative Degree 1 Case

The condition that $\hat{M}\,\hat{L}$ be SPR is considerably stronger than the condition that $\hat{M}\,\hat{L}$ simply be invertible (as required by the input error scheme, and guaranteed by (A2)). The relative degree of $\hat{L}^{-1}$ however is only required to be $n-m-1$, as compared to $n-m$ for proper invertibility. In the case when the relative degree $n-m$ of the model and of the plant is 1, $\hat{L}^{-1}$ is unnecessary along with the additional signal y_a. The output error direct adaptive control scheme then has a much simpler form, in which the error equation used for identification involves the output error $e_0 = y_p - y_m$ only. The simplicity of this scheme makes it attractive as an example of adaptive control. We assume now

(A6) Relative Degreee 1 Assumption

$n-m=1$

Output Error Direct Adaptive Control Algorithm, Relative Degree 1 - Practical Implementation

Assumptions

(A1)-(A3), (A6)

Data

n, $\hat{M}$ (i.e. k_m, $\hat{n}_m$, $\hat{d}_m$), k_p

Input

$r(t),\ y_p(t) \in \mathbf{R}$

Output

$u(t) \in \mathbf{R}$

Internal Signals

$w(t) \in \mathbf{R}^{2n}$ $(w^{(1)}(t),\ w^{(2)}(t) \in \mathbf{R}^{n-1})$

$\theta(t) \in \mathbf{R}^{2n}$ ($c_0(t), d_0(t) \in \mathbf{R}$, $c(t), d(t) \in \mathbf{R}^{n-1}$)

$y_m(t), e_0(t) \in \mathbf{R}$

Initial conditions are arbitrary

Design Parameters

Choose

- $\Lambda \in R^{n-1 \times n-1}$, $b_\lambda \in R^{n-1}$ in controllable canonical form, and such that

$\det(sI - \Lambda) = \hat{n}_m(s)$

- $g > 0$

Controller Structure

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\theta^T = (c_0, c^T, d_0, d^T)$$

$$w^T = (r, w^{(1)^T}, y_p, w^{(2)^T})$$

$$u = \theta^T w$$

Identifier Structure

$$y_m = \hat{M}(r)$$

$$e_0 = y_p - y_m$$

Gradient Algorithm

$$\dot{\theta} = -g\, e_0 w$$

□

Output Error Equation

The identifier error equation is (3.3.15), and is the *basic* SPR error equation of chapter 2. The high-frequency gain k_p (and consequently c_0) can be assumed to be unknown, but the sign of k_p must still be known to ensure that $c_0^* > 0$, so that $(1/c_0^*)\hat{M}$ is SPR.

3.3.3 Indirect Adaptive Control

In the indirect adaptive control scheme presented in this section, estimates of the plant parameters k_p, $\hat{n}_p$, and $\hat{d}_p$ are obtained using the standard identifier of chapter 2. The controller parameters c_0, $\hat{c}$, $\hat{d}$ are then computed using the relationships resulting from the matching equality (3.2.6).

Note that the dimension of the signals $w^{(1)}$, $w^{(2)}$ used for identification in chapter 2 is n, the order of the plant. For control, it is sufficient that this dimension be $n-1$. However, in order to share the observers for identification and control, we will let their dimension be n. Proposition 3.2.1 is still true then, but the degrees of the polynomials become respectively: $\partial\hat{\lambda} = n$, $\partial\hat{\lambda}_0 = n-m$, $\partial\hat{q} = n-m$, $\partial\hat{d} = n-1$, and $\partial\hat{c} = n-1$. Since $\partial\hat{d} = n-1$, it can be realized as $d^T(sI-\Lambda)^{-1}b$, without the direct gain d_0 from y_p. This a (minor) technical difference, and for simplicity, we will keep our previous notation. Thus, we define

$$\bar{\theta}^T := (c^T, d^T) \in \mathbf{R}^{2n} \qquad \bar{w}^T := (w^{(1)^T}, w^{(2)^T}) \in \mathbf{R}^{2n} \tag{3.3.20}$$

and

$$\theta^T := (c_0, \bar{\theta}^T) \in \mathbf{R}^{2n+1} \qquad w^T := (r, \bar{w}^T) \in \mathbf{R}^{2n+1} \tag{3.3.21}$$

The controller structure is otherwise completely identical to the controller structure described previously.

The identifier parameter is now different from the controller parameter θ. We will denote, in analogy with (2.2.17)

$$\pi^T := (a^T, b^T) := (a_1, \cdots a_{m+1}, 0, \cdots, b_1, \cdots b_n) \in \mathbf{R}^{2n} \tag{3.3.22}$$

Since the relative degree is assumed to be known, there is no need to update the parameters $a_{m+2}, \ldots$ so that we let these parameters be zero in (3.3.22). The corresponding components of $\bar{w}$ are thus not used for identification. We let $\tilde{w}$ be equal to $\bar{w}$ except for those components which are not used, and are thus set to zero, so that

$$\tilde{w}^T := (w_1^{(1)}, \cdots w_{m+1}^{(1)}, 0 \cdots, w^{(2)^T}) \in \mathbf{R}^{2n} \tag{3.3.23}$$

A consequence (that will be used in the stability proof in section 3.7) is that the relative degree of the transfer function from $u \to \bar{w}$ is at least 1, while the relative degree of the transfer function from $u \to \tilde{w}$ is at least $n-m$.

The nominal value of the identifier parameter π^* can be found from the results of chapter 2 through the polynomial equalities in (2.2.5), that is

$$\hat{a}^*(s) = a_1^* + a_2^* s + \cdots a_{m+1}^* s^m = k_p \hat{n}_p(s)$$
$$\hat{b}^*(s) = b_1^* + b_2^* s + \cdots b_n^* s^{n-1} = \hat{\lambda}(s) - \hat{d}_p(s) \tag{3.3.24}$$

The *identifier parameter error* is now denoted

$$\psi := \pi - \pi^* \quad \in \mathbf{R}^{2n} \tag{3.3.25}$$

The transformation $\pi \to \theta$ is chosen following a certainty equivalence principle to be the same as the transformation $\pi^* \to \theta^*$, as in (3.2.7)-(3.2.9). Note that our estimate of the high-frequency gain k_p is a_{m+1}. Since $c_0^* = k_m / k_p$, we will let $c_0 = k_m / a_{m+1}$. The control input u will be unbounded if a_{m+1} goes to zero, and to avoid this problem, we make the following assumption.

(A7) Bound on the High-Frequency Gain

Assume $k_p \geqslant k_{\min} > 0$.

The practical implementation of the indirect adaptive control algorithm is summarized hereafter.

Indirect Adaptive Control Algorithm - Practical Implementation

Assumptions

(A1)-(A3), (A7)

Data

$n, m, \hat{M}$ (i.e. $k_m, \hat{n}_m, \hat{d}_m$), $k_{\min}$

Input

$r(t), y_p(t) \in \mathbf{R}$

Output

$u(t) \in \mathbf{R}$

Internal Signals

$w(t) \in \mathbf{R}^{2n+1}$ ($w^{(1)}(t), w^{(2)}(t) \in \mathbf{R}^n$)

$\theta(t) \in \mathbf{R}^{2n+1}$ $(c_0(t) \in \mathbf{R}, c(t), d(t) \in \mathbf{R}^n)$

$\pi(t) \in \mathbf{R}^{2n}$ $(a(t), b(t) \in \mathbf{R}^n)$

$\tilde{w}(t) \in \mathbf{R}^{2n}$

$y_i(t), e_3(t) \in \mathbf{R}$

Initial conditions are arbitrary, except $a_{m+1}(0) > k_{\min}$

Design Parameters

Choose

- $\Lambda \in \mathbf{R}^{n \times n}$, $b_\lambda \in \mathbf{R}^n$ in controllable canonical form, such that $\det(sI - \Lambda) = \hat{\lambda}(s)$ is Hurwitz, and $\hat{\lambda}(s) = \hat{\lambda}_0(s)\hat{n}_m(s)$
- $g, \gamma > 0$

Controller Structure

$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$

$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$

$\theta^T = (c_0, c^T, d^T) = (c_0, c_1, \cdots c_n, d_1, \cdots d_n)$

$w^T = (r, w^{(1)^T}, w^{(2)^T})$

$u = \theta^T w$

Identifier Structure

$\pi^T = (a^T, b^T) = (a_1, \cdots a_{m+1}, 0 \cdots, b_1, \cdots b_n)$

$\tilde{w} = (w_1^{(1)}, \cdots w_{m+1}^{(1)}, 0 \cdots, w^{(2)^T})$

$y_i = \pi^T \tilde{w}$

$e_3 = \pi^T \tilde{w} - y_p$

Normalized Gradient Algorithm with Projection

$$\dot{\pi} = -g \frac{e_3 \tilde{w}}{1 + \gamma \tilde{w}^T \tilde{w}} \quad \text{if } a_{m+1} = k_{\min} \text{ and } \dot{a}_{m+1} < 0, \text{ then let } \dot{a}_{m+1} = 0$$

Transformation Identifier Parameter → Controller Parameter

Let the polynomials with time-varying coefficients

$\hat{a}(s) = a_1 + \ldots a_{m+1} s^m \qquad \hat{c}(s) = c_1 + \ldots c_n s^{n-1}$

$\hat{b}(s) = b_1 + \ldots b_n\, s^{n-1}$ $\qquad$ $\hat{d}(s) = d_1 + \ldots d_n\, s^{n-1}$

Divide $\hat{\lambda}_0 \hat{d}_m$ by $(\hat{\lambda} - \hat{b})$, and let $\hat{q}$ be the quotient

θ is given by the coefficients of the polynomials

$$\hat{c} = \hat{\lambda} - \frac{1}{a_{m+1}}\, \hat{q}\, \hat{a}$$

$$\hat{d} = \frac{1}{a_{m+1}} (\hat{q}\, \hat{\lambda} - \hat{q}\, \hat{b} - \hat{\lambda}_0 \hat{d}_m)$$

and by

$$c_0 = \frac{k_m}{a_{m+1}}$$

□

Transformation Identifier Parameter → Controller Parameter

We assumed that the transformation form the identifier parameter π to the controller parameter θ is performed *instantaneously*. Note that $\hat{\lambda} - \hat{b}$ is a monic polynomial, so that $\hat{q}$ is also a monic polynomial (of degree $n - m$). Its coefficients can be expressed as the sum of products of coefficients of $\hat{\lambda}_0 \hat{d}_m$ and $\hat{\lambda} - \hat{b}$. The same is true for $\hat{c}$, $\hat{d}$, and c_0 with an additional division by a_{m+1}. Therefore, given n and m, the transformation consists of a fixed number of multiplications, additions, and a division.

Note also that if the coefficients of $\hat{a}$ and $\hat{b}$ are bounded, and if a_{m+1} is bounded away from zero (as is guaranteed by the projection), then the coefficients of $\hat{q}$, $\hat{c}$, $\hat{d}$, and c_0 are bounded. Therefore, the transformation is also continuously differentiable, and has bounded derivatives.

3.3.4 Connections to Alternate Schemes

The input error scheme is closely related to the schemes presented in discrete time by Goodwin and Sin (1984), and in continuous time by Goodwin and Mayne (1985). Their identifier structure is identical to the structure used here, but their controller structure is somewhat different. In our notation, Goodwin and Mayne choose

$$\hat{M}(s) = k_m \frac{\hat{n}(s)}{\lambda(s)\, \hat{L}(s)} \qquad (3.3.26)$$

where $\hat{n}$, λ and $\hat{L}$ are *polynomials* of degree $\leq n$, n, and $n-m$ respectively. The polynomials $\hat{\lambda}$, $\hat{L}$ are used for similar purposes as in the input error scheme. However, except for possible pole-zero cancellations, $\hat{\lambda}\hat{L}$ now also defines the model poles in (3.3.26). The filtered reference input

$$\bar{r} = k_m \frac{\hat{n}(s)}{\hat{\lambda}(s)}(r) \qquad (3.3.27)$$

is used as input to the actual controller. Then, the transfer function $\hat{\bar{r}} \rightarrow \hat{y}_p$ is made to match $\hat{L}^{-1}$, so that the transfer function from $r \rightarrow y_p$ is $\hat{M}$. Thus, by prefiltering the input, the control problem of matching a transfer function $\hat{M}$ is altered to the problem of matching the arbitrary all-pole transfer function $\hat{L}^{-1}$.

The input error adaptive control scheme of section 3.3.1 can be used to achieve this new objective, and is represented on figure 3.6. This scheme is the one obtained by Goodwin and Mayne (up to a small remaining difference described hereafter). Since the new model is $\hat{L}^{-1}$, the new transfer function $\hat{M}\hat{L}$ is equal to 1. Note that, in this instance, the input and output errors are identical, and the input and output error schemes are very similar. The analysis is also considerably simplified.

Goodwin and Mayne's algorithms essentially control the plant by reducing the transfer function to an all-pole transfer function of relative degree $n-m$. The additional dynamics are provided by prefiltering the reference input. Thus, the input error scheme presented in section 3.3.1 is a more general scheme, allowing for the placement of all the closed-loop poles directly at the desired locations without prefiltering.

Note that since identification and control can be separated in the input error scheme, we may identify $1/c_0$ and $\bar{\theta}/c_0$, rather than c_0 and $\bar{\theta}$. This is shown in figure 3.6. By dividing the identifier error e_2 by c_0, the appropriate linear error equation may be found and used for identification.

The problems encountered are different depending whether we identify c_0 or $1/c_0$. If we identify $1/c_0$, as we did in the indirect scheme, the control input $u = c_0 r + \bar{\theta}^T \bar{w}$ will be unbounded if the estimate of $1/c_0$ goes to zero. To avoid the zero crossing, we require knowledge of the sign of $1/c_0$ (i.e. of k_p), and of a lower bound on $1/c_0$, i.e. a *lower* bound on k_p to be used with the projection algorithm.

If we identify c_0 directly, as we did in the input error scheme, a different problem appears. If $c_0 = 0$, and $\bar{\theta} = 0$, then $u = 0$, and $e_2 = 0$ (cf. figure 3.5). No adaptation will occur ($\dot{\phi} = 0$) although $y_p - y_m$ does not tend necessarily to zero, and may even be unbounded. This is an identification problem, since we basically lose information in the regression vector. To avoid it, we require the knowledge of the sign of c_0 (i.e. of k_p), and a lower bound on c_0, i.e. an *upper* bound on k_p, to be used by the projection algorithm.

3.4 The Stability Problem in Adaptive Control

Stability Definitions

Various definitions and concepts of stability have been proposed. A classical definition for systems of the form

$$\dot{x} = f(t, x) \tag{3.4.1}$$

is the stability in the sense of Lyapunov defined in chapter 1.

The adaptive systems described so far are of the special form

$$\dot{x} = f(t, x, r(t)) \tag{3.4.2}$$

where r is the input to the system, and x is the total state of the system, including the plant, the controller, and the identifier. For practical reasons, stability in the sense of Lyapunov is not sufficient for adaptive systems. As we recall, this definition is a local property, guaranteeing that the trajectories will remain arbitrarily close to the equilibrium, *when started sufficiently close*. In adaptive systems, we do not have any control on how close initial conditions are to equilibrium values. A natural stability concept is then the *bounded-input bounded-state* stability (BIBS): for any $r(.)$ bounded, and $x_0 \in R^n$, the solution $x(.)$ remains bounded. This is the concept of stability that will be used in this chapter.

The Problem of Proving Stability in Adaptive Control

The stability of the identifiers presented in chapter 2 was assessed in theorem 2.4.5. There, the stability of the plant was guaranteed independently. In adaptive control, the stability of the plant must be guaranteed by the identifier, which seriously complicates the problem. The stability of the *overall adaptive system*, which includes the plant, the controller, and the identifer, must then be considered.

To understand the nature of the problem, we will take a general approach in this section, and consider the generic model reference adaptive control system shown in figure 3.7. The signals and systems defined previously can be recognized. θ is the controller parameter, and π is the identifier parameter. In the case of direct control, $\theta = \pi$, i.e. the parameter being identified is directly the controller parameter. The identifier error may be the output error $e_0 = y_p - y_m$, the input error $e_i = r_p - r$, or any other error used for identification.

The problem of stability can be understood as follows. Intuitively, the plant with the control loop will be stable if θ is sufficiently close to the true value θ^*. However, as we saw in chapter 2, the convergence of the identifier is dependent on the stability and persistent excitation of signals originating from the control loop.

To break this circular argument, we must first express properties of the identifier that are independent of the stability and persistency of excitation of these signals. Such properties were already derived in chapter 2, and were expressed in terms of the identifier error. Recall that the identifier *parameter* error $\pi - \pi^*$ does not converge to zero, but that only the identifier error converges to zero in some sense. Thus, we cannot argue that for t sufficiently large, the controller parameter θ will be arbitrarily close to the nominal value that stabilizes the plant-control loop.

Instead of relying on the convergence of θ to θ^* to prove stability, we can express the control signal as a nominal control signal - that makes the controlled plant match the reference model -, plus a control error. The problem then is to transfer the properties of the identifier to the control loop, i.e the identifier error to the control error, and prove stability. Several difficulties are encountered here. First, the transformation $\theta(\pi)$ is usually nonlinear. In direct adaptive control, the transformation is the identity, and the proof is consequently simplified. Another difficulty arises however from the different signals v and w used for identification and control. A major step will be to transfer properties of the identifier involving v to properties of the controller involving w. Provided that the resulting control error is a "small" gain from plant signals, the proof of stability will basically be a *small gain theorem* type of proof, a generic proof to assess the stability of nonlinear time varying systems (cf. Desoer and Vidyasagar (1975)).

3.5 Analysis of the Model Reference Adaptive Control System

We now return to the model reference adaptive control system presented in sections 3.1-3.3. The results derived in this section are the basis for analyses presented in this and following chapters. Most identities involve signals which are not available in practice (since $\hat{P}$ is unknown), but are well-defined for the analysis. Most results also rely on the control input being defined by

$$u = \theta^T w$$

$$\theta^T = (c_0, c^T, d_0, d^T)$$

$$w^T = (r, w^{(1)T}, y_p, w^{(2)T}) \tag{3.5.1}$$

Error Formulation

It will be useful to represent the adaptive system in terms of its deviation with respect to the ideal situation when $\theta = \theta^*$, i.e. $\phi = 0$. This step is similar to transfering the equilibrium point of a differential equation as (3.4.1) to $x = 0$ by a change of coordinates.

Recall that we defined r_p in (3.3.1) as

$$r_p = \hat{M}^{-1}(y_p) \tag{3.5.2}$$

while

$$y_m = \hat{M}(r) \tag{3.5.3}$$

Applying $\hat{L}$ to (3.3.10), it follows, since θ^* is constant, that

$$u = c_0^* r_p + \bar{\theta}^{*T} \bar{w} \tag{3.5.4}$$

and, since u is given by (3.5.1),

$$r_p = r + \frac{1}{c_0^*} \phi^T w \tag{3.5.5}$$

Further, applying $\hat{M}$ to both sides of (3.5.5)

$$y_p = y_m + \frac{1}{c_0^*} \hat{M}(\phi^T w) \tag{3.5.6}$$

The signal $\phi^T w$ will be called the *control error*. We note that the *input error* $e_i = r_p - r$ is directly proportional to the *control error* $\phi^T w$ (cf (3.5.5)), while the *output error* $e_0 = y_p - y_m$ is related to the control error through the model transfer function $\hat{M}$ (cf (3.5.6)).

Since $y_p = \hat{P}(u) = \hat{M}(r_p)$, the control input can also be expressed in terms of the control error as

$$u = \hat{P}^{-1}\hat{M}(r_p) = \hat{P}^{-1}\hat{M}\left(r + \frac{1}{c_0^*}\phi^T w\right) \tag{3.5.7}$$

and the vector $\bar{w}$ is similarly expressed as

$$\bar{w} = \begin{bmatrix} w^{(1)} \\ y_p \\ w^{(2)} \end{bmatrix} = \begin{bmatrix} (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1}\hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} \left(r + \frac{1}{c_0^*}\phi^T w\right) \tag{3.5.8}$$

while v (cf. (3.3.9)) is given by

$$v = \hat{L}^{-1} \begin{bmatrix} 1 \\ (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1}\hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} \left(r + \frac{1}{c_0^*}\phi^T w\right) \tag{3.5.9}$$

For the purpose of the analysis alone, we will define

$$z := \hat{L}(v) = \begin{bmatrix} r_p \\ \bar{w} \end{bmatrix} = \begin{bmatrix} 1 \\ (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1}\hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} \left(r + \frac{1}{c_0^*}\phi^T w\right) \tag{3.5.10}$$

Note that the transfer functions appearing in (3.5.6)-(3.5.10) are all stable (using assumptions (A1)-(A2), and the definitions of Λ and $\hat{L}^{-1}$).

Model Signals

The *model signals* are defined as the signals corresponding to the plant signals when $\theta = \theta^*$, i.e. $\phi = 0$. As expected, the model signals corresponding to y_p and r_p are y_m and r respectively (cf (3.5.6) and (3.5.5)). Similarly, we define

$$\bar{w}_m := \begin{bmatrix} w_m^{(1)} \\ y_m \\ w_m^{(2)} \end{bmatrix} := \begin{bmatrix} (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1}\hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r)$$

$$:= \hat{H}_{r\bar{w}_m}(r) \tag{3.5.11}$$

and

$$v_m := \hat{L}^{-1}(z_m) := \hat{L}^{-1}\begin{pmatrix} 1 \\ (sI-\Lambda)^{-1}b_\lambda \hat{P}^{-1}\hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1}b_\lambda \hat{M} \end{pmatrix}(r) \tag{3.5.12}$$

By defining

$$w_m := \begin{pmatrix} r \\ \bar{w}_m \end{pmatrix} \tag{3.5.13}$$

we note the remarkable fact that

$$w_m = z_m \tag{3.5.14}$$

Since the transfer functions relating r to the model signals are all stable, and since r is bounded (assumption (A3)), it follows that all models signals are bounded functions of time. Consequently, if the differences between plant and model signals are bounded, the plant signals will be bounded.

State-Space Description

We now show how a state-space description of the overall adaptive system can be obtained. In particular, we will check that no cancellation of possibly unstable modes occurs when $\theta = \theta^*$.

The plant has a minimal state-space representation $[A_p, b_p, c_p^T]$ such that

$$\hat{P}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} = c_p^T (sI - A_p)^{-1} b_p \tag{3.5.15}$$

With the definitions of $w^{(1)}$, $w^{(2)}$ in (3.2.17)-(3.2.19), the plant with observer is described by

$$\begin{aligned} \dot{x}_p &= A_p x_p + b_p u \\ \dot{w}^{(1)} &= \Lambda w^{(1)} + b_\lambda u \\ \dot{w}^{(2)} &= \Lambda w^{(2)} + b_\lambda y_p = \Lambda w^{(2)} + b_\lambda c_p^T x_p \end{aligned} \tag{3.5.16}$$

The control input u can be expressed in terms of its desired value, plus the control error $\phi^T w$, as

$$u = \theta^T w = \theta^{*T} w + \phi^T w \tag{3.5.17}$$

so that

$$\begin{pmatrix} \dot{x}_p \\ \dot{w}^{(1)} \\ \dot{w}^{(2)} \end{pmatrix} = \begin{pmatrix} A_p + b_p d_0^* c_p^T & b_p c^{*T} & b_p d^{*T} \\ b_\lambda d_0^* c_p^T & \Lambda + b_\lambda c^{*T} & b_\lambda d^{*T} \\ b_\lambda c_p^T & 0 & \Lambda \end{pmatrix} \cdot \begin{pmatrix} x_p \\ w^{(1)} \\ w^{(2)} \end{pmatrix}$$

$$+ \begin{pmatrix} b_p \\ b_\lambda \\ 0 \end{pmatrix} \phi^T w + \begin{pmatrix} b_p \\ b_\lambda \\ 0 \end{pmatrix} c_0^* r$$

$$y_p = c_p^T x_p \tag{3.5.18}$$

Defining $x_{pw} \in \mathbf{R}^{3n-2}$ to be the total state of the plant and observer, this equation is rewritten

$$\dot{x}_{pw} = A_m x_{pw} + b_m \phi^T w + b_m c_0^* r$$

$$y_p = c_m^T x_{pw} \tag{3.5.19}$$

where $A_m \in \mathbf{R}^{3n-2 \times 3n-2}$, $b_m \in \mathbf{R}^{3n-2}$ and $c_m \in \mathbf{R}^{3n-2}$ are defined through (3.5.18). Since the transfer function from $r \to y_p$ is $\hat{M}$ when $\phi = 0$, we must have that $c_m^T (sI - A_m)^{-1} b_m = (1/c_0^*) \hat{M}(s)$, i.e. that $[A_m, b_m, c_m^T]$ is a representation of the model transfer function, divided by c_0^*. Therefore, we can also represent the model and its output by

$$\dot{x}_m = A_m x_m + b_m c_o^* r$$

$$y_m = c_m^T x_m \tag{3.5.20}$$

Note that although the transfer function $\hat{M}$ is stable, its representation is non-minimal, since the order of $\hat{M}$ is n, while the dimension of A_m is $3n-2$. It can be checked, using the Popov-Belevitch-Hautus rank test (Kailath (1980), p. 136), that (A_m, b_m) is a controllable pair. We can find where the unobservable modes are located by noting that the representation of the model is that of figure 3.8. Using standard transfer function manipulations, but avoiding cancellations, we get

$$c_m^T (sI - A_m)^{-1} b_m = \frac{\dfrac{k_p \hat{n}_p \hat{\lambda}}{(\hat{\lambda} - \hat{c}^*) \hat{d}_p}}{1 - \dfrac{k_p \hat{n}_p \hat{\lambda}}{(\hat{\lambda} - \hat{c}^*) \hat{d}_p} \dfrac{\hat{d}^*}{\hat{\lambda}}}$$

$$= \frac{k_p \hat{\lambda} \hat{n}_p \hat{\lambda}_0 \hat{n}_m}{\hat{\lambda}((\hat{\lambda} - \hat{c}^*)\hat{d}_p - k_p \hat{n}_p \hat{d}^*)} \tag{3.5.21}$$

and, using the matching equality (3.2.6)

$$c_m{}^T (sI - A_m)^{-1} b_m = \frac{1}{c_0^*} k_m \frac{\hat{n}_m}{\hat{d}_m} \frac{\hat{\lambda}\hat{\lambda}_0 \hat{n}_p}{\hat{\lambda}\hat{\lambda}_0 \hat{n}_p} = \frac{1}{c_0^*} \hat{M} \tag{3.5.22}$$

Thus, the unobservable modes are those of $\hat{\lambda}$, $\hat{\lambda}_0$, and $\hat{n}_p$, which are all stable by choice of $\hat{\lambda}$, $\hat{\lambda}_0$, and by assumption (A1). In other words, A_m is a stable matrix.

Since r is assumed to be bounded and A_m is stable, the state vector trajectory x_m is bounded. We can represent the plant states as their differences with the model states, letting the *state error* $e = x_{pw} - x_m \in \mathbf{R}^{3n-2}$, so that

$$\dot{e} = A_m e + b_m \phi^T w$$

$$e_0 = y_p - y_m = c_m^T e \tag{3.5.23}$$

and

$$e_0 = y_p - y_m = \frac{1}{c_0^*} \hat{M}(\phi^T w) = \hat{M}(\frac{1}{c_0^*} \phi^T w) \tag{3.5.24}$$

which is (3.5.6), derived above through a somewhat shorter path.

Note that (3.5.23) is not a linear differential equation representing the plant with controller, because w depends on e. This can be resolved by expressing the dependence of w on e as

$$w = w_m + Q e \tag{3.5.25}$$

where

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ c_p^T & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \in \begin{pmatrix} \mathbf{R}^{1\times n} & \mathbf{R}^{1\times n-1} & \mathbf{R}^{1\times n-1} \\ \mathbf{R}^{n-1\times n} & \mathbf{R}^{n-1\times n-1} & \mathbf{R}^{n-1\times n-1} \\ \mathbf{R}^{1\times n} & \mathbf{R}^{1\times n-1} & \mathbf{R}^{1\times n-1} \\ \mathbf{R}^{n-1\times n} & \mathbf{R}^{n-1\times n-1} & \mathbf{R}^{n-1\times n-1} \end{pmatrix} = \mathbf{R}^{2n\times 3n-2} \tag{3.5.26}$$

A differential equation representing the plant with controller is then

$$\dot{e} = A_m e + b_m \phi^T w_m + b_m \phi^T Q e$$

$$e_1 = c_m^T e \tag{3.5.27}$$

where w_m is an exogeneous, bounded input.

Complete Description - Output Error, Relative Degree 1 Case

To completely describe the adaptive system, one must simply add to this set of differential equations the set corresponding to the identifier. For example, in the case of the output error adaptive control scheme for relative degree 1 plants, the overall adaptive system (including the plant, controller, and identifier) is described by

$$\dot{e} = A_m\, e + b_m\, \phi^T\, w_m + b_m\, \phi^T\, Q\, e$$

$$\dot{\phi} = -g\, c_m^T\, e\, w_m - g\, c_m^T\, e\, Q\, e \tag{3.5.28}$$

As for all adaptive control schemes presented in this work, the adaptive control scheme is described by a nonlinear time varying ordinary differential equation. This specific case will be used in subsequent chapters as a convenient example.

3.6 Useful Lemmas

The following lemmas are useful to prove the stability of adaptive control schemes. Most lemmas are inspired from lemmas that are present in one form or another in existing stability proofs. In contrast with Sastry (1984), and Narendra, Annaswamy, and Singh (1985), we do not use any ordering of signals (order relations $o(.)$ and $O(.)$), but keep relationships between signals in terms of norm inequalities.

The systems considered in this section are of the general form

$$y = H\,(u) \tag{3.6.1}$$

where $H : L_{pe} \to L_{pe}$ is a SISO *causal* operator, that is, such that

$$y_t = (H\,(u_t\,))_t \tag{3.6.2}$$

for all $u \in L_{pe}$, and for all $t \geqslant 0$. Lemmas 3.6.1-3.6.5 further restrict the attention to LTI systems with proper transfer functions $\hat{H}(s)$.

Lemma 3.6.1 is a standard result in linear system theory, and relates the L_p norm of the output to the L_p norm of the input.

Lemma 3.6.1 BIBO Stability

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function. Let h be the impulse response corresponding to $\hat{H}$.

If $\hat{H}$ is stable

Then for all $p \in [1, \infty]$, and for all $u \in L_p$

$$\| y \|_p \leq \| h \|_1 \| u \|_p + \| \epsilon \|_p \tag{3.6.3}$$

for all $u \in L_{\infty e}$

$$| y(t) | \leq \| h \|_1 \| u_t \|_\infty + | \epsilon(t) | \qquad \text{for all } t \geq 0 \tag{3.6.4}$$

where $\epsilon(t)$ is an exponentially decaying term due to the initial conditions.

Proof of Lemma 3.6.1 cf. Desoer and Vidyasagar (1975), p. 241.

It is useful, although not standard, to obtain a result that is the converse of lemma 3.6.1, i.e. with u and y interchanged in (3.6.3)-(3.6.4). Such a lemma can be found in Narendra, Lin, and Valavani (1980), Narendra (1984), Sastry (1984), Narendra, Annaswamy, and Singh (1985), for $p = \infty$. Lemma 3.6.2 is a version that is valid for $p \in [1, \infty]$, with a completely different proof (see appendix).

Note that if $\hat{H}$ is minimum phase, and has relative degree zero, then it has a proper and stable inverse, and the converse result is true by lemma 3.6.1. If $\hat{H}$ is minimum phase, but has relative degree greater than zero, then the converse result will be true provided that additional conditions are placed on the input signal u. This is the result of lemma 3.6.2.

Lemma 3.6.2 BOBI Stability

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function. Let $p \in [1, \infty]$.

If $\hat{H}$ is minimum phase

For some $k_1, k_2 \geq 0$, and for all $t \geq 0$, $u_t, \dot{u}_t \in L_{pe}$, and

$$\| \dot{u}_t \|_p \leq k_1 \| u_t \|_p + k_2 \tag{3.6.5}$$

Then there exist $a_1, a_2 \geq 0$ such that

$$\| u_t \|_p \leq a_1 \| y_t \|_p + a_2 \tag{3.6.6}$$

for all $t \geqslant 0$.

Proof of Lemma 3.6.2 in appendix.

It is also interesting to note the following equivalence, related to L_∞ norms. For all $a, b \in L_{\infty e}$

$$|a(t)| \leqslant k_1 \|b_t\|_\infty + k_2 \quad \text{iff} \quad \|a_t\|_\infty \leqslant k_1 \|b_t\|_\infty + k_2 \tag{3.6.7}$$

The same is true if the right-hand side of the inequalities is replaced by any positive, monotonically increasing function of time. Therefore, for $p = \infty$, the assumption (3.6.5) of lemma 3.6.2 is that u is regular (cf. definition in (2.4.14)). In particular, lemma 3.6.2 shows that if u is regular and y is bounded, then u is bounded. Lemma 3.6.2 therefore leads to the following corollary.

Corollary 3.6.3 Properties of Regular Signals

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function. Let $\hat{H}$ be stable and minimum phase.

(a) *if* u is regular

then $|u(t)| \leqslant a_1 \|y_t\|_\infty + a_2$ for all $t \geqslant 0$

(b) *if* u is bounded, and $\hat{H}$ is strictly proper

then y is regular

(c) *if* u is regular

then y is regular

The properties are also valid if u and y are vectors such that each component y_i of y is related to the corresponding u_i through $y_i = \hat{H}(u_i)$.

Proof of Corollary 3.6.3 in appendix.

In chapter 2, a main property of the identification algorithms was obtained in terms of a gain belonging to L_2. Lemmas 3.6.4 is useful for such gains appearing in connection with systems with rational transfer function $\hat{H}$.

Lemma 3.6.4

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function.

If $\hat{H}$ is stable, $u \in L_{\infty e}$, and for some $x \in L_{\infty e}$

$$|u(t)| \leqslant \beta_1(t) \|x_t\|_\infty + \beta_2(t) \tag{3.6.8}$$

for all $t \geqslant 0$, and for some $\beta_1, \beta_2 \in L_2$

Then there exist $\gamma_1, \gamma_2 \in L_2$ such that, for all $t \geqslant 0$

$$|y(t)| \leqslant \gamma_1(t) \|x_t\|_\infty + \gamma_2(t) \tag{3.6.9}$$

If in addition, either $\hat{H}$ is strictly proper,

or $\beta_1, \beta_2 \in L_\infty$, and $\beta_1(t), \beta_2(t) \to 0$ as $t \to \infty$

Then $\gamma_1, \gamma_2 \in L_\infty$, and $\gamma_1(t), \gamma_2(t) \to 0$ as $t \to \infty$

Proof of Lemma 3.6.4 in appendix.

The following lemma is the so-called *swapping lemma* (Morse 1980), and is essential to the stability proofs presented in section 3.7.

Lemma 3.6.5 Swapping Lemma

Let $\phi, w : \mathbf{R}_+ \to \mathbf{R}^n$, and ϕ be differentiable. Let $\hat{H}$ be a proper, rational transfer function.

If $\hat{H}$ is stable, with a minimal realization

$$\hat{H} = c^T (sI - A)^{-1} b + d \tag{3.6.10}$$

Then

$$\hat{H}(w^T \phi) - \hat{H}(w^T)\phi = \hat{H}_c(\hat{H}_b(w^T)\dot{\phi}) \tag{3.6.11}$$

where

$$\hat{H}_b = (sI - A)^{-1} b \qquad \hat{H}_c = c^T (sI - A)^{-1} \tag{3.6.12}$$

Proof of Lemma 3.6.5 in appendix.

Lemma 3.6.6 is the so-called *small gain theorem* (Desoer and Vidyasagar (1975)), and concerns general nonlinear time-varying systems connected as shown in figure 3.9.

Roughly speaking, the small gain theorem states that the system of figure 3.9, with inputs u_1, u_2, and outputs y_1, y_2 is BIBO stable, provided that H_1 and H_2 are BIBO stable, and provided that the product of the gains of H_1 and H_2 is small enough (less than 1).

Lemma 3.6.6 Small Gain Theorem

Consider the system shown in figure 3.9. Let $p \in [1, \infty]$. Let H_1, $H_2 : L_{pe} \to L_{pe}$ be causal operators. Let $e_1, e_2 \in L_{pe}$, and define u_1, u_2 by

$$u_1 = e_1 + H_2(e_2)$$

$$u_2 = e_2 - H_1(e_1) \tag{3.6.13}$$

Suppose that there exist constants β_1, β_2, and $\gamma_1, \gamma_2 \geq 0$, such that

$$\| H_1(e_1)_t \| \leq \gamma_1 \| e_{1_t} \| + \beta_1$$

$$\| H_2(e_2)_t \| \leq \gamma_2 \| e_{2_t} \| + \beta_2 \qquad \text{for all } t \geq 0 \tag{3.6.14}$$

If $\quad \gamma_1 \cdot \gamma_2 < 1$

Then

$$\| e_{1_t} \| \leq (1 - \gamma_1\gamma_2)^{-1} \left(\| u_{1_t} \| + \gamma_2 \| u_{2_t} \| + \beta_2 + \gamma_2\beta_1 \right)$$

$$\| e_{2_t} \| \leq (1 - \gamma_1\gamma_2)^{-1} \left(\| u_{2_t} \| + \gamma_1 \| u_{1_t} \| + \beta_1 + \gamma_1\beta_2 \right) \qquad \text{for all } t \geq 0 \tag{3.6.15}$$

If in addition, $u_1, u_2 \in L_p$

Then $e_1, e_2, y_1 = H_1(e_1), y_1 = H_2(e_2) \in L_p$, and (3.6.15) is valid with all subscripts t dropped.

Proof of lemma 3.6.6 cf. Desoer and Vidyasagar (1975), p. 41.

3.7 Stability Proofs

3.7.1 Stability - Input Error Direct Adaptive Control

The following theorem is the main stability theorem for the input error direct adaptive control scheme. It shows that, given any initial condition, and any bounded input $r(t)$, the states of the adaptive system remain bounded (BIBS stability), and the output error tends to zero, as $t \to \infty$. Further, the error is bounded by an L_2 function.

We also obtain that the difference between the regressor vector v and the corresponding model vector v_m tend to zero as $t \to \infty$, and is in L_2. This result will be useful to prove exponential convergence in section 3.8.

We insist that initial conditions must be in some small B_h, because although the properties are valid for any initial conditions, the convergence of the error to zero, and the L_2 bounds are not uniform globally. For example, there does not exist a fixed L_2 function that bounds the output error, no matter how large the initial conditions are.

Theorem 3.7.1

Consider the input error direct adaptive control scheme described in section 3.3.1, with initial conditions in an arbitrary B_h.

Then

(a) all states of the adaptive system are bounded functions of time.

(b) the output error $e_o = y_p - y_m \in L_2$, and tends to zero as $t \to \infty$
the regressor error $v - v_m \in L_2$, and tends to zero as $t \to \infty$.

Comments

The proof of the theorem is organized to highlight the main steps that we described in section 3.4.

Although the theorem concerns the adaptive scheme with the gradient algorithm, examination of the proof shows that it only requires the standard identifier properties resulting from theorems 2.4.1-2.4.4. Therefore, theorem 3.7.1 is also valid if the normalized gradient algorithm is replaced by the normalized LS algorithm with covariance resetting.

Proof of Theorem 3.7.1

(a) *Derive properties of the identifier that are independent of the boundedness of the regressor*

These results were obtained in theorems 2.4.1-2.4.4, and led to

$$|\phi(t)\, v(t)| = \beta(t)\, \|v_t\|_\infty + \beta(t)$$

$$\beta \in L_2 \cap L_\infty$$

$$\phi \in L_\infty \qquad \dot{\phi} \in L_2 \cap L_\infty$$

$$c_0(t) \geq c_{\min} > 0 \qquad \text{for all } t \geq 0 \tag{3.7.1}$$

The inequality for $c_0(t)$ follows from the use of the projection in the update law.

(b) *Express the system states and inputs in term of the control error*

This was done in section 3.5, and led to the control error $\phi^T w$, with

$$r_p = r + \frac{1}{c_0^*} \phi^T w$$

$$u = \hat{P}^{-1} \hat{M} (r_p)$$

$$y_p = \hat{M}(r_p) = y_m + \frac{1}{c_0^*} \hat{M}(\phi^T w)$$

$$\bar{w} = \begin{pmatrix} (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{pmatrix} (r_p) = \hat{H}_{r\bar{w}_m} (r_p)$$

$$= \bar{w}_m + \hat{H}_{r\bar{w}_m} \left(\frac{1}{c_0^*} \phi^T w \right) \tag{3.7.2}$$

where the transfer functions $\hat{M}$ and $\hat{H}_{r\bar{w}_m}$ are stable and strictly proper.

(c) *Relate the identifier error to the control error*

The properties of the identifier are stated in terms of the error $\phi^T v = \phi^T L^{-1}(z)$, while the control error is $\phi^T w$. The relationship between the two can be examined in two steps.

(c1) Relate $\phi^T w$ to $\phi^T z$

Only the first component of w, namely r, is different from the first component of z, namely r_p. The two can be related using (3.5.4), that is

$$u = c_0^* r_p + \bar{\theta}^{*T} \bar{w} \tag{3.7.3}$$

and using the fact that the control input $u = c_0 r + \bar{\theta}^T \bar{w}$ to obtain

$$r_p = \frac{1}{c_0^*} (c_0 r + \bar{\phi}^T \bar{w}) = r + \frac{1}{c_0^*} \phi^T w \tag{3.7.4}$$

and

$$r = \frac{1}{c_0}\,(c_0^* \, r_p - \bar{\phi}^T\, \bar{w}) = r_p - \frac{1}{c_0}\,\phi^T\, w \tag{3.7.5}$$

It follows that

$$\phi^T\, w = (c_0 - c_0^*)\, r + \bar{\phi}^T\, \bar{w}$$

$$= \frac{c_0 - c_0^*}{c_0}\, c_0^*\, r_p - \frac{c_0 - c_0^*}{c_0}\, \bar{\phi}^T\, \bar{w} + \bar{\phi}^T\, \bar{w}$$

$$= \frac{c_0^*}{c_0}\,((c_0 - c_0^*)\, r_p + \bar{\phi}^T\, \bar{w}\,) = \frac{c_0^*}{c_0}\,\phi^T\, z \tag{3.7.6}$$

that is

$$\frac{1}{c_0^*}\,\phi^T\, w = \frac{1}{c_0}\,\phi^T\, z \tag{3.7.7}$$

(c2) Relate $\phi^T\, z$ to $\phi^T\, v = \phi^T\, \hat{L}^{-1}(z)$

This relationship is obtained through the swapping lemma (lemma 3.6.5). We have, with notation borrowed from the lemma

$$\hat{L}^{-1}(\,\frac{1}{c_0}\,\phi^T\, z\,) = \frac{1}{c_0}\,\phi^T\, v + \hat{L}_c^{-1}(\,\hat{L}_b^{-1}(z^T)\,(\,\frac{\dot{\phi}}{c_0}\,)\,) \tag{3.7.8}$$

and, using (3.7.7) with (3.7.8)

$$\frac{1}{c_0^*}\,\hat{M}\,(\phi^T\, w) = \hat{M}\,\hat{L}\,(\hat{L}^{-1}(\frac{1}{c_0^*}\,\phi^T\, w\,)) = \hat{M}\,\hat{L}\,(\hat{L}^{-1}(\frac{1}{c_0}\,\phi^T\, z\,))$$

$$= \hat{M}\,\hat{L}\,(\,\frac{1}{c_0}\,\phi^T\, v\,) + \hat{M}\,\hat{L}\,\hat{L}_c^{-1}(\,\hat{L}_b^{-1}(z^T)\,(\frac{\dot{\phi}}{c_0})\,) \tag{3.7.9}$$

With (3.7.2), this equation leads to figure 3.10. It represents the plant as the model transfer function with the control error $\phi^T\, w$ in feedback. The control error has now been expressed as a function of the identifier error $\phi^T\, v$ using (3.7.9).

The gain ϕ^T operating on v is equal to the gain β operating on $\| v_t \|_\infty$, and this gain belongs to L_2. On the other hand, $\dot{\phi} \in L_2$, so that any of its component is in L_2. In particular $\dot{c}_0 \in L_2$. Also, $c_0(t) \geq c_{\min}$, so that $1/\,c_0 \in L_\infty$. Thus, $(\frac{\dot{\phi}}{c_0}) \in L_2$. Therefore, in figure 3.10, the controlled plant appears as a stable transfer function $\hat{M}$ with an L_2 feedback gain.

(d) *Establish the regularity of the signals*

The need to establish the regularity of the signals can be understood from the following. We are not only concerned with the boundedness of the output y_p, but also of all the other signals present in the adaptive system. By ensuring the regularity of the signals in the loop, we guarantee, using lemma 3.6.2, that boundedness of one signal implies boundedness of all the others.

Now, note that since $\phi \in L_\infty$, the controller parameter θ is also bounded. It follows, from proposition 1.4.1, that all signals belong to $L_{\infty e}$.

Recall from (3.7.4) that

$$r_p = \frac{c_0}{c_0^*} r + \frac{1}{c_0^*} \bar{\phi}^T \bar{w} \tag{3.7.10}$$

Note that c_0 and r are bounded, by the results of (a), and by assumption (A3). $\bar{w}$ is related to r_p through a strictly proper, stable transfer function (cf (3.7.2)). Therefore, with (3.7.10), and lemma 3.6.1

$$|\bar{w}| \leq k \, \|(\bar{\phi}^T \bar{w})_t\|_\infty + k$$

$$|\frac{d}{dt} \bar{w}| \leq k \, \|(\bar{\phi}^T \bar{w})_t\|_\infty + k \tag{3.7.11}$$

for some constant $k \geq 0$. To prevent proliferation of constants, we will hereafter use the single symbol k, whenever such inequality is valid for some positive constant.

Since ϕ is bounded, the last inequality implies that

$$|\frac{d}{dt} \bar{w}| \leq k \, \|\bar{w}_t\|_\infty + k \tag{3.7.12}$$

i.e. that $\bar{w}$ is regular.

Similarly, since ϕ and $\dot{\phi}$ are bounded, and using (3.7.11)

$$|\frac{d}{dt} (\bar{\phi}^T \bar{w})| \leq |(\frac{d}{dt} \bar{\phi}^T) \bar{w}| + |\bar{\phi}^T (\frac{d}{dt} \bar{w})|$$

$$\leq k \, \|(\bar{\phi}^T \bar{w})_t\|_\infty + k \tag{3.7.13}$$

so that $\bar{\phi}^T \bar{w}$ is also regular.

The output y_p is given by, using (3.7.10)

$$y_p = \hat{M}(r_p) = \frac{1}{c_0^*}\ \hat{M}(c_0 r) + \frac{1}{c_o^*}\ \hat{M}\ (\bar{\phi}^T\ \bar{w}) \tag{3.7.14}$$

where $\hat{M}(c_0 r)$ is bounded. Using lemma 3.6.2, with the fact that $\bar{\phi}^T\ \bar{w}$ is regular, and then (3.7.14)

$$\begin{aligned}|\bar{\phi}^T\ \bar{w}\ | &\leqslant k \|(\hat{M}(\bar{\phi}^T \bar{w}))_t\|_\infty + k \\ &\leqslant k\ \|y_{p_t}\|_\infty + k\ \|(\hat{M}(c_0 r))_t\|_\infty + k \\ &\leqslant k\ \|y_{p_t}\|_\infty + k\end{aligned} \tag{3.7.15}$$

hence, with (3.7.10) and (3.7.11)

$$\begin{aligned}|r_p\ | &\leqslant k\ \|(\bar{\phi}^T\ \bar{w})_t\|_\infty + k\ \leqslant k\ \|y_{p_t}\|_\infty + k \\ |\bar{w}\ | &\leqslant k\ \|y_{p_t}\|_\infty + k\end{aligned} \tag{3.7.16}$$

Inequalities in (3.7.16) show that the boundedness of y_p implies the boundedness of $r_p, \bar{w}, u, \cdots$ therefore of all the states of the adaptive system.

It also follows that v is regular, since it is the sum of two regular signals, specifically

$$v = \hat{L}^{-1}(z) = \begin{pmatrix} \hat{L}^{-1} r_p \\ \hat{L}^{-1} \bar{w} \end{pmatrix} = \begin{pmatrix} \hat{L}^{-1} \frac{c_0}{c_0^*}\ r \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{L}^{-1} \frac{1}{c_0^*}\ \bar{\phi}^T\ \bar{w} \\ \hat{L}^{-1} \bar{w} \end{pmatrix} \tag{3.7.17}$$

where the first term is the output of $\hat{L}^{-1}$ (a stable and striclty proper, minimum phase LTI system) with bounded input, while the second term is the output of $\hat{L}^{-1}$ with a regular input (cf. corollary 3.6.3).

(e) *Stability proof*

Since v is regular, theorem 2.4.6 shows that $\beta \to 0$ as $t \to \infty$. From (3.7.2) and (3.7.9)

$$\begin{aligned}y_p &= y_m + \frac{1}{c_0^*}\ \hat{M}\ (\phi^T\ w\ ) \\ &= y_m + \hat{M}\ \hat{L}\ (\frac{1}{c_0}\ \phi^T\ v\ ) + \hat{M}\ \hat{L}\ \hat{L}_c^{-1}(\ \hat{L}_b^{-1}(z^T)(\frac{\dot{\phi}}{c_0}\ ))\end{aligned} \tag{3.7.18}$$

We will now use the single symbol β in inequalities satisfied for some function satisfying the same conditions as β, that is $\beta \in L_2 \cap L_\infty$, and $\beta(t) \to 0$, as $t \to \infty$.

The transfer functions $\hat{M}\,\hat{L}$, $\hat{L}_b^{-1}$, $\hat{L}_c^{-1}$, are all stable, and the last two are strictly proper. The gain $\frac{1}{c_0}$ is bounded by (3.7.2), because of the projection in the update law. Therefore, using results obtained so far, and lemmas 3.6.1 and 3.6.4

$$
\begin{aligned}
|y_p - y_m| &\leq \beta \| v_t \|_\infty + \beta \| z_t \|_\infty + \beta \\
&\leq \beta \| r_{p_t} \|_\infty + \beta \| \bar{w}_t \|_\infty + \beta \\
&\leq \beta \| y_{p_t} \|_\infty + \beta \\
&\leq \beta \| (y_p - y_m)_t \|_\infty + \beta \qquad (3.7.19)
\end{aligned}
$$

Recall that since $\theta \in L_\infty$, all signals in the adaptive system belong to $L_{\infty e}$. On the other hand, for T sufficiently large, $\beta(t \geq T) < 1$. Therefore, application of the small gain theorem (lemma 3.6.6) with (3.7.19) shows that $y_p - y_m$ is bounded for $t \geq T$. But since $y_p, y_m \in L_{\infty e}$, it follows that $y_p \in L_\infty$. Consequently, all signals belong to L_∞.

From (3.7.19), it also follows that $e_0 = y_p - y_m \in L_2$, and tends to zero as $t \to \infty$. Similarly, using (3.5.9), (3.5.12), and (3.7.9)

$$
v = v_m + \begin{bmatrix} 1 \\ (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} \left(\frac{1}{c_0} \phi^T v + \hat{L}_c^{-1} (\hat{L}_b^{-1} (z^T) (\frac{\dot{\phi}}{c_0})) \right) \qquad (3.7.20)
$$

so that $v - v_m$ also belongs to L_2, and tends to zero, as $t \to \infty$.

3.7.2 Stability - Output Error Direct Adaptive Control

Theorem 3.7.2

Consider the output error direct adaptive control scheme described in section 3.3.2, with initial conditions in an arbitrary B_h.

Then

(a) all states of the adaptive system are bounded functions of time.

(b) the output error $e_0 = y_p - y_m \in L_2$ and tends to zero as $t \to \infty$

the regressor error $\hat{L}^{-1}(w) - \hat{L}^{-1}(w_m) \in L_2$ and tends to zero as $t \to \infty$.

Proof of Theorem 3.7.2

The proof is very similar to the proof for the input error scheme, and is just sketched here, following the steps of the proof of theorem 3.7.1.

(a) we now have, instead

$$e_1, \dot{\phi} \in L_2$$

$$e_1, \phi \in L_\infty \tag{3.7.21}$$

Note that these results are valid, although the realization of $\hat{M}$ is not minimal (but is stable).

(b) as in theorem 3.7.1.

(c) since $c_0 = c_0^*$, (3.7.9) becomes

$$\frac{1}{c_0^*} \hat{M}(\phi^T \bar{w}) = \frac{1}{c_0^*} \hat{M} \hat{L} (\phi^T \bar{v}) + \frac{1}{c_0^*} \hat{M} \hat{L} (\hat{L}_c^{-1} (\hat{L}_b^{-1}(\bar{w}^T) \dot{\phi})) \tag{3.7.22}$$

(d) as in theorem 3.7.1.

(e) Recall, from (3.3.16) and the definition of the gradient update law, that

$$\frac{1}{c_0^*} \hat{M} \hat{L} (\phi^T \bar{v}) = e_1 + \frac{\gamma}{c_0^*} \hat{M} \hat{L} (\bar{v}^T \bar{v} e_1) = e_1 - \frac{\gamma}{g c_0^*} \hat{M} \hat{L} (\dot{\phi}^T \bar{v}) \tag{3.7.23}$$

so that, with (3.7.22)

$$y_p - y_m = \frac{1}{c_0^*} \hat{M} \hat{L} (\phi^T \bar{v}) + \frac{1}{c_o^*} \hat{M} \hat{L} (\hat{L}_c^{-1} (\hat{L}_b^{-1} (\bar{w}^T)) \dot{\phi}))$$

$$= e_1 - \frac{\gamma}{g c_0^*} \hat{M} \hat{L} (\dot{\phi} \bar{v}) + \frac{1}{c_0^*} \hat{M} \hat{L} (\hat{L}_c^{-1} (\hat{L}_b^{-1} (\bar{w}^T) \dot{\phi}) \tag{3.7.24}$$

Recall that e_1 is bounded (part (a)), and that $\hat{M} \hat{L}$ is strictly proper (in the output error scheme). The proof can then be completed as in theorem 3.7.1. □

3.7.3 Stability - Indirect Adaptive Control

Theorem 3.7.3

Consider the indirect adaptive control scheme described in section 3.3.3, with initial conditions in an arbitrary B_h .

Then

(a) all states of the adaptive system are bounded functions of time.

(b) the output error $e_0 = y_p - y_m \in L_2$, and tends to zero as $t \to \infty$

the regressor error $\tilde{w} - \tilde{w}_m \in L_2$, and tends to zero as $t \to \infty$.

Comments

Compared with previous proofs, the proof of theorem 3.7.3 presents additional complexities due to the transformation $\pi \to \theta$. A major step is to relate the identification error $\psi^T \tilde{w}$ to the control error $\phi^T w$.

To understand the idea of the proof, assume that the parameters π and θ are fixed in time, and that k_p is known. For simplicity, let $k_p = a_{m+1} = k_m = 1$. The nominal values of the identifier parameters are then given by

$$\hat{a}^* = \hat{n}_p$$

$$\hat{b}^* = \hat{\lambda} - \hat{d}_p$$

The controller parameters are given as a function of the identifier parameters through

$$\hat{c} = \hat{\lambda} - \hat{q}\,\hat{a}$$

$$\hat{d} = \hat{q}\,\hat{\lambda} - \hat{q}\,\hat{b} - \hat{\lambda}_0 \hat{d}_m \tag{3.7.25}$$

while the nominal values are given by

$$\hat{c}^* = \hat{\lambda} - \hat{q}^*\,\hat{a}^* = \hat{\lambda} - \hat{q}^*\,\hat{n}_p$$

$$\hat{d}^* = \hat{q}^*\,\hat{\lambda} - \hat{q}^*\,\hat{b}^* - \hat{\lambda}_0 \hat{d}_m = \hat{q}^*\,\hat{d}_p - \hat{\lambda}_0 \hat{d}_m \tag{3.7.26}$$

It follows that

$$\hat{q}\,\hat{a} - \hat{q}\,\hat{a}^* = (\hat{\lambda} - \hat{c}) - \hat{q}\,\hat{n}_p = -(\hat{c} - \hat{c}^*) + (\hat{\lambda} - \hat{c}^*) - \hat{q}\,\hat{n}_p$$

$$= -(\hat{c} - \hat{c}^*) + (\hat{q}^* - \hat{q})\,\hat{n}_p \tag{3.7.27}$$

and

$$\hat{q}\,\hat{b}-\hat{q}\,\hat{b}^{*}=\hat{q}\,\lambda-\hat{d}-\lambda_0\hat{d}_m-\hat{q}\,\lambda+\hat{q}\,\hat{d}_p$$
$$=-(\hat{d}-\hat{d}^{*})+(-\hat{d}^{*}-\lambda_0\hat{d}_m+\hat{q}\,\hat{d}_p)$$
$$=-(\hat{d}-\hat{d}^{*})+(\hat{q}-\hat{q}^{*})\hat{d}_p \tag{3.7.28}$$

Therefore

$$\hat{q}\left(\frac{\hat{a}-\hat{a}^{*}}{\hat{\lambda}}+\frac{\hat{b}-\hat{b}^{*}}{\hat{\lambda}}\,\frac{\hat{n}_p}{\hat{d}_p}\right)=-\left(\frac{\hat{c}-\hat{c}^{*}}{\hat{\lambda}}+\frac{\hat{d}-\hat{d}^{*}}{\hat{\lambda}}\,\frac{\hat{n}_p}{\hat{d}_p}\right) \tag{3.7.29}$$

This equality of polynomial ratios can be interpreted as an operator equality in the Laplace transform domain, since we assumed that the parameters were fixed in time. If we apply the operator equality to the input u, it leads to (with the definitions of section 3.3)

$$\hat{q}\,(\psi^T\,\tilde{w})=-\bar{\phi}^T\,\bar{w} \tag{3.7.30}$$

and consequently

$$y_p-y_m=-\frac{1}{c_0^{*}}\,\hat{M}\,\hat{q}\,(\psi^T\,\tilde{w}) \tag{3.7.31}$$

Since the degree of $\hat{q}$ is at most equal to the relative degree of the plant, the transfer function $\hat{M}\,\hat{q}$ is proper and stable. The techniques used in the proof of theorem 3.7.1, and the properties of the identifier would then lead to a stability proof.

Two difficulties arise when using this approach to prove the stability of the indirect adaptive system. The first is related to the unknown high-frequency gain, but only requires more complex manipulations. The real difficulty comes from the fact that the polynomials $\hat{q}$, $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ vary as functions of time. Eqn (3.7.29) is still valid as a polynomial equality, but transforming it to an operator equality leading to (3.7.30) requires some care.

To make sense of time varying polynomials as operators in the Laplace transform domain, we define

$$\hat{s}_n=\begin{pmatrix}1\\ s\\ \cdot\\ \cdot\\ s^{n-1}\end{pmatrix} \tag{3.7.32}$$

so that

$$\hat{a}(s) = a^T \hat{s}_n \qquad \frac{\hat{a}(s)}{\hat{\lambda}(s)} = a^T \cdot \left(\frac{\hat{s}_n}{\hat{\lambda}} \right) \tag{3.7.33}$$

Consider the following equality of polynomial ratios

$$\frac{\hat{a}(s)}{\hat{\lambda}(s)} = \frac{\hat{b}(s)}{\hat{\lambda}(s)} \tag{3.7.34}$$

where $\hat{a}$ and $\hat{b}$ vary with time, but $\hat{\lambda}$ is a constant polynomial. Equality (3.7.34) implies the following operator equality

$$a^T \left(\frac{\hat{s}_n}{\hat{\lambda}} (.) \right) = b^T \left(\frac{\hat{s}_n}{\hat{\lambda}} (.) \right) \tag{3.7.35}$$

Similarly, consider the product

$$\frac{\hat{a}(s)}{\hat{\lambda}(s)} \cdot \frac{\hat{b}(s)}{\hat{\lambda}(s)} \tag{3.7.36}$$

This can be interpreted as an operator by multiplying the coefficients of the polynomials to lead to a ratio of higher order polynomials, and then interpreting it as previously. We note that the product of polynomials can be expressed as

$$\hat{a}(s)\hat{b}(s) = a^T (\hat{s}_n \hat{s}_n^T) b \tag{3.7.37}$$

so that the operator corresponding to (3.7.36) is

$$a^T \left(\frac{\hat{s}_n}{\hat{\lambda}} \cdot \frac{\hat{s}_n^T}{\hat{\lambda}} (.) \right) b \tag{3.7.38}$$

i.e. by first operating the matrix transfer function on the argument, and then multiplying by a and b in the time domain. Note that this operator is different from the operator

$$a^T \left(\frac{\hat{s}_n}{\hat{\lambda}} \left(\frac{\hat{s}_n^T}{\hat{\lambda}} (.) b \right) \right) \tag{3.7.39}$$

but the two operators can be related using the swapping lemma (lemma 3.6.5).

Proof of Theorem 3.7.3

The proof follows the steps of the proof of theorem 3.7.1, and is only sketched here.

(a) *Derive properties of the identifier that are independent of the boundedness of the regressor*

The properties of the identifier are the standard properties obtained in theorems 2.4.1-2.4.4

$$|\psi^T(t)\tilde{w}(t)| = \beta(t)\|\tilde{w}_t\|_\infty + \beta(t)$$

$$\beta \in L_2 \cap L_\infty$$

$$\psi \in L_\infty \qquad \dot{\psi} \in L_2 \cap L_\infty$$

$$a_{m+1}(t) \geq k_{\min} > 0 \qquad \text{for all } t \geq 0 \tag{3.7.40}$$

The inequality for $a_{m+1}(t)$ follows from the use of the projection in the update law.

We also noted, in section 3.3, that if π is bounded, and a_{m+1} is bounded away from zero, then θ is also bounded, and the transformation has bounded derivatives. The vector q of coefficients of the polynomial $\hat{q}$ is also bounded. By definition of the transformation, $\theta(\pi^*) = \theta^*$. Therefore, $\psi \in L_\infty$, $\dot{\psi} \in L_2 \cap L_\infty$ implies that $\phi \in L_\infty$, $\dot{\phi} \in L_2 \cap L_\infty$. Also, we have that $(k_m / \|a_{m+1}\|_\infty) \leq c_0(t) \leq k_m / k_{\min}$, for all $t \geq 0$.

(b) *Express the system states and inputs in term of the control error*

As in theorem 3.7.1.

(c) *Relate the identifier error to the control error*

We first establish an equality of ratios of polynomials, then transform it to an operator equality. Using a similar approach as in the comments before the proof, we have that

$$\begin{aligned}
\hat{q}\,\hat{a} - \hat{q}\,\hat{a}^* &= a_{m+1}(\hat{\lambda} - \hat{c}) - \hat{q}\,k_p\,\hat{n}_p \\
&= -a_{m+1}(\hat{c} - \hat{c}^*) + a_{m+1}(\hat{\lambda} - \hat{c}^*) - k_p\,\hat{q}\,\hat{n}_p \\
&= -a_{m+1}(\hat{c} - \hat{c}^*) + (a_{m+1}\hat{q}^* - k_p\,\hat{q}\,)\hat{n}_p
\end{aligned} \tag{3.7.41}$$

$$\begin{aligned}
\hat{q}\,\hat{b} - \hat{q}\,\hat{b}^* &= \hat{q}\,\hat{\lambda} - a_{m+1}\hat{d} - \hat{\lambda}_0\hat{d}_m - \hat{q}\,\hat{\lambda} + \hat{q}\,\hat{d}_p \\
&= -a_{m+1}(\hat{d} - \hat{d}^*) + (-a_{m+1}\hat{d}^* - \hat{\lambda}_0\hat{d}_m + \hat{q}\,\hat{d}_p)
\end{aligned}$$

$$= -a_{m+1}(\hat{d} - \hat{d}^*)$$

$$+ (-\frac{a_{m+1}}{k_p} \hat{q}^* + \frac{a_{m+1}}{k_p} \hat{\lambda}_0 \frac{\hat{d}_m}{\hat{d}_p} + \hat{q} - \hat{\lambda}_0 \frac{\hat{d}_m}{\hat{d}_p}) \hat{d}_p \qquad (3.7.42)$$

Therefore

$$\frac{\hat{q}}{\hat{\lambda}_0} \left[\frac{\hat{a} - \hat{a}^*}{\hat{\lambda}} \right] + \frac{\hat{q}}{\hat{\lambda}_0} \left[\frac{\hat{b} - \hat{b}^*}{\hat{\lambda}} \right] \frac{k_p \, \hat{n}_p}{\hat{d}_p}$$

$$= -a_{m+1} \left[\frac{\hat{c} - \hat{c}^*}{\hat{\lambda}\hat{\lambda}_0} + \frac{\hat{d} - \hat{d}^*}{\hat{\lambda}\hat{\lambda}_0} \hat{P} \right] - (k_p - a_{m+1}) \frac{\hat{\lambda}_0}{\hat{\lambda}} \frac{\hat{d}_m \hat{n}_p}{\hat{d}_p} \frac{1}{\hat{\lambda}_0}$$

$$= -\frac{k_m}{c_0} \left[\frac{\hat{c} - \hat{c}^*}{\hat{\lambda}\hat{\lambda}_0} + \frac{\hat{d} - \hat{d}^*}{\hat{\lambda}\hat{\lambda}_0} \hat{P} + (c_0 - c_0^*) \hat{M}^{-1} \frac{\hat{P}}{\hat{\lambda}_0} \right] \qquad (3.7.43)$$

where we divided by $\hat{\lambda}\hat{\lambda}_0$ to obtain proper stable transfer functions. The polynomial $\hat{\lambda}_0$ is Hurwitz, and q is bounded, so that the operator $q^T \hat{s}_r / \hat{\lambda}_0$ is a bounded operator.

We now transform this polynomial equality into an operator equality as in the comments before the proof. Applying both sides of (3.7.43) to u

$$-q^T \frac{\hat{s}_r}{\hat{\lambda}_0} (\tilde{w}^T) \psi = \frac{k_m}{c_0} \left[(c_0 - c_0^*) \hat{M}^{-1} \frac{\hat{P}}{\hat{\lambda}_0} (u) + \bar{\phi}^T \frac{1}{\hat{\lambda}_0} (\bar{w}) \right] \qquad (3.7.44)$$

The right-hand side is very reminiscent of the signal z obtained in the input error scheme. A filtered version of the signal $\hat{M}^{-1} \hat{P}(u) = r_p$ appears, instead of r, with the error $c_0 - c_0^*$. From proposition 3.3.1, with $\hat{L} = \hat{\lambda}_0$ (cf. (3.3.10))

$$c_0^* \hat{M}^{-1} \hat{P} \frac{1}{\hat{\lambda}_0} (u) = \frac{1}{\hat{\lambda}_0} (u) - \frac{1}{\hat{\lambda}_0} (\bar{\theta}^{*T} \bar{w}) \qquad (3.7.45)$$

and since $u = c_0 r + \bar{\theta}^T \bar{w}$, it follows that

$$\hat{M}^{-1} \frac{\hat{P}}{\hat{\lambda}_0} (u) = \frac{1}{c_0^*} (\frac{1}{\hat{\lambda}_0} (c_0 r) + \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w})) \qquad (3.7.46)$$

The right-hand side of (3.7.44) becomes, using (3.7.46) followed by the swapping lemma (and using the notation of the swapping lemma)

$$\frac{k_m}{c_0^*} \left[\frac{c_0 - c_0^*}{c_0} \frac{1}{\hat{\lambda}_0} (c_0 r) + \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w}) + \frac{c_0^*}{c_0} (\bar{\phi}^T \frac{1}{\hat{\lambda}_0} (\bar{w}) - \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w})) \right]$$

$$= \frac{k_m}{c_0^*} \left[\frac{1}{\hat{\lambda}_0} ((c_0 - c_0^*) r) - \hat{\Lambda}_{oc} (\hat{\Lambda}_{oc} (c_0 r) (\frac{\dot{c}_0 - c_0^*}{c_0})) \right.$$

$$+\frac{1}{\hat{\lambda}_0}(\phi^T\,\bar{w}) - \frac{c_0^*}{c_0}\,\hat{\Lambda}_{oc}\,(\hat{\Lambda}_{ob}(\bar{w}^T)\,\dot{\phi})\Bigg] \tag{3.7.47}$$

On the other hand, using again the swapping lemma, the left-hand side of (3.7.44) becomes

$$q^T\,\frac{\hat{s}_r}{\hat{\lambda}_0}\,(\tilde{w}^T)\,\psi = q^T\,\frac{\hat{s}_r}{\hat{\lambda}_0}\,(\tilde{w}^T\,\psi) - q^T\,\hat{S}_{rc}\,(\hat{S}_{rb}\,(\tilde{w}^T)\,\dot{\psi}) \tag{3.7.48}$$

where the transfer functions $\hat{\Lambda}_{ob}$, $\hat{\Lambda}_{oc}$, $\hat{S}_{rb}$, and $\hat{S}_{rc}$ result from the application of the swapping lemma. The output error is then equal to (using (3.7.2), (3.7.44), (3.7.47), (3.7.48))

$$\begin{aligned}
y_p - y_m &= \frac{1}{c_0^*}\,\hat{M}\,((c_0 - c_0^*)\,r + \phi^T\,\bar{w}) \\
&= \frac{1}{k_m}\,\hat{M}\,\hat{\lambda}_0\left[\frac{k_m}{c_0^*}\,\frac{1}{\hat{\lambda}_0}\,((c_0 - c_0^*)\,r\,) + \frac{1}{\hat{\lambda}_0}\,(\phi^T\,\bar{w})\right] \\
&= \frac{1}{k_m}\,\hat{M}\,\hat{\lambda}_0\Bigg[-q^T\,\frac{\hat{s}_r}{\hat{\lambda}_0}\,(\tilde{w}^T\,\psi) + q^T\,\hat{S}_{rc}\,(\hat{S}_{rb}\,(\tilde{w}^T)\,\dot{\psi}) \\
&\quad + \frac{k_m}{c_0^*}\,\hat{\Lambda}_{oc}\,(\hat{\Lambda}_{ob}\,(c_0\,r)\,(\frac{\dot{\overline{c_0 - c_0^*}}}{c_0}\,)) + \frac{k_m}{c_0}\,\hat{\Lambda}_{oc}\,(\hat{\Lambda}_{ob}(\bar{w}^T)\,\dot{\phi})\Bigg]
\end{aligned} \tag{3.7.49}$$

(d) *Establish the regularity of the signals*

As in theorem 3.7.1.

(e) *Stability Proof*

$\hat{M}\,\hat{\lambda}_0$ is a stable transfer function, and since q^T is bounded, $q^T\,\hat{s}_r/\hat{\lambda}_0$ is a bounded operator. We showed that $\dot{\psi}$, $\dot{\phi}$, $\dot{c}_0 \in L_2$, so that, from (3.7.49) and part (a), an inequality such as (3.7.19) can be obtained. As before $\tilde{w}$ regular implies that $\beta \to 0$ as $t \to \infty$. The boundedness of all signals in the adaptive system then follows as in theorem 3.7.1. Similarly, $y_p - y_m \in L_2$ and tends to zero as $t \to \infty$. Since the relative degree of the transfer function from $u \to \tilde{w}$ is the same as the relative degree of $\hat{P}$, $\hat{M}$, and therefore $\hat{L}^{-1}$, the same result is true for $\tilde{w} - \tilde{w}_m$.

3.8 Exponential Parameter Convergence

Exponential convergence of the identification algorithms under persistency of excitation conditions was established in sections 2.5 and 2.6. Consider now the input error direct adaptive control scheme of section 3.3.1. Using theorem 2.5.3, it would be straightforward to show that the parameters of the adaptive system converge exponentially to their nominal values, provided that the regressor v is persistently exciting. However, such result is useless, since the signal v is generated inside the adaptive system, and is unknown *a priori*. Theorem 3.8.1 shows that it is sufficient for the *model* signal w_m to be persistently exciting to guarantee exponential convergence.

Note that in the case of adaptive control, we are not only interested in the convergence of the parameter error to zero, but also in the convergence of the errors between plant states and model states. In other words, we are concerned with the exponential stability of the overall adaptive system.

Theorem 3.8.1

Consider the input error direct adaptive control scheme of section 3.3.1.

If w_m is PE

Then the adaptive system is exponentially stable in any closed ball.

Proof of Theorem 3.8.1

Since w_m, $\dot{w}_m$ are bounded, lemma 2.6.6 implies that $v_m = \hat{L}^{-1}(z_m) = \hat{L}^{-1}(w_m)$ is PE. In theorem 3.7.1, we found that $v - v_m \in L_2$. Therefore, using lemma 2.6.5, v_m PE implies that v is PE. Finally, since v is PE, by theorem 2.5.3, the parameter error ϕ converges exponentially to zero.

Recall that in section 3.5, it was established that the errors between the plant and the model signals are the outputs of stable transfer functions with input $\phi^T w$. Since w is bounded (by theorem 3.7.1), $\phi^T w$ converges exponentially to zero. Therefore, all errors between plant and model signals converge to zero exponentially fast. □

Comments

Although theorem 3.8.1 establishes exponential stability in any closed ball, it does not prove global exponential stability. This is because $v - v_m$ is not bounded by a unique L_2 function for any initial condition. Results in section 4.5 will actually show that the adaptive control system is not globally exponentially stable.

The various theorems and lemmas used to prove theorem 3.8.1 can be used to obtain estimates of the convergence rates of the parameter error. It is, however, doubtful that these estimates would be of any practical use, due to their complexity and to their conservatism. A more successful approach is that of chapter 4, using averaging techniques.

The result of theorem 3.8.1 has direct parallels for the other adaptive control algorithms presented in section 3.3.

Theorem 3.8.2

Consider the output error direct adaptive control scheme of section 3.3.2 (or the indirect scheme of section 3.3.3)

If w_m is PE ($\tilde{w}_m$ is PE)

Then the adaptive system is exponentially stable in any closed ball.

Proof of Theorem 3.8.2

The proof of theorem 3.8.2 is completely analogous to the proof of theorem 3.8.1, and is omitted here. □

3.9 Conclusions

In this chapter, we derived three model reference adaptive control schemes. All had a similar controller structure, but had different identification structures. The first two schemes were direct adaptive control schemes, where the parameters updated by the identifier were the same as those used by the controller. The third scheme was an indirect scheme, where the parameters updated by the identifier were the same as those of the basic identifier of chapter 2. Then, the controller parameters were obtained from the identifier parameters through a nonlinear transformation resulting from the model reference control objective.

We investigated the connections between the adaptive control schemes, and also with other known schemes. The difficulties related to the unknown high-frequency gain were also discussed. The stability of the model reference adaptive control schemes was proved, together with the result that the error between the plant and the reference model converged to zero as t approached infinity. Although the proofs relied strongly on known results, we used a unified framework, and an identical step-by-step procedure for all three schemes. We proved - with original or reviewed proofs - basic lemmas that are fundamental to the stability proofs, and we emphasized a basic intuitive idea of the proof of stability, that was the existence of a small loop gain appearing in the adaptive system.

The exponential parameter convergence was established, with the additional assumption of the persistency of excitation of a model regressor vector. This condition was to be satisfied by an exogeneous model signal, influenced by the designer, and was basically a condition on the reference input.

An interesting conclusion is that the stability and convergence properties are identical for all three adaptive control schemes. Further, the normalized gradient identification algorithm can be replaced by the least squares algorithm with projection without altering the results. Differences appear between the schemes however, in connection with the high-frequency gain, and with other practical considerations.

The input error direct adaptive control scheme and the indirect scheme are attractive because they lead to linear error equations, and do not involve SPR conditions. Another advantage is that they allow for a decoupling of identification and control useful in practice. The indirect scheme is quite more intuitive than the input error direct scheme, although more complex in implementation, and especially as far as the analysis is concerned. The end result shows however that stability is not an argument to prefer one over the other.

The various model reference adaptive control schemes also showed that the model reference approach is not bound to the choice of a direct adaptive control scheme, to the use of the output error in the identification algorithm, or to SPR conditions on the reference model.

Chapter 4 Parameter Convergence Using Averaging Techniques

4.1 Introduction

The method of averaging is a method of analysis of differential equations of the form

$$\dot{x} = \epsilon f(t,x) \tag{4.1.1}$$

and relates properties of the solutions of system (4.1.1) to properties of the solutions of the so-called *averaged* system

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \tag{4.1.2}$$

where

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t,x)\,dt \tag{4.1.3}$$

provided that the parameter ϵ is sufficiently small. The method was proposed originally by Bogoliuboff and Mitropolskii (1961), developed subsequently by Volosov (1962), Sethna (1970), Balachandra and Sethna (1975), Hale (1980), and stated in a geometric form in Arnold (1982), and Guckenheimer and Holmes (1983).

Averaging methods were introduced for the stability analysis of deterministic adaptive systems in the work of Astrom (1983), Astrom (1984), Riedle and Kokotovic (1985) and (1986), Mareels et al (1986), and Anderson et al (1986). We also find early informal use of averaging in Astrom and Wittenmark (1973), and, in a stochastic context, in Ljung and Soderstrom (1983).

Averaging is very valuable to assess the stability of adaptive systems in the presence of unmodeled dynamics, and to understand mechanisms of instability. However, it is not only useful in stability problems, but in general as an *approximation* method, allowing one to replace a system of nonautonomous differential equations by an autonomous system. This aspect was emphasized in Fu, Bodson, and Sastry (1985), Bodson et al (1986), and theorems were derived for one-time scale, and two-time scale systems such as those arising in identification and control. These results are reviewed here, together with their application to the adaptive systems described in previous chapters.

4.2 Averaging Theory - One-Time Scale

In this section, we consider differential equations of the form

$$\dot{x} = \epsilon f(t, x, \epsilon) \qquad x(0)=x_0 \tag{4.2.1}$$

where $x \in \mathbf{R}^n$, $t \geq 0$, $0 < \epsilon \leq \epsilon_0$, and f is piecewise continuous with respect to t. We will concentrate our attention on the behavior of the solutions in some closed ball B_h of radius h, centered at the origin.

For small ϵ, the variation of x with time is slow, as compared to the rate of time variation of f. The *method of averaging* relies on the assumption of the existence of the mean value of $f(t, x, 0)$ defined by the limit

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x, 0)\, d\tau \tag{4.2.2}$$

assuming that the limit exists uniformly in t_0 and x. This is formulated more precisely in the following definition.

Definition Mean Value of a Function, Convergence Function

The function $f(t, x, 0)$ is said to have mean value $f_{av}(x)$ if there exists a continuous function $\gamma(T)$: $\mathbf{R}_+ \to \mathbf{R}_+$, strictly decreasing, such that $\gamma(T) \to 0$ as $T \to \infty$, and

$$\left| \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x, 0)\, d\tau - f_{av}(x) \right| \leq \gamma(T) \tag{4.2.3}$$

for all $t_0 \geq 0, T \geq 0$, $x \in B_h$.

The function $\gamma(T)$ is called the *convergence function*.

Note that the function $f(t, x, 0)$ has mean value $f_{av}(x)$ if and only if the function

$$d(t, x) = f(t, x, 0) - f_{av}(x) \tag{4.2.4}$$

has zero mean value.

The following definition (Hahn (1967), p. 7) will be useful.

Definition Class K Function

A function $\alpha(\epsilon): \mathbf{R}_+ \rightarrow \mathbf{R}_+$ belongs to class K $(\alpha(\epsilon) \in K)$, if it is continuous, strictly increasing, and $\alpha(0)=0$.

It is common, in the literature on averaging, to assume that the function $f(t,x,\epsilon)$ is periodic in t, or almost periodic in t. Then, the existence of the mean value is guaranteed, without further assumption (Hale (1980), theorem 6, p. 344). Here, we do not make the assumption of (almost) periodicity, but consider instead the assumption of the existence of the mean value as the starting point of our analysis.

Note that if the function $d(t,x)$ is periodic in t, and is bounded, then the integral of the function $d(t,x)$ is also a bounded function of time. This is equivalent to saying that there exists a convergence function $\gamma(T)=a/T$ (i.e. of the order of $1/T$) such that (4.2.3) is satisfied. On the other hand, if the function $d(t,x)$ is bounded, and is not required to be periodic but almost periodic, then the integral of the function $d(t,x)$ need not be a bounded function of time, even if its mean value is zero (Hale (1980), p. 346). The function $\gamma(T)$ is bounded (by the same bound as $d(t,x)$), and converges to zero as $T \rightarrow \infty$, but the convergence function need not be bounded by a/T as $T \rightarrow \infty$ (it may be of order $1/\sqrt{T}$ for example). In general, a zero mean function need not have a bounded integral, although the converse is true. In this paper, we do not make the distinction between the periodic, and the almost periodic case, but we do distinguish the bounded integral case from the general case, and indicate the importance of the function $\gamma(T)$ in the subsequent developments.

System (4.2.1) will be called the *original system* and, assuming the existence of the mean value for the original system, the *averaged system* is defined to be

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \qquad x_{av}(0)=x_0 \tag{4.2.5}$$

Note that the averaged system is autonomous and, for T fixed and ϵ varying, the solutions over intervals $[0,T/\epsilon]$ are identical, modulo a simple time scaling by ϵ.

We address the following two questions

(a) the closeness of the response of the original and averaged systems on intervals $[0, T/\epsilon]$,

(b) the relationships between the stability properties of the two systems.

To compare the solutions of the original and of the averaged system, it is convenient to transform the original system in such a way that it becomes a *perturbed* version of the averaged system. An important lemma that leads to this result is attributed to Bogoliuboff and Mitropolskii ((1961), p. 450, and Hale (1980), lemma 4, p. 346). We state a generalized version of this lemma.

Lemma 4.2.1 Approximate Integral of a Zero Mean Function

If $d(t,x): \mathbf{R}_+ \times B_h \to \mathbf{R}^n$ is a bounded function, piecewise continuous with respect to t, and has zero mean value with convergence function $\gamma(T)$

Then there exists $\xi(\epsilon) \in K$, and a function $w_\epsilon(t,x): \mathbf{R}_+ \times B_h \to \mathbf{R}^n$ such that

$$|\epsilon w_\epsilon(t,x)| \leqslant \xi(\epsilon) \tag{4.2.6}$$

$$\left| \frac{\partial w_\epsilon(t,x)}{\partial t} - d(t,x) \right| \leqslant \xi(\epsilon) \tag{4.2.7}$$

for all $t \geqslant 0$, $x \in B_h$. Moreover, $w_\epsilon(0,x)=0$, for all $x \in B_h$.

If, moreover $\gamma(T) = a/T^r$ for some $a \geqslant 0$, $r \in (0,1]$,

Then the function $\xi(\epsilon)$ can be chosen to be $2a\epsilon^r$.

Proof of Lemma 4.2.1 in appendix.

Comments

The construction of the function $w_\epsilon(t,x)$ in the proof is identical to that in Bogoliuboff and Mitropolskii (1961), but the proof of (4.2.6), (4.2.7) is different, and leads to the relationship between the convergence function $\gamma(T)$ and the function $\xi(\epsilon)$.

The main point of lemma 4.2.1 is that, although the exact integral of $d(t,x)$ may be an unbounded function of time, there exists a bounded function $w_\epsilon(t,x)$, whose first partial derivative with respect to t is arbitrarily close to $d(t,x)$. Although the bound on $w_\epsilon(t,x)$ may increase as $\epsilon \to 0$, it increases slower than $\xi(\epsilon)/\epsilon$, as indicated by (4.2.6).

It is necessary to obtain a function $w_\epsilon(t,x)$, as in lemma 4.2.1, that has some additional smoothness properties. A useful lemma is given by Hale ((1980), lemma 5, p. 349). At the price of additional assumptions on the function $d(t,x)$, the following lemma leads to stronger conclusions that are useful in the sequel.

Lemma 4.2.2 Smooth Approximate Integral of a Zero Mean Function

If $d(t,x)$: $\mathbf{R}_+ \times B_h \to \mathbf{R}^n$ is piecewise continuous with respect to t, has bounded and continuous first partial derivatives with respect to x, and $d(t,0)=0$ for all $t \geq 0$. Moreover, $d(t,x)$ has zero mean value, with convergence function $\gamma(T)|x|$, and $\frac{\partial d(t,x)}{\partial x}$ has zero mean value, with convergence function $\gamma(T)$

Then there exists $\xi(\epsilon) \in K$, and a function $w_\epsilon(t,x)$: $\mathbf{R}_+ \times B_h \to \mathbf{R}^n$, such that

$$|\epsilon w_\epsilon(t,x)| \leq \xi(\epsilon)|x| \tag{4.2.8}$$

$$\left|\frac{\partial w_\epsilon(t,x)}{\partial t} - d(t,x)\right| \leq \xi(\epsilon)|x| \tag{4.2.9}$$

$$\left|\epsilon \frac{\partial w_\epsilon(t,x)}{\partial x}\right| \leq \xi(\epsilon) \tag{4.2.10}$$

for all $t \geq 0$, $x \in B_h$. Moreover, $w_\epsilon(0,x)=0$, for all $x \in B_h$.

If, moreover $\gamma(T)=a/T^r$ for some $a \geq 0$, $r \in (0,1]$,

Then the function $\xi(\epsilon)$ can be chosen to be $2a\epsilon^r$.

Proof of Lemma 4.2.2 in appendix.

Comments

The difference between this lemma and lemma 4.2.1 is in the condition on the partial derivative of $w_\epsilon(t,x)$ with respect to x in (4.2.10), and the dependence on $|x|$ in (4.2.8), (4.2.9).

Note that if the original system is linear, i.e.

$$\dot{x} = A(t)x \qquad x(0)=x_0 \tag{4.2.11}$$

for some $A(t)$: $\mathbf{R}_+ \to \mathbf{R}^{n \times n}$, then the main assumption of lemma 4.2.2 is that there exists A_{av} such that $A(t)-A_{av}$ has zero mean value.

The following assumptions will now be in effect.

Assumptions

For some $h>0$, $\epsilon_0>0$

(A1) $x=0$ is an equilibrium point of system (4.2.1), i.e. $f(t,0,0)=0$ for all $t\geqslant 0$. $f(t,x,\epsilon)$ is Lipschitz in x, i.e. for some $l_1\geqslant 0$

$$|f(t,x_1,\epsilon)-f(t,x_2,\epsilon)|\leqslant l_1|x_1-x_2| \tag{4.2.12}$$

for all $t\geqslant 0$, $x_1,x_2\in B_h$, $\epsilon\leqslant\epsilon_0$.

(A2) $f(t,x,\epsilon)$ is Lipschitz in ϵ, linearly in x, i.e. for some $l_2\geqslant 0$

$$|f(t,x,\epsilon_1)-f(t,x,\epsilon_2)|\leqslant l_2|x||\epsilon_1-\epsilon_2| \tag{4.2.13}$$

for all $t\geqslant 0$, $x\in B_h$, $\epsilon_1,\epsilon_2\leqslant\epsilon_0$.

(A3) $f_{av}(0)=0$, and $f_{av}(x)$ is Lipschitz in x, i.e. for some $l_{av}\geqslant 0$

$$|f_{av}(x_1)-f_{av}(x_2)|\leqslant l_{av}|x_1-x_2| \tag{4.2.14}$$

for all $x_1,x_2\in B_h$.

(A4) the function $d(t,x)=f(t,x,0)-f_{av}(x)$ satisfies the conditions of lemma 4.2.2.

Lemma 4.2.3 Perturbation Formulation of Averaging

If the original system (4.2.1) and the averaged system (4.2.5) satisfy assumptions (A1)-(A4)

Then there exist functions $w_\epsilon(t,x)$, $\xi(\epsilon)$ as in lemma 4.2.2, and $\epsilon_1>0$ such that the transformation

$$x=z+\epsilon\, w_\epsilon(t,z) \tag{4.2.15}$$

is a homeomorphism in B_h for all $\epsilon\leqslant\epsilon_1$, and

$$|x-z|\leqslant\xi(\epsilon)|z| \tag{4.2.16}$$

Under the transformation, system (4.2.1) becomes

$$\dot z=\epsilon\, f_{av}(z)+\epsilon\, p(t,z,\epsilon) \qquad z(0)=x_0 \tag{4.2.17}$$

where $p(t,z,\epsilon)$ satisfies

$$|p(t,z,\epsilon)| \leqslant \psi(\epsilon)|z| \qquad (4.2.18)$$

for some $\psi(\epsilon) \in K$. Further, $\psi(\epsilon)$ is of the order of $\epsilon+\xi(\epsilon)$.

Proof of Lemma 4.2.3 in appendix.

Comments

a) A similar lemma can be found in Hale (1980) (lemma 3.2, p. 192). Inequality (4.2.18) is a Lipschitz type of condition on $p(t,z,\epsilon)$, which is not found in Hale (1980), and results from the stronger conditions and conclusions of lemma 4.2.2.

b) Lemma 4.2.3 is fundamental to the theory of averaging presented hereafter. It separates the error in the approximation of the original system by the averaged system $(x-x_{av})$ into two components: $x-z$ and $z-x_{av}$. The first component results from a pointwise (in time) transformation of variable. This component is guaranteed to be small by inequality (4.2.16). For ϵ sufficiently small ($\epsilon \leqslant \epsilon_1$), the transformation $z \to x$ is invertible, and as $\epsilon \to 0$, it tends to the identity transformation. The second component is due to the perturbation term $p(t,z,\epsilon)$. Inequality (4.2.18) guarantees that this perturbation is small as $\epsilon \to 0$.

c) At this point, we can relate the convergence of the function $\gamma(T)$ to the order of the two components of the error $x-x_{av}$ in the approximation of the original system by the averaged system. The relationship between the functions $\gamma(T)$ and $\xi(\epsilon)$ was indicated in lemma 4.2.1. Lemma 4.2.3 relates the function $\xi(\epsilon)$ to the error due to the averaging. If $d(t,x)$ has a bounded integral (i.e. $\gamma(T) \sim 1/T$), then both $x-z$ and $p(t,z,\epsilon)$ are of the order of ϵ with respect to the main term $f_{av}(z)$. If $d(t,x)$ has zero mean but unbounded integral, these terms go to zero as $\epsilon \to 0$, but possibly more slowly than linearly (as $\sqrt{\epsilon}$ for example). The proof of lemma 4.2.1 provides a direct relationship between the order of the convergence to the mean value, and the order of the error terms.

We now focus attention on the approximation of the original system by the averaged system. Consider first the following assumption.

(A5) x_0 is sufficiently small so that, for fixed T, and some $h'<h$, $x_{av}(t) \in B_{h'}$ for all $t \in [0, T/\epsilon]$ (this is possible, using the Lipschitz assumption (A3), and proposition 1.4.1).

Theorem 4.2.4 Basic Averaging Theorem

If the original system (4.2.1) and the averaged system (4.2.5) satisfy assumptions (A1)-(A5)

Then there exists $\psi(\epsilon)$ as in lemma 4.2.3 such that, given $T \geqslant 0$

$$|x(t) - x_{av}(t)| \leqslant \psi(\epsilon) b_T \tag{4.2.19}$$

for some $b_T \geqslant 0$, $\epsilon_T > 0$, and for all $t \in [0, T/\epsilon]$, and $\epsilon \leqslant \epsilon_T$.

Proof of Theorem 4.2.4

We apply the transformation of lemma 4.2.3, so that

$$|x - z| \leqslant \xi(\epsilon) |z| \leqslant \psi(\epsilon) |z| \tag{4.2.20}$$

for $\epsilon \leqslant \epsilon_1$. On the other hand, we have that

$$\frac{d}{dt}(z - x_{av}) = \epsilon (f_{av}(z) - f_{av}(x_{av})) + \epsilon\, p(t, z, \epsilon) \quad z(0) - x_{av}(0) = 0 \tag{4.2.21}$$

for all $t \in [0, T/\epsilon]$, $x_{av} \in B_{h'}$, $h' < h$.

We will now show that, on this time interval, and for as long as $x, z \in B_h$, the errors $(z - x_{av})$ and $(x - x_{av})$ can be made arbitrarily small by reducing ϵ.

Integrating (4.2.21)

$$|z(t) - x_{av}(t)| \leqslant \epsilon l_{av} \int_0^t |z(\tau) - x_{av}(\tau)| d\tau + \epsilon\, \psi(\epsilon) \int_0^t |z(\tau)| d\tau \tag{4.2.22}$$

Using the Bellman-Gronwall lemma (lemma 1.4.2)

$$|z(t) - x_{av}(t)| \leqslant \epsilon \psi(\epsilon) \int_0^t |z(\tau)|\, e^{\epsilon l_{av}(t-\tau)} d\tau \leqslant \psi(\epsilon) h \left(\frac{e^{\epsilon l_{av} T} - 1}{l_{av}} \right)$$

$$:= \psi(\epsilon)\, a_T \tag{4.2.23}$$

Combining (4.2.20), (4.2.23)

$$\begin{aligned} |x(t) - x_{av}(t)| &\leqslant |x(t) - z(t)| + |z(t) - x_{av}(t)| \\ &\leqslant \psi(\epsilon) |x_{av}(t)| + (1 + \psi(\epsilon)) |z(t) - x_{av}(t)| \\ &\leqslant \psi(\epsilon) (h + (1 + \psi(\epsilon_1))\, a_T) \end{aligned}$$

$$:= \psi(\epsilon) b_T \tag{4.2.24}$$

By assumption, $|x_{av}(t)| \leqslant h' < h$. Let ϵ_T (with $0 < \epsilon_T \leqslant \epsilon_1$) such that $\psi(\epsilon_T) b_T < h - h'$. It follows, from a simple contradiction argument, that $x(t) \in B_h$, and that the estimate in (4.2.24) is valid for all $t \in [0, T/\epsilon]$, whenever $\epsilon \leqslant \epsilon_T$. □

Comments

Theorem 4.2.4 establishes that the trajectories of the original system and of the averaged system are arbitrarily close on intervals $[0, T/\epsilon]$, as ϵ is sufficiently small. The error is of the order of $\psi(\epsilon)$, and the order is related to the order of convergence of $\gamma(T)$. If $d(t, x)$ has a bounded integral (i.e. $\gamma(T) \sim 1/T$), then the error is of the order of ϵ.

It is important to remember that, although the intervals $[0, T/\epsilon]$ are unbounded, theorem 4.2.4 does not state that

$$|x(t) - x_{av}(t)| \leqslant \psi(\epsilon) b \tag{4.2.25}$$

for all $t \geqslant 0$, and some b. Consequently, theorem 4.2.4 does not allow us to relate the stability of the original and of the averaged system. This relationship is investigated in theorem 4.2.5.

Theorem 4.2.5 Exponential Stability Theorem

If the original system (4.2.1) and the averaged system (4.2.5) satisfy assumptions (A1)-(A5), the function $f_{av}(x)$ has continuous and bounded first partial derivatives in x, and $x = 0$ is an exponentially stable equilibrium point of the averaged system

Then the equilibrium point $x = 0$ of the original system is exponentially stable for ϵ sufficiently small.

Proof of Theorem 4.2.5

The proof relies on the converse theorem of Lyapunov for exponentially stable systems (theorem 1.4.3). Under the hypotheses, there exists a function $v(x_{av})$: $\mathbf{R}^n \to \mathbf{R}_+$, and strictly positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that, for all $x_{av} \in B_h$

$$\alpha_1 |x_{av}|^2 \leqslant v(x_{av}) \leqslant \alpha_2 |x_{av}|^2 \tag{4.2.26}$$

$$\dot{v}(x_{av})\Big|_{(4.2.5)} \leqslant -\epsilon\, \alpha_3 |x_{av}|^2 \tag{4.2.27}$$

$$\left| \frac{\partial v}{\partial x_{av}} \right| \leqslant \alpha_4 |x_{av}| \tag{4.2.28}$$

The derivative in (4.2.27) is to be taken along the trajectories of the averaged system (4.2.5).

The function v is now used to study the stability of the perturbed system (4.2.17), where $z(x)$ is defined by (4.2.15). Considering $v(z)$, inequalities (4.2.26) and (4.2.28) are still verified, with z replacing x_{av}. The derivative of $v(z)$ along the trajectories of (4.2.17) is given by

$$\dot{v}(z)\Big|_{(4.2.17)} = \dot{v}(z)\Big|_{(4.2.5)} + \left(\frac{\partial v}{\partial z} \right) (\epsilon\, p(t, z, \epsilon)) \tag{4.2.29}$$

and, using previous inequalities (including those from lemma 4.2.3)

$$\begin{aligned} \dot{v}(z)\Big|_{(4.2.17)} &\leqslant -\epsilon\alpha_3 |z|^2 + \epsilon\alpha_4 \psi(\epsilon) |z|^2 \\ &\leqslant -\epsilon \left(\frac{\alpha_3 - \psi(\epsilon)\alpha_4}{\alpha_2} \right) v(z) \end{aligned} \tag{4.2.30}$$

for all $\epsilon \leqslant \epsilon_1$. Let ϵ'_2 be such that $\alpha_3 - \psi(\epsilon'_2)\alpha_4 > 0$, and define $\epsilon_2 = \min(\epsilon_1, \epsilon'_2)$. Denote

$$\alpha(\epsilon) := \frac{\alpha_3 - \psi(\epsilon)\alpha_4}{2\alpha_2} \tag{4.2.31}$$

Consequently, (4.2.30) implies that

$$v(z) \leqslant v(z(t_0))\, e^{-2\epsilon\alpha(\epsilon)(t - t_0)} \tag{4.2.32}$$

and

$$|z(t)| \leqslant \left(\frac{\alpha_2}{\alpha_1} \right)^{1/2} |z(t_0)|\, e^{-\epsilon\alpha(\epsilon)(t - t_0)} \tag{4.2.33}$$

Since $\alpha(\epsilon) > 0$ for all $\epsilon \leqslant \epsilon_2$, system (4.2.17) is exponentially stable. Using (4.2.16), it follows that

$$|x(t)| \leqslant \frac{1 + \xi(\epsilon)}{1 - \xi(\epsilon)} \left(\frac{\alpha_2}{\alpha_1} \right)^{1/2} |x(t_0)|\, e^{-\epsilon\alpha(\epsilon)(t - t_0)} \tag{4.2.34}$$

for all $t \geqslant t_0 \geqslant 0$, $\epsilon \leqslant \epsilon_2$, and $x(t_0)$ sufficiently small that all signals remain in B_h. In other words, the original system is exponentially stable, with rate of convergence (at least) $\epsilon\alpha(\epsilon)$. □

Comments

a) Theorem 4.2.5 is a *local* exponential stability result. The original system will be *globally* exponentially stable, if the averaged system is globally exponentially stable, and provided that *all* assumptions are valid globally.

b) The proof of theorem 4.2.5 gives a useful bound on the rate of convergence of the original system. As ϵ tends to zero, $\epsilon\alpha(\epsilon)$ tends to $\frac{\epsilon}{2}\frac{\alpha_3}{\alpha_2}$, which is the bound on the rate of convergence of the averaged system that one would obtain using (4.2.26)-(4.2.27). In other words, the proof provides a bound on the rate of convergence, and this bound gets arbitrarily close to the corresponding bound for the averaged system, provided that ϵ is sufficiently small. This is a useful conclusion because it is in general very difficult to obtain a guaranteed rate of convergence for the original, nonautonomous system. The proof assumes the existence of a Lyapunov function satisfying (4.2.26)-(4.2.28), but does not depend on the specific function chosen. Since the averaged system is autonomous, it is usually easier to find such a function for it than for the original system, and any such function will provide a bound on the rate of convergence of the original system for ϵ sufficiently small.

c) The conclusion of theorem 4.2.5 is quite different from the conclusion of theorem 4.2.4. Since both x and x_{av} go to zero exponentially with t, the error $x - x_{av}$ also goes to zero exponentially with t. Yet, theorem 4.2.5 does not relate the bound on the error to ϵ. It is possible, however, to combine theorem 4.2.4 and theorem 4.2.5 to obtain a uniform approximation result, with an estimate similar to (4.2.25).

4.3 Application to Identification

To apply the averaging theory to the identifier described in chapter 2, we will study the case when $g = \epsilon > 0$, and the update law is given by (cf. (2.4.1))

$$\dot{\phi}(t) = -\epsilon\, e_1(t)\, w(t) \qquad \phi(0) = \phi_0 \tag{4.3.1}$$

The evolution of the parameter error is described by

$$\dot{\phi}(t) = -\epsilon\, w(t)\, w^T(t)\, \phi(t) \qquad \phi(0) = \phi_0 \tag{4.3.2}$$

In theorem 2.5.1, we found that system (4.3.2) is exponentially stable, provided that w is *persistently exciting*, i.e., there exist constants $\alpha_1, \alpha_2, \delta > 0$, such that

$$\alpha_2 I \geq \int_{t_0}^{t_0+\delta} w(\tau) w^T(\tau)\, d\tau \geq \alpha_1 I \quad \text{for all } t_0 \geq 0 \tag{4.3.3}$$

On the other hand, the averaging theory presented above leads us to the following definition.

Definition Stationarity, Autocovariance

A signal $z : \mathbf{R}_+ \to \mathbf{R}^n$ is said to be *stationary* if the following limit exists, uniformly in t_0

$$R_z(t) := \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} z(\tau) z^T(t+\tau)\, d\tau \quad \in \mathbf{R}^{n \times n} \tag{4.3.4}$$

in which instance, the limit $R_z(t)$ is called the *autocovariance* of z.

Frequency Domain Analysis

We now review some results from Boyd and Sastry (1984) and (1985). Many results have direct parallels with results in the stochastic literature, but are obtained in a completely deterministic framework.

The autocovariance matrix of a stationary signal w is a positive semidefinite function of t. Therefore, $R_w(t)$ can be written as the inverse Fourier transform of the positive *spectral measure* $S_w(d\omega)$

$$R_w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} S_w(d\omega) \tag{4.3.5}$$

If the input r of a proper stable transfer function $\hat{H}_{rw}$ is stationary, then the output w is also stationary. Its spectrum is related to the spectrum of r through

$$S_w(d\omega) = \hat{H}_{rw}(j\omega) \hat{H}_{rw}^{*T}(j\omega)\, s_r(d\omega) \tag{4.3.6}$$

and, using (4.3.5) and (4.3.6), we have that

$$R_w(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}_{rw}(j\omega) \hat{H}_{rw}^{*T}(j\omega)\, s_r(d\omega) \tag{4.3.7}$$

In the context of the identifier considered here, $\hat{H}_{rw}$ is given by (cf. (2.2.16)-(2.2.17))

$$\hat{H}_{rw}(s) = \begin{bmatrix} (sI-\Lambda)^{-1} b_\lambda \\ (sI-\Lambda)^{-1} b_\lambda \hat{P}(s) \end{bmatrix} \in \mathbb{R}^{2n}(s) \tag{4.3.8}$$

It can be shown (cf. Boyd and Sastry (1984) and (1985)) that when w is stationary, w is PE if and only if $R_w(0)$ is positive definite. From (4.3.7) and (4.3.8), it follows that this is true if the support of $s_r(d\,\omega)$ is greater than or equal to $2n$ points (the dimension of w = the number of unknown parameters = $2n$). Note that a DC component in $r(t)$ contributes one point to the support of $s_r\,(d\,\omega)$, while a sinusoidal component contributes two points.

With these definitions, the averaged system corresponding to (4.3.2) is simply

$$\dot{\phi}_{av} = -\epsilon\, R_w(0)\, \phi_{av} \qquad\qquad \phi_{av}(0) = \phi_0 \tag{4.3.9}$$

This system is particularly easy to study, since it is linear.

Convergence Analysis

When w is persistently exciting, $R_w(0)$ is a positive definite matrix. A natural Lyapunov function for (4.3.9) is

$$v(\phi_{av}) = \frac{1}{2}\,|\phi_{av}|^2 = \frac{1}{2}\,\phi_{av}^T\,\phi_{av} \tag{4.3.10}$$

and

$$-\epsilon\lambda_{\min}(R_w(0))\,|\phi_{av}|^2 \leqslant -\dot{v}(\phi_{av}) \leqslant -\epsilon\lambda_{\max}(R_w(0))\,|\phi_{av}|^2 \tag{4.3.11}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are respectively the minimum and maximum eigenvalues of $R_w(0)$. Thus, the rate of exponential convergence of the averaged system is at least $\epsilon\lambda_{\min}(R_w(0))$, and at most $\epsilon\lambda_{\max}(R_w(0))$. We can conclude that the rate of convergence of the original system for ϵ small enough is close to the interval $[\epsilon\lambda_{\min}(R_w(0)), \epsilon\lambda_{\max}(R_w(0))]$.

Equation (4.3.7) gives an interpretation of $R_w(0)$ in the frequency domain, and also a mean of computing an estimate of the rate of convergence of the adaptive algorithm, given the spectral content of the reference input. If the input r is periodic or almost periodic

$$r(t) = \sum_k r_k\, e^{j\,\omega_k\, t} \tag{4.3.12}$$

then the integral in (4.3.7) may be replaced by a summation

$$R_w(0) = \sum_k \hat{H}_{rw}(j\omega_k)\,\hat{H}_{rw}^{*T}(j\omega_k)\,r_k^2 \tag{4.3.13}$$

Since the transfer function $\hat{H}_{rw}$ depends on the unknown plant being identified, the use of (4.3.11) to determine the rate of convergence is limited. With knowledge of the plant, it could be used to determine the spectral content of the reference input that will optimize the rate of convergence of the identifier, given the physical constraints on r. Such a procedure is very reminiscent of the procedure indicated in Goodwin and Payne (1977) (chapter 6), for the design of input signals in identification. The autocovariance matrix defined here is similar to the *average information matrix* defined in Goodwin and Payne (1977) (p. 134). Our interpretation is, however, in terms of rates of parameter convergence of the averaged system rather than in terms of parameter error covariance in a stochastic framework.

Note that the proof of exponential stability of theorem 2.5.1 was based on the Lyapunov function of theorem 1.4.1, that was an average of the norm along the trajectories of the system. In this chapter, we averaged the *differential equation* itself, and found that the norm becomes a Lyapunov function to prove exponential stability.

It is also interesting to compare the convergence rate obtained through averaging with the convergence rate obtained in chapter 2. We found, in the proof of exponential convergence of theorem 2.5.1, that the estimate of the convergence rate tends to $g\,\alpha_1/\delta$ when the adaptation gain g (denoted ϵ in this section) tends to zero. The constants α_1, δ resulted from the PE condition (2.5.3), i.e. (4.3.3). By comparing (4.3.3) and (4.3.4), we find that the estimates provided by direct proof and by averaging are essentially identical for $g = \epsilon$ small.

Example

To illustrate the conclusions of this section, we consider the following example

$$\hat{P}(s) = \frac{k_p}{s + a_p} \tag{4.3.14}$$

The filter is chosen to be $\hat{\lambda}(s) = l_1 / s + l_2$ (where $l_1 = 10.05$, $l_2 = 10$ are arbitrarily chosen such that $|\hat{\lambda}(j1)| = 1$). Although $\hat{\lambda}$ is not monic, the gain l_1 can easily be taken into account.

Since the number of unknown parameters is 2, parameter convergence will occur when the support of $s_r(d\,\omega)$ is greater than or equal to 2 points. We consider an input of the form $r = r_0 \sin(\omega_0 t)$, so that the support consists of exactly 2 points.

The averaged system can be found by using (4.3.8).

$$\dot{\phi}_{av} = -\epsilon\,\frac{r_0^2}{2}\,\frac{l_1^2}{l_2^2+\omega_0^2}\begin{bmatrix} 1 & \dfrac{a_p k_p}{\omega_0^2+a_p^2} \\ \dfrac{a_p k_p}{\omega_0^2+a_p^2} & \dfrac{k_p^2}{\omega_0^2+a_p^2}\end{bmatrix}\cdot\phi_{av} \qquad \phi_{av}(0)=\phi_0 \qquad (4.3.15)$$

With $r_0 = 1$, $\omega_0 = 1$, $a_p = 1$, $k_p = 2$, the eigenvalues of the averaged system (4.3.15) are computed to be $-\dfrac{3+\sqrt{5}}{4}\,\epsilon = -1.309\,\epsilon$, and $-\dfrac{3-\sqrt{5}}{4}\,\epsilon = -0.191\,\epsilon$. The nominal parameter $\theta^{*T} = (k_p / l_1, (l_2 - a_p)/ l_1)$. We let $\theta(0) = 0$, so that $\phi^T(0) = (-0.199, -0.9)$.

Figures 4.1 to 4.4 show the plots of the parameter errors ϕ_1 and ϕ_2, for both the original and averaged systems, and with two different adaptation gains $\epsilon = 1$, and $\epsilon = 0.1$. We notice the closeness of the approximation for $\epsilon = 0.1$.

Figures 4.5 and 4.6 are plots of the Lyapunov function (4.3.10) for $\epsilon = 1$ and $\epsilon = 0.1$, using a logarithmic scale. We observe the two slopes, corresponding to the two eigenvalues. The closeness of the estimate of the convergence rate by the averaged system can also be appreciated from these figures.

Figure 4.7 represents the two components of ϕ, one as a function of the other when $\epsilon = 0.1$. It shows the two subspaces corresponding to the small and large eigenvalues: the parameter error first moves fast along the direction of the eigenvector corresponding to the large eigenvalue. Then, it slowly moves along the direction corresponding to the small eigenvalue.

4.4 Averaging Theory - Two-Time Scales

We now consider a more general class of differential equations arising in the adaptive control schemes presented in chapter 3.

4.4.1 Separated Time Scales

We first consider the system of differential equations

$$\dot{x} = \epsilon f(t, x, y) \tag{4.4.1}$$

$$\dot{y} = A(x)y + \epsilon g(t, x, y) \tag{4.4.2}$$

where $x(0) = x_0$, $y(0) = y_0$, $x \in \mathbf{R}^n$, and $y \in \mathbf{R}^m$.

The state vector is divided in a fast state vector y, and a slow state vector x, whose dynamics are of the order of ϵ with respect to the fast dynamics. The dominant term in (4.4.2) is linear in y, but is itself allowed to vary as a function of the slow state vector.

As previously, we define

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x, 0)\, d\tau \tag{4.4.3}$$

and the system

$$\dot{x}_{av} = f_{av}(x_{av}) \qquad x_{av}(0) = x_0 \tag{4.4.4}$$

is the *averaged system* corresponding to (4.4.1)-(4.4.2). We make the following additional assumption.

Definition Uniform Exponential Stability of a Family of Square Matrices

The family of matrices $A(x) \in \mathbf{R}^{m \times m}$ is *uniformly exponentially stable for all* $x \in B_h$, if there exist $m, \lambda, m', \lambda' > 0$, such that, for all $x \in B_h$ and $t \geq 0$

$$m' e^{-\lambda' t} \leq \| e^{A(x)t} \| \leq m\, e^{-\lambda t} \tag{4.4.5}$$

Comments

This definition is equivalent to require that the solutions of the system $\dot{y} = A(x)y$ are bounded above and below by decaying exponentials, independently of the parameter x.

It is also possible to show that the definition is equivalent to requiring that there exist p_1, p_2, q_1, $q_2 > 0$, such that for all $x \in B_h$, there exists $P(x)$ satisfying $p_1 I \leq P(x) \leq p_2 I$, and $-q_2 I \leq A^T(x)P(x) + P(x)A(x) \leq -q_1 I$.

We will make the following assumptions.

Assumptions

For some $h > 0$

(B1) The functions f and g are piecewise continuous functions of time, and continuous functions of x and y. Moreover, $f(t,0,0)=0$, $g(t,0,0)=0$ for all $t \geqslant 0$, and for some $l_1, l_2, l_3, l_4 \geqslant 0$

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \leqslant l_1|x_1-x_2|+l_2|y_1-y_2|$$
$$|g(t,x_1,y_1)-g(t,x_2,y_2)| \leqslant l_3|x_1-x_2|+l_4|y_1-y_2| \tag{4.4.6}$$

for all $t \geqslant 0$, $x_1, x_2 \in B_h$, $y_1, y_2 \in B_h$. Also assume that $f(t,x,0)$ has continuous and bounded first partial derivatives with respect to x, for all $t \geqslant 0$, and $x \in B_h$.

(B2) The function $f(t,x,0)$ has average value $f_{av}(x)$. Moreover, $f_{av}(0)=0$, and $f_{av}(x)$ has continuous and bounded first partial derivatives with respect to x, for all $x \in B_h$, so that for some $l_{av} \geqslant 0$

$$|f_{av}(x_1)-f_{av}(x_2)| \leqslant l_{av}|x_1-x_2| \tag{4.4.7}$$

for all $x_1, x_2 \in B_h$.

(B3) Let $d(t,x)=f(t,x,0)-f_{av}(x)$, so that $d(t,x)$ has zero average value. Assume that the convergence function can be written as $\gamma(T)|x|$, and that $\frac{\partial d(t,x)}{\partial x}$ has zero average value, with convergence function $\gamma(T)$.

(B4) A(x) is uniformly exponentially stable for all $x \in B_h$ and, for some $k_a \geqslant 0$

$$\left\| \frac{\partial A(x)}{\partial x} \right\| \leqslant k_a \qquad \text{for all } x \in B_h \tag{4.4.8}$$

(B5) For some $h' < h$, $|x_{av}(t)| \in B_{h'}$ on the time intervals considered, and for some h_0, $y_0 \in B_{h_0}$ (where h', h_0 are constants to be defined later). This assumption is technical, and will allow us to guarantee that all signals remain in B_h.

As for one-time scale systems, we first obtain the following preliminary lemma, similar to lemma 4.2.3.

Lemma 4.4.1 Perturbation Formulation of Averaging - Two-Time Scales

If the original system (4.4.1)-(4.4.2) and the averaged system (4.4.4) satisfy assumptions (B1)-(B3)

Then there exist functions $w_\epsilon(t,x)$, $\xi(\epsilon)$ as in lemma 4.2.2, and $\epsilon_1 > 0$, such that the transformation

$$x = z + \epsilon\, w_\epsilon(t,z) \tag{4.4.9}$$

is a homeomorphism in B_h for all $\epsilon \leqslant \epsilon_1$, and

$$|x - z| \leqslant \xi(\epsilon)|z| \tag{4.4.10}$$

Under the transformation, system (4.4.1) becomes

$$\dot{z} = \epsilon\, f_{av}(z) + \epsilon\, p_1(t,z,\epsilon) + \epsilon\, p_2(t,z,y,\epsilon) \qquad z(0) = x_0 \tag{4.4.11}$$

where

$$|p_1(t,z,\epsilon)| \leqslant \xi(\epsilon)\, k_1 |z| \qquad \text{and} \qquad |p_2(t,z,y,\epsilon)| \leqslant k_2 |y| \tag{4.4.12}$$

for some k_1, k_2 depending on l_1, l_2, l_{av}.

Proof of Lemma 4.4.1 in appendix.

We are now ready to state the first averaging theorem concerning the differential system (4.4.1)-(4.4.2). Theorem 4.4.2 is an approximation theorem similar to theorem 4.2.4, and guarantees that the trajectories of the original and averaged system are arbitrarily close on compact intervals, when ϵ tends to zero.

Theorem 4.4.2 Basic Averaging Theorem

If the original system (4.4.1)-(4.4.2) and the averaged system (4.4.4) satisfy assumptions (B1)-(B5)

Then there exists $\psi(\epsilon)$ as in lemma 4.2.3 such that, given $T \geqslant 0$

$$|x(t) - x_{av}(t)| \leqslant \psi(\epsilon)\, b_T \tag{4.4.13}$$

for some $b_T \geqslant 0$, $\epsilon_T > 0$, and for all $t \in [0, T/\epsilon]$, and $\epsilon \leqslant \epsilon_T$.

Proof of Theorem 4.4.2

The proof assumes that for all $t \in [0, T/\epsilon]$, the solutions $x(t)$, $y(t)$, and $z(t)$ (to be defined) remain in B_h. Since this is not guaranteed a priori, the steps of the proof are only valid for as long as the condition is verified. By assumption, $x_{av}(t) \in B_{h'}$, with $h' < h$. We will show that by letting ϵ and h_0 sufficiently small, we can let $x(t)$ be arbitrarily close to $x_{av}(t)$, and $y(t)$ arbitrarily small. It then follows, from a contradiction argument, that $x(t)$, $y(t) \in B_h$ for all $t \in [0, T/\epsilon]$, provided that ϵ and h_0 are sufficiently small.

Using lemma 4.4.1, we transform the original system (4.4.1),(4.4.2) into the system (4.4.11),(4.4.2). A bound on the error $|z(t) - x_{av}(t)|$ can be calculated by integrating the difference (4.4.11)-(4.4.4), and by using (4.4.7) and (4.4.12)

$$|z(t) - x_{av}(t)| \leqslant \epsilon\, l_{av} \int_0^t |z(\tau) - x_{av}(\tau)| d\tau + \epsilon\, \xi(\epsilon)\, k_1 \int_0^t |z(\tau)| d\tau$$

$$+ \epsilon\, k_2 \int_0^t |y(\tau)| d\tau \tag{4.4.14}$$

Bound on $|y(t)|$

To obtain a bound on $|y(t)|$, we divide the interval $[0, T/\epsilon]$ in intervals $[t_i, t_{i+1}]$ of length ΔT (the last interval may be of smaller length, and ΔT will be defined later). The differential equation for y is

$$\dot{y} = A(x)y + \epsilon\, g(t, x, y) \tag{4.4.15}$$

and is rewritten on the time interval $[t_i, t_{i+1}]$ as follows

$$\dot{y} = A_{x_i} y + \epsilon\, g(t, x, y) + (A_{x_t} - A_{x_i}) y \tag{4.4.16}$$

where $A_{x_t} = A(x(t))$, and $A_{x_i} = A(x(t_i))$, so that the solution $y(t)$, for $t \in [t_i, t_{i+1}]$, is given by

$$y(t) = e^{A_{x_i}(t - t_i)} y_i + \epsilon \int_{t_i}^t e^{A_{x_i}(t-\tau)} g(\tau, x, y)\, d\tau$$

$$+ \int_{t_i}^t e^{A_{x_i}(t-\tau)} (A_{x_\tau} - A_{x_i}) y(\tau)\, d\tau \tag{4.4.17}$$

where $y_i = y(t_i)$. From the assumptions, it follows that

$$|A_{x_\tau} - A_{x_i}| \leq k_a \, |\dot{x}| (\tau - t_i) \leq \epsilon \, (l_1 + l_2) h \; k_a \; \Delta T \tag{4.4.18}$$

and, using the uniform exponential stability assumption on $A(x)$

$$|y(t)| \leq m \; |y_i| e^{-\lambda(t - t_i)} + \epsilon \frac{m}{\lambda} \; h \; ((l_3 + l_4) + (l_1 + l_2) k_a \; \Delta T) \tag{4.4.19}$$

Let the last term in (4.4.19) be denoted by $\epsilon \, k_b$, and use (4.4.19) as a recursion formula for y_i, so that

$$|y_i| \leq (m \; e^{-\lambda \Delta T})^i \; |y_0| + \epsilon \, k_b \sum_{j=0}^{i-1} (m \; e^{-\lambda \Delta T})^j \tag{4.4.20}$$

Choose ΔT sufficiently large that

$$m \; e^{-\lambda \Delta T} \leq e^{-\lambda \Delta T / 2} \qquad \text{i.e.} \qquad \Delta T \geq \frac{2}{\lambda} \ln m \tag{4.4.21}$$

It follows that

$$\sum_{j=0}^{i-1} \left(m \; e^{-\lambda \Delta T} \right)^j \leq \sum_{j=0}^{\infty} \left(e^{-\lambda \Delta T / 2} \right)^j = \frac{1}{1 - e^{-\lambda \Delta T / 2}} \tag{4.4.22}$$

Combining (4.4.20)-(4.4.22), and using the assumption $y_0 \in B_{h_0}$

$$|y_i| \leq e^{-\lambda \Delta T \, i / 2} h_0 + \frac{\epsilon \, k_b}{1 - e^{-\lambda \Delta T / 2}} := e^{-\lambda t_i / 2} h_0 + \epsilon \, k_c \tag{4.4.23}$$

Using this result in (4.4.19), it follows that for all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} |y(t)| &\leq m \; e^{-\lambda t_i / 2} h_0 e^{-\lambda(t - t_i)} + m \; \epsilon \; k_c \; e^{-\lambda(t - t_i)} + \epsilon \, k_b \\ &\leq m \; h_0 e^{-\lambda t / 2} + \epsilon \, (m \; k_c + k_b) \end{aligned} \tag{4.4.24}$$

Since the last inequality does not depend on i, it gives a bound on $|y(t)|$ for all $t \in [0, T / \epsilon]$.

Bound on $z(t) - x_{av}(t)$

We now return to (4.4.14), and to the approximation error, using the bound on $|y(t)|$

$$\begin{aligned} |z(t) - x_{av}(t)| \leq \; &\epsilon \, l_{av} \int_0^t |z(\tau) - x_{av}(\tau)| d\tau + \epsilon \, \xi(\epsilon) k_1 \int_0^t h \; d\tau \\ &+ \epsilon \, k_2 \int_0^t (m \; h_0 e^{-\lambda \tau / 2} + \epsilon \, (m \; k_c + k_b)) \, d\tau \end{aligned} \tag{4.4.25}$$

so that, using the Bellman-Gronwall lemma (lemma 1.4.2)

$$|z(t) - x_{av}(t)| \leqslant \int_0^t (\xi(\epsilon) k_1 h + k_2 m\, h_0 e^{-\lambda \tau / 2} + k_2 \epsilon\, (m\, k_c + k_b))\, \epsilon\, e^{\epsilon\, l_{av}\, (t - \tau)} d\tau$$

$$\leqslant (\epsilon + \xi(\epsilon))\, (k_1 h + \frac{k_2 m\, h_0 l_{av}}{\lambda / 2 + \epsilon\, l_{av}} + k_2 (m\, k_c + k_b))\, \frac{e^{l_{av} T}}{l_{av}}$$

$$:= \psi(\epsilon)\, a_T \tag{4.4.26}$$

and, using (4.4.10)

$$|x(t) - x_{av}(t)| \leqslant \psi(\epsilon)\, b_T \tag{4.4.27}$$

for some b_T.

Assumptions

We assumed in the proof that all signals remained in B_h. By assumption, $x_{av}(t) \in B_{h'}$, for some $h' < h$. Let h_0, and ϵ_T be sufficiently small so that, for all $\epsilon \leqslant \epsilon_T \leqslant \epsilon_1$, we have that $m\, h_0 + \epsilon\, (m\, k_c + k_b) \leqslant h$ (cf (4.4.24)), and that $\psi(\epsilon)\, b_T \leqslant h - h'$ (cf (4.4.27)). It follows, from a simple contradiction argument, that the solutions $x(t)$, $y(t)$, and $z(t)$ remain in B_h for all $t \in [0, T/\epsilon]$, so that all steps of the proof are valid, and (4.4.27) is in fact satisfied over the whole time interval. □

Theorem 4.4.3 Exponential Stability Theorem

If the original system (4.4.1)-(4.4.2) and the averaged system (4.4.4) satisfy assumptions (B1)-(B5), the function $f_{av}(x)$ has continuous and bounded first partial derivatives in x, and $x = 0$ is an exponentially stable equilibrium point of the averaged system

Then the equilibrium point $x = 0$, $y = 0$ of the original system is exponentially stable for ϵ sufficiently small.

Proof of Theorem 4.4.3

The proof relies on the converse theorem of Lyapunov for exponentially stable systems (theorem 1.4.3). Under the hypotheses, there exists a function $v(x_{av}): \mathbf{R}^n \to \mathbf{R}_+$, and strictly positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that, for all $x_{av} \in B_h$

$$\alpha_1 |x_{av}|^2 \leqslant v(x_{av}) \leqslant \alpha_2 |x_{av}|^2 \tag{4.4.28}$$

$$\dot{v}(x_{av}) \Big|_{(4.4.4)} \leqslant -\epsilon\, \alpha_3 |x_{av}|^2 \tag{4.4.29}$$

$$\left| \frac{\partial v}{\partial x_{av}} \right| \leqslant \alpha_4 \,|\, x_{av} \,| \tag{4.4.30}$$

The derivative in (4.4.29) is to be taken along the trajectories of the averaged system (4.4.4).

We now study the stability of the original system (4.4.1),(4.4.2), through the transformed system (4.4.11),(4.4.2), where $x(z)$ is defined in (4.4.9). Consider the following Lyapunov function

$$v_1(z,y) = v(z) + \frac{\alpha_2}{p_2}\, y^T P(x(z))\, y \tag{4.4.31}$$

where $P(x)$, p_2 are defined in the comments after the definition of uniform exponential stability of $A(x)$. Defining $\alpha'_1 = \min(\alpha_1, \dfrac{\alpha_2}{p_2}\, p_1)$, it follows that

$$\alpha'_1 (|z|^2 + |y|^2) \leqslant v_1(z,y) \leqslant \alpha_2 (|z|^2 + |y|^2) \tag{4.4.32}$$

The derivative of v_1 along the trajectories of (4.4.11)-(4.4.2) can be bounded, using above inequalities

$$\begin{aligned} \dot{v}_1(z,y) \leqslant & -\epsilon\, \alpha_3 |z|^2 + \epsilon\, \xi(\epsilon) k_1 \alpha_4 |z|^2 + \epsilon\, k_2 \alpha_4 |z||y| \\ & + \frac{\alpha_2}{p_2} \left\| \frac{\partial P(x)}{\partial x} \right\| \left\| \frac{\partial x}{\partial z} \right\| |\dot{z}|\, |y|^2 - \frac{\alpha_2}{p_2}\, q_1 |y|^2 \\ & + 4\epsilon\, l_3 \alpha_2 |z|\,|y| + 2\epsilon\, l_4 \alpha_2 |y|^2 \end{aligned} \tag{4.4.33}$$

for $\epsilon \leqslant \epsilon_1$ (so that the transformation $x \to z$ is well-defined, and $|x| \leqslant 2|z|$). We now calculate bounds on the terms in (4.4.33).

Bound on $|\,\partial P / \partial x\,|$

Note that $P(x)$ can be defined by

$$P(x) = \int_0^\infty e^{A^T(x)t}\, Q\, e^{A(x)t}\, dt \tag{4.4.34}$$

so that

$$\frac{\partial P(x)}{\partial x_i} = \int_0^\infty \left\{ \left(\frac{\partial}{\partial x_i} e^{A^T(x)t} \right) Q\, e^{A(x)t} + e^{A^T(x)t} Q \left(\frac{\partial}{\partial x_i} e^{A(x)t} \right) \right\} dt \quad (4.4.35)$$

The partial derivatives in parentheses solve the differential equation

$$\frac{d}{dt} \left(\frac{\partial}{\partial x_i} e^{A(x)t} \right) = A(x) \left(\frac{\partial}{\partial x_i} e^{A(x)t} \right) + \frac{\partial A(x)}{\partial x_i} e^{A(x)t} \quad (4.4.36)$$

with zero initial conditions, so that

$$\frac{\partial}{\partial x_i} e^{A(x)t} = \int_0^t e^{A(x)(t-\tau)} \frac{\partial A(x)}{\partial x_i} e^{A(x)\tau} d\tau \quad (4.4.37)$$

From the boundedness of $\frac{\partial A(x)}{\partial x_i}$, and from the exponential stability of $A(x)$, it follows that

$$\left\| \frac{\partial}{\partial x} e^{A(x)t} \right\| \leq m^2 k_a\, t\, e^{-\lambda t} \quad (4.4.38)$$

With (4.4.35), this implies that $\| \partial P(x) / \partial x \|$ is bounded by some $k_p \geq 0$.

Bound on $\| \partial x / \partial z \|$, *and* $|\dot{z}|$

On the other hand, using (4.4.9), (4.2.8) and (4.4.12)

$$\left\| \frac{\partial x}{\partial z} \right\| < 1 + \xi(\epsilon) < 2 \qquad \text{and} \qquad |\dot{z}| \leq \epsilon h\, (l_{av} + \xi(\epsilon) k_1 + k_2) \quad (4.4.39)$$

Using these results in (4.4.33), and noting the fact that, for all $y, z \in \mathbf{R}$

$$\epsilon |z|\, |y| \leq \frac{1}{2} (\epsilon^{4/3} |z|^2 + \epsilon^{2/3} |y|^2) \quad (4.4.40)$$

it follows that

$$\begin{aligned} \dot{v}_1(z, y) \leq & -\epsilon \left(\alpha_3 - \xi(\epsilon) k_1 \alpha_4 - \epsilon^{1/3} \frac{k_2 \alpha_4}{2} - 2 \epsilon^{1/3} l_3 \alpha_2 \right) |z|^2 \\ & - \left(\frac{\alpha_2}{p_2} q_1 - 2 \epsilon l_4 \alpha_2 - \epsilon^{2/3} \frac{k_2 \alpha_4}{2} - 2 \epsilon^{2/3} l_3 \alpha_2 \right. \\ & \left. + 2 \epsilon \frac{\alpha_2}{p_2} k_p h\, (l_{av} + \xi(\epsilon) k_1 + k_2) \right) |y|^2 \\ := & -2 \epsilon \alpha_2 \alpha(\epsilon) |z|^2 - q(\epsilon) |y|^2 \end{aligned} \quad (4.4.41)$$

Note that, with this definition, $\alpha(\epsilon) \to \frac{1}{2} \frac{\alpha_3}{\alpha_2}$ as $\epsilon \to 0$, while $q(\epsilon) \to \frac{\alpha_2}{p_2} q_1$.

Let $\epsilon \leqslant \epsilon_1$ be sufficiently small that $\alpha(\epsilon) > 0$, and $2\epsilon\alpha_2\alpha(\epsilon) \leqslant q(\epsilon)$. Then

$$\dot{v}_1(z,y) \leqslant -2\epsilon\alpha(\epsilon)v_1(z,y) \tag{4.4.42}$$

so that the z,y system is exponentially stable with rate of convergence $\epsilon\alpha(\epsilon)$ (v_1 being bounded above and below by the *square* of the norm of the state). The same conclusion holds for the x,y system, given the transformation (4.4.9), with (4.4.10). Also, for ϵ, h_0 sufficiently small, all signals are actually guaranteed to remain in B_h so that all assumptions are valid. □

Comments

As for theorem 4.2.5, the proof of theorem 4.4.3 gives a useful bound on the rate of convergence of the nonautonomous system. As $\epsilon \to 0$, the rate tends to $\frac{\epsilon}{2}\frac{\alpha_3}{\alpha_2}$, which is the bound on the rate of convergence of the averaged system that one would obtain using the Lyapunov function $v(x_{av})$. Since the averaged system is autonomous, it is usually easier to obtain such a Lyapunov function for the averaged system than for the original nonautonomous system, and conclusions about its exponential convergence can be applied to the nonautonomous system for ϵ sufficiently small.

4.4.2 Mixed Time Scales

We now discuss a more general class of two-time scale systems, arising in adaptive control

$$\dot{x} = \epsilon f'(t,x,y') \tag{4.4.43}$$

$$\dot{y}' = A(x)y' + h(t,x) + \epsilon g'(t,x,y') \tag{4.4.44}$$

We will show that system (4.4.43)-(4.4.44) can be transformed into the system (4.4.1)-(4.4.2). In this case, x is a slow variable, but y' has both a fast, and a slow component.

The averaged system corresponding to (4.4.43), (4.4.44) is obtained as follows. Define the function

$$v(t,x) := \int_0^t e^{A(x)(t-\tau)}h(\tau,x)d\tau \tag{4.4.45}$$

and assume that the following limit exists uniformly in t and x

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f'(\tau, x, v(\tau, x)) d\tau \tag{4.4.46}$$

Intuitively, $v(t,x)$ represents the steady-state value of the variable y' with x frozen and $\epsilon=0$ in (4.4.44). Then, f is averaged with $v(t,x)$ replacing y' in (4.4.43).

Consider now the transformation

$$y = y' - v(t,x) \tag{4.4.47}$$

Since $v(t,x)$ satisfies

$$\frac{\partial}{\partial t} v(t,x) = A(x)v(t,x) + h(t,x) \qquad v(t,0)=0 \tag{4.4.48}$$

we have that

$$\dot{y} = A(x)y + \epsilon \left(-\frac{\partial v(t,x)}{\partial x} f'(t,x,y+v(t,x)) + g'(t,x,y+v(t,x)) \right) \tag{4.4.49}$$

so that (4.4.43), (4.4.49) is of the form of (4.4.1), (4.4.2) when

$$f(t,x,y) = f'(t,x,y+v(t,x)) \tag{4.4.50}$$

$$g(t,x,y) = -\frac{\partial v(t,x)}{\partial x} f'(t,x,y+v(t,x)) + g'(t,x,y+v(t,x)) \tag{4.4.51}$$

The averaged system is obtained by averaging the right-hand side of (4.4.50) with $y=0$, so that the definitions (4.4.46), and (4.4.3) (with f given by (4.4.50)) agree.

To apply theorems 4.4.2 and 4.4.3, we require assumptions (B1)-(B5) to be satisfied. In particular, we assume similar Lipschitz conditions on f', g', and the following assumption on $h(t,x)$

(B6) $h(t,0)=0$ for all $t \geq 0$, and $\| \partial h(t,x) / \partial x \|$ is bounded for all $t \geq 0$, $x \in B_h$.

This new assumption implies that $v(t,0)=0$. It also implies that $\| \frac{\partial v(t,x)}{\partial x} \|$ is bounded for all $t \geq 0$, $x \in B_h$, since

$$\frac{\partial v(t,x)}{\partial x_i} = \int_0^t \left(e^{A(x)(t-\tau)} \frac{\partial h(\tau,x)}{\partial x_i} + \frac{\partial}{\partial x_i} \left(e^{A(x)(t-\tau)} \right) h(\tau,x) \right) d\tau \tag{4.4.52}$$

and using the fact that $e^{A(x)(t-\tau)}$ and $\frac{\partial}{\partial x} e^{A(x)(t-\tau)}$ are bounded by exponentials ((4.4.5) and (4.4.38)).

4.5 Applications to Adaptive Control

For illustration, we apply the previous results to the output error direct adaptive control algorithm for the relative degree 1 case.

We established the complete description of the adaptive system in section 3.5 with (3.5.28), i.e.

$$\dot{e}(t) = A_m e(t) + b_m \phi^T(t) w_m(t) + b_m \phi^T(t) Q\, e(t)$$

$$\dot{\phi}(t) = -\epsilon\, c_m^T e(t)\, w_m(t) - \epsilon\, c_m^T e(t)\, Q\, e(t) \tag{4.5.1}$$

where ϵ is the adaptation gain. With the exception of the last terms (quadratic in e and ϕ), (4.5.1) is a set of linear time varying differential equations. They describe the adaptive control system, linearized around the equilibrium $e = 0$, $\phi = 0$. We first study these equations, then turn to the nonlinear equations.

4.5.1 Linearized Equations

The linearized equations, describing the adaptive system for small values of e and ϕ, are

$$\dot{e}(t) = A_m e(t) + b_m\, w_m^T(t)\phi(t)$$

$$\dot{\phi}(t) = -\epsilon\, w_m(t)\, c_m{}^T\, e(t) \tag{4.5.2}$$

Since w_m is bounded, it is easy to see that (4.5.2) is of the form of (4.4.43), (4.4.44) with the functions $f\,'$ and h satisfying the conditions of section 4.4. Recall that A_m is a stable matrix.

The function $v(t,\phi)$ defined in (4.4.45) is now

$$v(t,\phi) = \Big[\int_0^t e^{A_m(t-\tau)} b_m w_m^T(\tau)\, d\tau\Big]\phi \tag{4.5.3}$$

and f_{av} is given by

$$f_{av}(\phi) = -\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} w_m(t)\, c_m{}^T \Big[\int_0^t e^{A_m(t-\tau)} b_m w_m^T(\tau)\, d\tau\Big] dt\;\; \phi \tag{4.5.4}$$

Frequency Domain Analysis

To derive frequency domain expressions, we assume that r is stationary. Since the transfer function from $r \to w_m$ is stable, this implies that w_m is stationary. The spectral measure of w_m is related to that of r by

$$S_{w_m}(d\,\omega) = \hat{H}_{rw_m}(j\omega)\,\hat{H}^{*T}_{rw_m}(j\omega)\,s_r(d\,\omega) \tag{4.5.5}$$

where the transfer function from $r \to w_m$ is given by (using (3.5.11))

$$\hat{H}_{rw_m} = \begin{bmatrix} 1 \\ (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} \tag{4.5.6}$$

which is a stable transfer function.

Define now a filtered version of w_m to be

$$w_{mf}(t) = \int_0^t c_m^T e^{A_m(t-\tau)} b_m w_m(\tau)\,d\tau = \frac{1}{c_0^*}\hat{M}(w_m) \tag{4.5.7}$$

where the last equality follows from (3.5.22). Note that the signal w_f was also used in the direct proof of exponential convergence in chapter 2 (cf. (2.6.17)).

Since $c_m^T(sI - A_m)^{-1}b_m = \frac{1}{c_0^*}\hat{M}(s)$ is stable, $w_{mf}(t)$ is stationary. We let

$$R_{w_m w_{mf}}(0) := \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} w_m(t)\,w_{mf}^T(t)\,dt \tag{4.5.8}$$

which is called the *cross correlation* between w_m and w_{mf} evaluated at 0. Consequently, we may use (4.5.7) and (4.5.8) to obtain a frequency domain expression for $R_{w_m w_{mf}}(0)$ as

$$R_{w_m w_{mf}}(0) = \frac{1}{2\pi c_0^*}\int_{-\infty}^{\infty} \hat{H}_{rw_m}(j\omega)\,\hat{H}^{*T}_{rw_m}(j\omega)\,\hat{M}(j\omega)\,s_r(d\,\omega) \tag{4.5.9}$$

With (4.5.7) and (4.5.8), (4.5.4) shows that the averaged system is a LTI system

$$\dot{\phi}_{av} = -\epsilon\,R_{w_m w_{mf}}(0)\,\phi_{av} \qquad \phi_{av}(0) = \phi_0 \tag{4.5.10}$$

Convergence Analysis

Since $\hat{M}(s)$ is strictly positive real, the matrix $R_{w_m w_{mf}}(0)$ is a positive semidefinite matrix (cf. (4.5.9)). Unlike the matrix $R_w(0)$ of section 4.3, $R_{w_m w_{mf}}(0)$ need not be symmetric, so that its eigenvalues need not be real. However, the real parts are guaranteed to be positive, and a natural Lyapunov function is again

$$v(\phi_{av}) = |\phi_{av}|^2 = \phi_{av}^T \phi_{av} \tag{4.5.11}$$

and

$$-\dot{v}(\phi_{av}) = \epsilon \, \phi_{av}^T \left(R_{w_m w_{mf}}(0) + R_{w_m w_{mf}}^T(0) \right) \phi_{av} \tag{4.5.12}$$

The matrix in parentheses is symmetric positive semidefinite. As previously, it is positive definite if w_m is PE.

When the reference input r is almost periodic, i.e.

$$r(t) = \sum_k r_k e^{j \omega_k t} \tag{4.5.13}$$

an expression for $R_{w_m w_{mf}}(0)$ is

$$R_{w_m w_{mf}}(0) = \frac{1}{c_0^*} \sum_k \hat{H}_{rw_m}(j\omega_k) \hat{H}_{rw_m}^{*T}(j\omega_k) \hat{M}(j\omega_k) r_k^2 \tag{4.5.14}$$

Example

As an illustration of the preceding results, we consider the following example of a first order plant with an unknown pole and an unknown gain

$$\hat{P}(s) = \frac{k_p}{s + a_p} \tag{4.5.15}$$

We will choose values of the parameters corresponding to the "Rohrs example" (Rohrs (1982), see also section 5.1) , when no unmodeled dynamics are present.

The adaptive process is to adjust the feedforward gain c_0 and the feedback gain d_0 so as to make the closed-loop transfer function match the model transfer function

$$\hat{M}(s) = \frac{k_m}{s + a_m} \tag{4.5.16}$$

To guarantee persistency of excitation, we use a sinusoidal input signal of the form

$$r(t) = r_0 \sin(\omega_0 t) \tag{4.5.15}$$

Thus, (4.5.2) becomes

$$\dot{e}(t) = -a_m e(t) + k_p \left(\phi_r(t) r(t) + \phi_y(t) y_m(t) \right)$$

$$\dot{\phi}_r(t) = -\epsilon\, e(t) r(t)$$

$$\dot{\phi}_y(t) = -\epsilon\, e(t) y_m(t) \tag{4.5.18}$$

where

$$\phi_r(t) = c_0(t) - c_0^*$$

$$\phi_y(t) = d_0(t) - d_0^* \tag{4.5.19}$$

It can be checked, using (4.5.14), that the averaged system defined in (4.5.10) is now

$$\dot{\phi}_{av} = -\epsilon \frac{r_0^2}{2} \frac{k_p}{k_m} \begin{pmatrix} \dfrac{a_m k_m}{(a_m^2 + \omega_0^2)} & \dfrac{k_m^2 (a_m^2 - \omega_0^2)}{(a_m^2 + \omega_0^2)^2} \\ \dfrac{k_m^2}{(a_m^2 + \omega_0^2)} & \dfrac{a_m k_m^3}{(a_m^2 + \omega_0^2)^2} \end{pmatrix} \phi_{av} \tag{4.5.20}$$

With $a_m = 3$, $k_m = 3$, $a_p = 1$, $k_p = 2$, $r_0 = 1$, $\omega_0 = 1$, $\epsilon = 1$, the two eigenvalues of the averaged system are computed to be $-0.0163\,\epsilon$ and $-0.5537\,\epsilon$, and are both real negative. The nominal parameter $\theta^{*T} = (k_m / k_p, (a_p - a_m)/ k_p)$. We let $\theta(0) = 0$, so that $\phi^T(0) = (-1.5, 1)$.

Figures 4.8, 4.9 and 4.10 show the plots of the parameter errors ϕ_y (ϕ_r) for the original and averaged system, with three different frequencies ($\omega_0 = 1, 3, 5$). Figure 4.10 corresponds to a frequency of the input signal $\omega_0 = 5$, such that the eigenvalues of the matrix $R_{w_m w_{mf}}(0)$ are complex: $(-0.0553 \pm j\, 0.05076)\epsilon$. This explains the oscillatory behavior of the original and averaged systems observed in the figure, which did not exist in the previous examples of section 4.3.

4.5.2 Nonlinear Equations

We now return to the complete, nonlinear differential equations

$$\dot{e}(t) = A_m\, e(t) + b_m\, \phi^T(t)\, w_m(t) + b_m\, \phi^T(t)\, Q\, e(t)$$

$$\dot{\phi}(t) = -\epsilon\, w_m(t)\, c_m^T\, e(t) - \epsilon\, Q\, e(t)\, c_m^T\, e(t) \tag{4.5.21}$$

From (4.4.45)

$$v(t,\phi) = \int_0^t e^{(A_m + b_m \phi^T Q)(t-\tau)}\, b_m\, \phi^T\, w_m(\tau)\, d\tau \tag{4.5.22}$$

so that the averaged system is

$$\dot{\phi}_{av} = \epsilon\, f_{av}(\phi_{av}) \qquad \phi_{av}(0) = \phi(0) \tag{4.5.23}$$

where f_{av} is defined by the limit

$$f_{av}(\phi) = -\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(w_m(t)\, c_m^T\, v(t,\phi) + Q\, v(t,\phi)\, c_m^T\, v(t,\phi) \right) dt \tag{4.5.24}$$

The assumptions of the theorems will be satisfied if the limit in (4.5.24) is uniform in the sense of (B3), and provided that the matrix $A_m + b_m\, \phi^T\, Q$ is uniformly exponentially stable for $\phi \in B_h$. This means that if the controller parameters are frozen at any point of the trajectory the resulting time invariant system must be closed-loop stable. Naturally, this precludes consideration of adaptation from initial parameter values which define an unstable closed-loop system.

Frequency Domain Analysis

The expression of f_{av} in (4.5.24) can be translated into the frequency domain, noting that w_m is related to r through the vector transfer function $\hat{H}_{rw_m}$

$$f_{av}(\phi) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\hat{H}_{rw_m}(j\omega) + Q\, (j\omega I - A_m - b_m\, \phi^T\, Q)^{-1}\, b_m\, \phi^T\, \hat{H}_{rw_m}(j\omega) \right]$$

$$\cdot \left[c_m^T\, (-j\omega I - A_m - b_m\, \phi^T\, Q)^{-1}\, b_m\, \phi^T\, \hat{H}_{rw_m}(-j\omega) \right] s_r(d\omega) \tag{4.5.25}$$

where $s_r(d\omega)$ is the spectral measure of r. Note that f_{av} can be factored as

$$f_{av}(\phi) = -A_{av}(\phi) \cdot \phi \tag{4.5.26}$$

where $A_{av} : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n \times 2n}$ is similar to $R_{w_m w_{mf}}(0)$ in section 4.5.1, but now depends nonlinearily on ϕ. The expression in (4.5.25) is more complex than in the linear case, but some manipulations will allow us to find a more interesting result.

Recall that (4.5.21) was obtained from the differential equation

$$\dot{e}(t) = A_m e(t) + b_m \phi^T(t) w(t)$$
$$\dot{\phi}(t) = -\epsilon w(t) c_m^T e(t) \qquad (4.5.27)$$

by noting that $w(t) = w_m(t) + Q\, e(t)$. In general, (4.5.27) is of limited use, precisely because w depends on e. The signal w is not an external signal, but depends on internal variables. On the other hand, w_m is an exogeneous signal, related to r through a stable transfer function.

In the context of averaging, the differential equation describing the fast variable (i.e. e) is averaged, assuming that the slow variable (i.e. ϕ) is constant. However, when ϕ is constant, w *is* related to r through a linear time invariant system, with a transfer function depending on ϕ. If $\det(sI - A_m - b_m \phi^T Q)$ is Hurwitz (as we assume to apply averaging), this transfer function is stable. Therefore, *assuming that ϕ is fixed*, we can write

$$\hat{w} = \hat{H}_{rw}(j\omega, \phi) . \hat{r} \qquad (4.5.28)$$

so that using (4.5.27), (4.5.25) can be replaced by (4.5.26), with an expression similar to the expression of $R_{w_m w_{mf}}(0)$ in (4.5.9), i.e.

$$A_{av}(\phi) = \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \hat{H}_{rw}(j\omega, \phi) \hat{H}_{rw}^{*T}(j\omega, \phi) \hat{M}(j\omega)\, s_r(d\omega) \qquad (4.5.29)$$

Explicit expression of $\hat{H}_{rw}(j\omega, \phi)$

Recall that $\overline{w}_m$ is related to r through the transfer function $\hat{H}_{r\overline{w}_m}$, whose poles are the zeros of $\det(sI - A_m)$. Let

$$\hat{\chi}_m(s) = \det(sI - A_m) \qquad (4.5.30)$$

and write the transfer function $\hat{H}_{r\overline{w}_m}$ as the ratio of a vector polynomial $\hat{n}(s)$, and a characteristic polynomial $\hat{\chi}_m(s)$, i.e.

$$\hat{H}_{r\bar{w}_m}(s) = \frac{\hat{n}(s)}{\hat{\chi}_m(s)} \tag{4.5.31}$$

We found in section 3.5 (cf. (3.5.8), (3.5.11)) that

$$\bar{w} = \frac{\hat{n}(s)}{\hat{\chi}_m(s)}\,(r + \frac{1}{c_0^*}\,\phi^T w) \tag{4.5.32}$$

Denote $\phi_{c_0} = c_0 - c_0^*$, so that $\phi^T w = \phi_{c_0} r + \bar{\phi}^T \bar{w}$. Assuming that ϕ is constant, (4.5.32) becomes

$$\bar{w} = (\hat{\chi}_m(s).\mathrm{I} - \frac{1}{c_0^*}\,\hat{n}(s)\bar{\phi}^T\,)^{-1}\,\hat{n}(s)\,((\,1 + \frac{\phi_{c_0}}{c_0^*})\,r\,)$$

$$= \frac{\hat{n}(s)}{\hat{\chi}_m(s) - \frac{1}{c_0^*}\,\bar{\phi}^T\hat{n}(s)}\,((\,1 + \frac{\phi_{c_0}}{c_0^*})\,r\,) \tag{4.5.33}$$

Denote

$$\hat{\chi}_\phi(s) := \hat{\chi}_m(s) - \frac{1}{c_0^*}\,\bar{\phi}^T\hat{n}(s) \tag{4.5.34}$$

$\hat{\chi}_\phi(s)$ is closed-loop characteristic polynomial, giving the poles of the adaptive system with feedback θ, i.e. the poles of the model transfer function with feedback ϕ. Therefore, $\hat{\chi}_\phi(s)$ is also given by

$$\hat{\chi}_\phi(s) = \det(sI - A_m - b_m\phi^T Q) \tag{4.5.35}$$

With this notation, (4.5.33) can be written

$$\bar{w} = \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\,.\hat{H}_{r\bar{w}_m}(r + \frac{\phi_{c_0}}{c_0^*}r)$$

$$= \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\,(\bar{w}_m) + \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\,(\frac{\phi_{c_0}}{c_0^*}\,\bar{w}_m\,) \tag{4.5.36}$$

On the other hand

$$r = \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\,(\,1 - \frac{\bar{\phi}^T\hat{n}}{c_0^*\hat{\chi}_m}\,).(r)$$

$$= \frac{\hat{\chi}_m}{\hat{\chi}_\phi}(r) - \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\,(\frac{\bar{\phi}^T}{c_0^*}\,.\,\bar{w}_m\,) \tag{4.5.37}$$

Define

$$B(\phi) := \begin{pmatrix} 0 & -\frac{1}{c_0^*}\,\bar{\phi}^T \\ 0 & \frac{\phi_{c_0}}{c_0^*}\,.\,I \end{pmatrix} \in \begin{pmatrix} \mathbf{R}^{1\times 1} & \mathbf{R}^{1\times 2n-1} \\ \mathbf{R}^{2n-1\times 1} & \mathbf{R}^{2n-1\times 2n-1} \end{pmatrix} = \mathbf{R}^{2n\times 2n} \qquad (4.5.38)$$

so that (4.5.36)-(4.5.37) can be written

$$w = \begin{pmatrix} r \\ \bar{w} \end{pmatrix} = \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\begin{pmatrix} r \\ \bar{w}_m \end{pmatrix} + \frac{\hat{\chi}_m}{\hat{\chi}_\phi}\,B(\phi)\,.\begin{pmatrix} r \\ \bar{w}_m \end{pmatrix} \qquad (4.5.39)$$

The vector transfer function $\hat{H}_{rw}$ can therefore be expressed in terms of the vector transfer function $\hat{H}_{rw_m}$ by

$$\hat{H}_{rw}(s,\phi) = \frac{\hat{\chi}_m(s)}{\hat{\chi}_\phi(s)}(I + B(\phi))\hat{H}_{rw_m}(s) \qquad (4.5.40)$$

and, as expected

$$\hat{H}_{rw}(s,0) = \hat{H}_{rw_m}(s) \qquad (4.5.41)$$

Convergence Analysis

With (4.5.40), A_{av} can be written

$$A_{av}(\phi) = \frac{1}{2\pi c_0^*}\int_{-\infty}^{\infty}\left|\frac{\hat{\chi}_m(j\omega)}{\hat{\chi}_\phi(j\omega)}\right|^2(I + B(\phi))\hat{H}_{rw_m}(j\omega)\cdot$$

$$\hat{H}^{*T}_{rw_m}(j\omega)(I + B^T(\phi))\hat{M}(j\omega)\ s_r(d\omega) \qquad (4.5.42)$$

Consider now the trajectories of the averaged system, and let $v(\phi_{av}) = |\phi_{av}|^2 = \phi_{av}^T\phi_{av}$. Note that by choice of $B(\phi)$, it follows that

$$\phi^T\,.\,B(\phi) = 0 \qquad \text{for all } \phi \qquad (4.5.43)$$

Denote

$$R(\phi_{av}) := \frac{1}{2\pi c_0^*}\int_{-\infty}^{\infty}\left|\frac{\hat{\chi}_m(j\omega)}{\hat{\chi}_{\phi_{av}}(j\omega)}\right|^2\hat{H}_{rw_m}(j\omega)\hat{H}^{*T}_{rw_m}(j\omega)\hat{M}(j\omega)\,s_r(d\omega) \qquad (4.5.44)$$

It follows that the derivative of v is given by

$$-\dot{v}(\phi_{av}) = \epsilon\, \phi_{av}^T (R(\phi_{av}) + R^T(\phi_{av}))\, \phi_{av} \tag{4.5.45}$$

which is identical to the expression for the linear case (4.5.12), provided that $R(\phi_{av})$ given in (4.5.44) replaces $R_{w_m w_{mf}}(0)$ given in (4.5.9). It is remarkable that this result differs from the expression obtained by linearization followed by averaging in section 4.5.1 only by the *scalar* weighting factor $|\hat{\chi}_m \,/\, \hat{\chi}_\phi|^2$. This term is strictly positive, given any ϕ bounded, and it approaches unity continuously as ϕ approaches zero.

Since $\hat{M}(s)$ is strictly positive real, $R(\phi_{av})$ is at least positive semidefinite. As in the linearized case, it is positive definite if w_m is persistently exciting. Using the Lyapunov function $v(\phi_{av})$, this argument itself constitutes a proof of exponential stability of the averaged system, using (4.5.45). By theorem 4.4.3, the exponential stability of the original system is also guaranteed for ϵ sufficiently small.

Rates of convergence can also be determined, using the Lyapunov function $v(\phi_{av})$, so that

$$\begin{aligned} -\dot{v} &= \epsilon\, \phi_{av}^T\, (\, R(\phi_{av}) + R^T(\phi_{av})\,)\,)\, \phi_{av} \\ &\geqslant \epsilon \inf_{\phi_{av} \in B_h} (\lambda_{\min}(\, R(\phi_{av}) + R^T(\phi_{av})\,)\,)\, v := 2\,\epsilon\,\alpha\, v \end{aligned} \tag{4.5.46}$$

and the guaranteed rate of parameter convergence of the averaged adaptive system is $\epsilon\,\alpha$. The rate of convergence of the original system can be estimated by the same value, for ϵ sufficiently small.

It is interesting to note that, as $|\phi_{av}|$ increases, $\lambda_{\min}(\, R(\phi_{av}) + R^T(\phi_{av})\,)\,)$ tends to zero in some directions. This indicates that the adaptive control system is *not* globally exponentially stable.

Example

We consider the previous two parameter example. The adaptive system is described by

$$\begin{aligned} \dot{e}(t) &= -a_m\, e(t) + k_p\, (\phi_r(t)\, r(t) + \phi_y(t)\, e(t) + \phi_y(t)\, y_m(t)\,) \\ \dot{\phi}_r(t) &= -\epsilon\, e(t)\, r(t) \\ \dot{\phi}_y(t) &= -\epsilon\, e^2(t) - \epsilon\, e(t)\, y_m(t) \end{aligned} \tag{4.5.47}$$

Consider the case when $r = r_0 \sin(\omega_0 t)$. The averaged system can be computed using (4.5.42). We can also verify the expression using (4.5.47), and the definition of the averaged system (4.5.22). After some manipulations, we obtain, for the averaged system (dropping the "av" subscripts for simplicity)

$$\dot{\phi}_r = -\epsilon\, k_p \frac{r_0^2}{2} \frac{1}{\omega_0^2 + (a_m - k_p \phi_y)^2} \Bigg[(a_m - k_p \phi_y)\phi_r + \Big(\frac{a_m^2 - \omega_0^2}{\omega_0^2 + a_m^2} k_m \Big) \phi_y - \frac{k_p a_m k_m}{\omega_0^2 + a_m^2} \phi_y^2 \Bigg] \tag{4.5.48}$$

$$\dot{\phi}_y = -\epsilon\, k_p \frac{r_0^2}{2} \frac{1}{\omega_0^2 + (a_m - k_p \phi_y)^2} \Bigg[k_m \phi_r + \frac{a_m k_m^2}{\omega_0^2 + a_m^2} \phi_y + k_p \phi_r^2 + \frac{k_p a_m k_m}{\omega_0^2 + a_m^2} \phi_r \phi_y \Bigg] \tag{4.5.49}$$

Using this result, or using (4.5.42)-(4.5.43), we find that for $v = \phi^T \phi$

$$-\dot{v} = 2\epsilon \left[\frac{\omega_o^2 + a_m^2}{\omega_0^2 + (a_m - k_p \phi_y)^2} \right] \frac{r_0^2}{2} \frac{k_p}{k_m} \phi^T \begin{bmatrix} \dfrac{a_m k_m}{\omega_0^2 + a_m^2} & \dfrac{k_m^2 (a_m^2 - \omega_0^2)}{(\omega_0^2 + a_m^2)^2} \\ \dfrac{k_m^2}{\omega_0^2 + a_m^2} & \dfrac{a_m k_m^3}{(\omega_0^2 + a_m^2)^2} \end{bmatrix} \phi \tag{4.5.50}$$

It can easily be checked that when the first term in brackets is equal to 1 (i.e. with ϕ_y replaced by zero), the result is the same as the result obtained by first linearizing the system, then averaging it (cf. (4.5.20)). In fact, it can be seen, from the expressions of the averaged systems ((4.5.10) with (4.5.9), and (4.5.23) with (4.5.26), (4.5.38), and (4.5.42)) that the system obtained by linearization followed by averaging is *identical* to the system obtained by averaging followed by linearization. Also, given any prescribed B_h (but such that $\det(sI - A_m - b_m \phi^T Q)$ is Hurwitz), (4.5.50) can be used to obtain estimates of the rates of convergence of the *nonlinear* system.

We reproduce here simulations for the following values of the parameters: $a_m = 3$, $k_m = 3$, $a_p = 1$, $k_p = 2$, $r_0 = 1$, $\omega_0 = 1$, $\epsilon = 1$. The first set of figures is a simulation for initial conditions $\phi_r(0) = -0.5$, and $\phi_y(0) = 0.5$. Figure 4.11 represents the time variation of the function $\ln(v = \phi^T \phi)$ for the original, averaged, and linearized-averaged systems (the minimum slope of the curve gives the rate of convergence). It shows the close approximation of the original system by the averaged system. The slope for the linearized-averaged system is asymptotically identical to that of the averaged system, since parameters eventually get arbitrarily close to their nominal values. Figures 4.12 and 4.13 show the approximation of the trajectories of ϕ_r, and ϕ_y.

Figure 4.14 represents the logarithm of the Lyapunov function for a simulation with identical parameters, but initial conditions $\phi_r(0) = 0.5$, $\phi_y(0) = -0.5$. Due to the change of sign in $\phi_y(0)$, the rate of convergence of the nonlinear system is less now than the rate of the linearized system, while it was larger in the previous case. These simulations demonstrate the close approximation by the averaged system, and it should be noted that this is achieved despite an adaptation gain ϵ equal to 1. This shows that the averaging method is useful for values of ϵ which are not necessarily infinitesimal (i.e. not necessarily for very slow adaptation), but for values which are often practical ones.

Figure 4.15 shows the state-space trajectory $\phi_y(\phi_r)$, corresponding to figure 4.10, that is with initial conditions $\phi_r(0) = -1.5$, $\phi_y(0) = 1$, and parameters as above except $\omega_0 = 5$. Figure 4.15 shows the distortion of the trajectories in the state-space, due to the nonlinearity of the differential system.

4.6 Conclusions

Averaging is a powerful tool to approximate nonautonomous differential equations by autonomous differential equations. In this chapter, we introduced averaging as a method of analysis of adaptive systems. Although averaging was studied previously as a method of analysis of differential equations, we have established here results that are better suited to our purposes.

The approximation of parameter convergence rates using averaging was justified by general results concerning a class of systems including the adaptive systems described in chapter 2 and chapter 3. The analysis had the interesting feature of considering nonlinear differential equations, as well as linear ones. Therefore, the application was not restricted to linear or linearized systems, but extended to all adaptive systems considered in this work, including adaptive control systems. The results were also interesting in that they did not require the traditional almost periodicity condition, but instead a stationarity condition.

The application to adaptive systems included useful parameter convergence rates estimates for identification and adaptive control systems. The rates depended strongly on the reference input, and a frequency domain analysis related the frequency content of the reference input to the convergence rates, even in the nonlinear adaptive control case. These results are useful for the optimum design of reference input. They have the limitation of depending on unknown plant parameters, but an approximation of the complete parameter trajectory is obtained, and the understanding of the dynamical behavior of the parameter error is much increased using averaging. For example, it was found that the trajectory of the parameter error corresponding to the linear error equation could be approximated by an LTI system with real negative eigenvalues, while for the SPR error equation it had possibly complex eigenvalues.

Besides requiring stationarity of input signals, averaging also required slow parameter adaptation. We showed however, through simulations, that the approximation by the averaged system was good for values of the adaptation gain that were close to 1 (that is, not necessarily infinitesimal), and for acceptable time constants in the parameter variations. In fact, it appeared that a basic condition is simply that parameters vary slower than other states and signals of the adaptive system.

Chapter 5 Robustness

5.1 The Rohrs Examples

Despite the existence of stability proofs for adaptive control systems (cf. chapter 3), Rohrs, et al (1982), (1985) showed that several algorithms can become unstable when some of the assumptions required by the stability proofs are not satisfied. Especially concerned are the assumptions of the knowledge of

- the order of the plant
- the relative degree of the plant

In practice, plants cannot be modeled exactly with finite dimensional models, and the robustness problem is to guarantee that the adaptive system remains stable despite the presence of high frequency dynamics, and measurement noise.

While Rohrs, et al considered several continuous and discrete time algorithms, the results are qualitatively similar for the various schemes. We consider one of these schemes here, which is the output error direct adaptive control scheme of section 3.3.2, assuming that the degree and the relative degree of the plant are 1.

Rohrs Examples

The adaptive control scheme of Rohrs examples is designed assuming a first order plant with transfer function

$$\hat{P}(s) = \frac{k_p}{s + a_p} \tag{5.1.1}$$

and the SPR reference model

$$\hat{M}(s) = \frac{k_m}{s + a_m} = \frac{3}{s + 3} \tag{5.1.2}$$

The output error adaptive control scheme (cf. section 3.3.2) is described by

$$u = c_0 r + d_0 y_p \tag{5.1.3}$$

$$e_0 = y_p - y_m \tag{5.1.4}$$

$$\dot{c}_0 = -g\, r\, e_0 \tag{5.1.5}$$

$$\dot{d}_0 = -g\, y_p\, e_0 \tag{5.1.6}$$

In a first step, we assume that the plant transfer function is given by (5.1.1), with $k_p = 2, a_p = 1$. The nominal values of the controller parameters are then

$$c_0^* = \frac{k_m}{k_p} = 1.5 \tag{5.1.7}$$

$$d_0^* = \frac{a_p - a_m}{k_p} = -1 \tag{5.1.8}$$

The behavior of the adaptive system is then studied, assuming that the *actual* plant does not satisfy exactly the assumptions on which the adaptive control system is based. The actual plant is only *approximately* a first order plant, and has the third order transfer function

$$\frac{2}{s+1} \cdot \frac{229}{s^2 + 30s + 229} \tag{5.1.9}$$

In analogy with nonadaptive control terminology, the second term is called the *unmodeled dynamics*. The poles of the unmodeled dynamics are located at $-15 \pm j2$, and, at low frequencies, this term is approximately equal to 1.

In Rohrs examples, the measured output $y_p(t)$ is also affected by a measurement noise $n(t)$. The actual plant with the reference model and the controller are shown in figure 5.1.

An important aspect of Rohrs examples is that the modes of the actual plant and those of the model are well within the stability region. Moreover, the unmodeled dynamics are well-damped, stable modes. From a traditional control design standpoint, they would be considered rather innocuous.

At the outset, Rohrs, et al (1982) showed through simulations that, without measurement noise or unmodeled dynamics, the adaptive scheme is stable, and the output error converges to zero, as predicted by the stability analysis.

However, *with unmodeled dynamics*, three different mechanisms of instability appear.

(R1) With a *large, constant* reference input, and no measurement noise, the output error initially converges to zero, but eventually diverges to infinity, along with the controller parameters c_0 and d_0.

Figures 5.2 and 5.3 show a simulation with $r(t) = 4.3$, $n(t) = 0$, that illustrates this behavior ($c_0(0) = 1.14$, $d_0(0) = -0.65$, and other initial conditions are zero).

(R2) With a reference input having a *small constant* component, and a *large high frequency* component, the output error diverges at first slowly, and then more rapidly to infinity, along with the controller parameters c_0 and d_0.

Figures 5.4 and 5.5 show a simulation with $r(t) = 0.3 + 1.85 \sin 16.1t$, $n(t) = 0$ ($c_0(0) = 1.14$, $d_0(0) = -0.65$, and other initial conditions are zero).

(R3) With a moderate *constant input* and a small *output disturbance*, the output error initially converges to zero. After staying in the neighborhood of zero for an extended period of time, it diverges to infinity. On the other hand, the controller parameters c_0 and d_0 drift apparently at a constant rate, until they suddenly diverge to infinity.

Figures 5.6 and 5.7 show a simulation with $r(t) = 2$, $n(t) = 0.5 \sin 16.1t$ ($c_0(0) = 1.14$, $d_0(0) = -0.65$, and other initial conditions are zero).

Although this simulation corresponds to a comparatively high value of $n(t)$, simulations show that when smaller values of the output disturbance $n(t)$ are present, instability still appears, but after a longer period of time. The controller parameters simply drift at a slower rate. Instability is also observed with other frequencies of the disturbance, including a constant $n(t)$.

Rohrs examples stimulated much research about the robustness of adaptive systems. Examination of the mechanisms of instability in Rohrs examples show that the instabilities are related to the identifier. In identification, such instabilities involve computed signals, while in adaptive control, variables associated with the plant are also involved. This justifies a more careful consideration of robustness issues in the context of adaptive control.

5.2 Robustness of Adaptive Algorithms with Persistency of Excitation

Rohrs examples show that the BIBS stability property obtained in chapter 3 is not robust to uncertainties. In some cases, an arbitrary small disturbance can destabilize an adaptive system, which is otherwise proved to be BIBS stable. In this section, we will show that the property of *exponential stability* is robust, in the sense that exponentially stable systems can tolerate a certain amount of disturbances. Thus, provided that the nominal adaptive system is exponentially stable (guaranteed by a PE condition), we will obtain robustness margins, i.e. bounds on disturbances and unmodeled dynamics that do not destroy the stability of the adaptive system. Of course, the practical notion of robustness is that stability should be preserved in the presence of actual disturbances present in the system. Robustness margins must include actual disturbances for the adaptive system to be robust in that sense.

The main difference from classical LTI control system robustness margins is that robustness does not depend only on the plant and control system, but also on the *reference input*, which must guarantee persistent excitation of the nominal adaptive system (i.e. without disturbances or unmodeled dynamics).

5.2.1 Exponential Convergence and Robustness

In this section, we consider properties of a so-called *perturbed* system

$$\dot{x} = f(t, x, u) \qquad x(0) = x_0 \tag{5.2.1}$$

and relate its properties to those of the *unperturbed* system

$$\dot{x} = f(t, x, 0) \qquad x(0) = x_0 \tag{5.2.2}$$

where $t \geqslant 0$, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$. Depending on the interpretation, the signal u will be considered either a disturbance, or an input.

We restrict our attention to solutions x and inputs u belonging to some arbitrary balls $B_h \in \mathbf{R}^n$ and $B_c \in \mathbf{R}^m$.

Theorem 5.2.1 Small Signal I/O Stability

Consider the perturbed system (5.2.1) and the unperturbed system (5.2.2). Let $x = 0$ be an equilibrium point of (5.2.2), i.e. $f(t, 0, 0) = 0$, for all $t \geq 0$. Let f be piecewise continuous in t, and have continuous and bounded first partial derivatives in x, for all $t \geq 0$, $x \in B_h$, $u \in B_c$. Let f be Lipschitz in u, with Lipschitz constant l_u, for all $t \geq 0$, $x \in B_h$, $u \in B_c$. Let $u \in L_\infty$.

If $x = 0$ is an exponentially stable equilibrium point of the unperturbed system

Then (a)

the perturbed system is *small-signal* L_∞*-stable*, i.e. there exist γ_∞, $c_\infty > 0$, such that $\| u \|_\infty < c_\infty$ implies that

$$\| x \|_\infty \leq \gamma_\infty \| u \|_\infty \tag{5.2.3}$$

where x is the solution of (5.2.1) starting at $x_0 = 0$;

(b)

there exists $m \geq 1$ such that, for all $|x_0| < h / m$, $\| u \|_\infty < c_\infty$ implies that $x(t)$ converges to a B_δ ball of radius $\delta = \gamma_\infty \| u \|_\infty < h$, that is: for all $\epsilon > 0$, there exists $T \geq 0$ such that

$$| x(t) | \leq (1 + \epsilon) \delta \tag{5.2.4}$$

for all $t \geq T$, along the solutions of (5.2.1) starting at x_0.

Also, for all $t \geq 0$, $x(t) \in B_h$.

Comments

Part (a) of theorem 5.2.1 is a direct extension of theorem 1 of Vidyasagar & Vannelli (1982) (see also Hill & Moylan (1980)) to the non autonomous case. Part (b) further extends it to non zero initial conditions.

Theorem 5.2.1 relates *internal* exponential stability to *external* input/output stability (the output is here identified with the state). In contrast with the definition of BIBS stability of section 3.4, we require a linear relationship between the norms in (5.2.3) for L_∞ stability.

Although lack of exponential stability does not imply input/output instability, it is known that simple stability, and even (non uniform) asymptotic stability are *not* sufficient conditions to guarantee I/O stability (see e.g. Kalman & Bertram (1960) Ex. 5 p. 379).

Proof of Theorem 5.2.1

The differential equation (5.2.2) satisfies the conditions of theorem 1.4.3, so that there exists a Lyapunov function $v(t,x)$ satisfying the following inequalities

$$\alpha_1 |x|^2 \leqslant v(t,x) \leqslant \alpha_2 |x|^2 \tag{5.2.5}$$

$$\left.\frac{dv(t,x)}{dt}\right|_{(5.2.2)} \leqslant -\alpha_3 |x|^2 \tag{5.2.6}$$

$$\left|\frac{\partial v(t,x)}{\partial x}\right| \leqslant \alpha_4 |x| \tag{5.2.7}$$

for some strictly positive constants $\alpha_1 \ldots \alpha_4$, and for all $t \geqslant 0$, $x \in B_h$.

If we consider the same function to study the perturbed differential equation (5.2.1), inequalities (5.2.5) and (5.2.7) still hold, while (5.2.6) is modified, since the derivative is now to be taken along the trajectories of (5.2.1), instead of (5.2.2). The two derivatives are related through

$$\left.\frac{dv(t,x)}{dt}\right|_{(5.2.1)} = \frac{\partial v(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial v(t,x)}{\partial x_i} f_i(t,x,u)$$

$$= \left.\frac{dv(t,x)}{dt}\right|_{(5.2.2)} + \sum_{i=1}^{n} \frac{\partial v(t,x)}{\partial x_i} \Big(f_i(t,x,u) - f_i(t,x,0) \Big) \tag{5.2.8}$$

Using (5.2.5)-(5.2.7), and the Lipschitz condition on f

$$\left.\frac{dv(t,x)}{dt}\right|_{(5.2.1)} \leqslant -\alpha_3 |x|^2 + \alpha_4 |x| \, l_u \, \|u\|_\infty \tag{5.2.9}$$

Define

$$\gamma_\infty := \frac{\alpha_4}{\alpha_3} \, l_u \left(\frac{\alpha_2}{\alpha_1} \right)^{1/2} \tag{5.2.10}$$

$$\delta := \gamma_\infty \|u\|_\infty \tag{5.2.11}$$

$$m := \left(\frac{\alpha_2}{\alpha_1} \right)^{1/2} \geqslant 1 \tag{5.2.12}$$

Inequality (5.2.9) can now be written

$$\left.\frac{dv(t,x)}{dt}\right|_{(5.2.1)} \leqslant -\alpha_3 |x| \left(|x| - \frac{\delta}{m} \right) \tag{5.2.13}$$

This inequality is the basis of the proof.

Part (a) Consider the situation when $|x_0| \leqslant \delta/m$ (this is true in particular if $x_0=0$). We show that this implies that $x(t) \in B_\delta$ for all $t \geqslant 0$ (note that $\delta/m \leqslant \delta$, since $m \geqslant 1$).

Suppose, for the sake of contradiction, that it was not true. Then, by continuity of the solutions, there would exist $T_0, T_1 (T_1 > T_0 \geqslant 0)$, such that $|x(T_0)| = \delta/m$, $|x(T_1)| > \delta$, and for all $t \in [T_0, T_1] : |x(t)| \geqslant \delta/m$. Consequently, inequality (5.2.13) shows that, in $[T_0, T_1]$, $dv/dt \leqslant 0$. However, this contradicts the fact that $v(T_0, x(T_0)) \leqslant \alpha_2 (\delta/m)^2 = \alpha_1 \delta^2$, and $v(T_1, x(T_1)) > \alpha_1 \delta^2$.

Part (b) Assume now that $|x_0| > \delta/m$. We show the result in two steps.

(b1) for all $\epsilon > 0$, there exists $T \geqslant 0$ such that $|x(T)| = (\delta/m)(1+\epsilon)$.

Suppose it was not true. Then, for some $\epsilon > 0$, and for all $t \geqslant 0$, $|x(t)| > (\delta/m)(1+\epsilon)$ and, from (5.2.13), $dv/dt < -\alpha_3 (\delta/m)^2 (1+\epsilon)\epsilon$, which is a strictly negative constant. However, this contradicts the fact $v(0, x_0) \leqslant \alpha_2 |x_0|^2$, and $v(t, x(t)) > \alpha_1 (\delta/m)^2 (1+\epsilon)^2$ for all $t \geqslant 0$.

(b2) for all $t \geqslant T$, $|x(t)| \leqslant \delta(1+\epsilon)$. This follows directly from (b1), using an argument identical to the one used to prove (a).

Finally, recall that the assumptions require that $x(t) \in B_h$, $u(t) \in B_c$, for all $t \geqslant 0$. This is also guaranteed, using an argument similar to (a), provided that $|x_0| < h/m$ and $\|u\|_\infty < c_\infty$, where m is defined in (5.2.12), and

$$c_\infty := \min(c, h/\gamma_\infty) \tag{5.2.14}$$

(5.2.14) implies that $\delta < h$, and $|x_0| < h/m \leqslant h$ implies that $|x(t)| \leqslant m|x_0| < h$ for all $t \geqslant 0$.

Note that although part (a) of the proof is, in itself, a result for non zero initial conditions, the size of the ball $B_{\delta/m}$ involved decreases when the amplitude of the input decreases, while the size of $B_{h/m}$ is independent of it. □

Additional Comments

a) The proof of the theorem gives an interesting interpretation of the interaction between the exponential convergence of the original system, and the effect of the disturbances on the perturbed system. To see this, consider (5.2.9): the term $-\alpha_3 |x|^2$ acts like a restoring force bringing the state vector back to the origin. This term originates from the exponential stability of the unperturbed system. The term $\alpha_4 |x| l_u \|u\|_\infty$ acts like a disturbing force, pulling the state *away* from the origin. This term is caused by the input u (i.e. by the disturbance acting on the system). While the first term is proportional to the norm squared, the second is only proportional to the norm, so that when $|x|$ is sufficiently large, the restoring force equilibrates the disturbing force. In the form (5.2.13), we see that this happens when $|x| = \delta / m = \gamma_\infty / m \|u\|_\infty$.

b) If the assumptions are valid *globally*, then the results are valid globally too. The system remains stable, and has finite I/O gain, independent of the size of the input. In the example of section 5.2.2, and for a wide category of nonlinear systems (bilinear systems for example), the Lipschitz condition is not verified globally. Yet, given *any* balls B_h, B_c, the system satisfies a Lipschitz condition with constant l_u depending on the size of the balls (actually increasing with it). The balls B_h, B_c are consequently arbitrary in that case, but the values of γ_∞ (the L_∞ gain) and c_∞ (the stability margin) will vary with them. In general, it can be expected that c_∞ will remain bounded despite the freedom left in the choice of h and c, so that the I/O stability will only be local.

c) *Explicit* values of γ_∞ and c_∞ can be obtained from parameters of the differential equation, using equations (5.2.10) and (5.2.14). Note that if we used the Lyapunov function satisfying (5.2.5)-(5.2.7) to obtain a convergence rate for the unperturbed system, this rate would be $\alpha_3 / 2\alpha_1$. Therefore, it can be verified that, with other parameters remaining identical, the L_∞ gain is decreased, and the stability margin c_∞ is increased, *when the rate of exponential convergence is increased*.

5.2.2 Robustness of an Adaptive Control Scheme

For the purpose of illustration, we consider the output error direct adaptive control algorithm of section 3.3.2, when the relative degree of the plant is 1. This example contains the specific cases of the Rohrs examples.

In section 3.5, we showed that the overall output error adaptive scheme for the relative degree 1 case is described by (cf. (3.5.28))

$$\dot{e}(t) = A_m e(t) + b_m \phi^T(t) w_m(t) + b_m \phi^T(t) Q\, e(t)$$

$$\dot{\phi}(t) = -g\, c_m^T e(t) w_m(t) - g\, c_m^T e(t) Q\, e(t) \tag{5.2.15}$$

where $e(t) \in \mathbf{R}^{3n-2}$, and $\phi(t) \in \mathbf{R}^{2n}$. A_m is a stable matrix, and $w_m(t) \in \mathbf{R}^{2n}$ is bounded for all $t \geqslant 0$. (5.2.15) is a nonlinear ordinary differential equation (actually it is bilinear) of the form

$$\dot{x} = f(t, x) \qquad x(0) = x_0 \tag{5.2.16}$$

which is of the form (5.2.2), where

$$x := \begin{bmatrix} e \\ \phi \end{bmatrix} \in R^{5n-2} \tag{5.2.17}$$

Recall that we also found, in section 3.8, that (5.2.15) (i.e. (5.2.16)) is exponentially stable in any closed ball, provided that w_m is PE.

Robustness to Output Disturbances

Consider the case when the measured output is affected by a measurement moise $n(t)$, as in figure 5.1. Denote by y_p^* the output of the plant $\hat{P}(s)$ (i.e. the output without measurement noise), and by $y_p(t)$, the measured output, affected by noise, so that

$$y_p(t) = y_p^*(t) + n(t) = \hat{P}(u) + n(t) \tag{5.2.18}$$

To find a description of the adaptive system in the presence of the measurement noise $n(t)$, we return to the derivation of (5.2.15) (that is (3.5.28)) in section 3.5. The plant $\hat{P}$ has a minimal state-space representation $[A_p, b_p, c_p^T]$ such that

$$\dot{x}_p = A_p x_p + b_p u$$

$$y_p^* = c_p^T x_p \tag{5.2.19}$$

The observers are described by

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p = \Lambda w^{(2)} + b_\lambda c_p^T x_p + b_\lambda n \tag{5.2.20}$$

and the control input is given by $u = \theta^T w = \phi^T w + \theta^{*T} w$.

As previously, we let $x_{pw}^T = (x_p^T, w^{(1)^T}, w^{(2)^T})$. Using the definition of A_m, b_m, and c_m in (3.5.18)-(3.5.19), the description of the plant with controller is now

$$\begin{aligned} \dot{x}_{pw} &= A_m\, x_{pw} + b_m\, \phi^T w + b_m c_0^* \, r + b_n\, n \\ y_p^* &= c_m^T\, x_{pw} \end{aligned} \tag{5.2.21}$$

where we defined $b_n^T = (0, 0, b_\lambda^T) \in (\mathbf{R}^n, \mathbf{R}^{n-1}, \mathbf{R}^{n-1}) = \mathbf{R}^{3n-2}$.

As previously, we represent the model and its output by

$$\begin{aligned} \dot{x}_m &= A_m\, x_m + b_m c_0^* \, r \\ y_m &= c_m^T\, x_m \end{aligned} \tag{5.2.22}$$

and we let $e = x_{pw} - x_m$.

The update law is given by

$$\begin{aligned} \dot{\phi} &= -g\, (y_p - y_m)\, w \\ &= -g\, c_m^T\, e\, w - g\, n\, w \end{aligned} \tag{5.2.23}$$

and the regressor is now related to the state e by

$$w = \begin{pmatrix} r \\ w^{(1)} \\ y_p \\ w^{(2)} \end{pmatrix} = w_m + \begin{pmatrix} 0 \\ w^{(1)} - w_m^{(1)} \\ y_p^* - y_m \\ w^{(2)} - w_m^{(2)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ n \\ 0 \end{pmatrix}$$

$$= w_m + Q\, e + q_n\, n \tag{5.2.24}$$

where we defined $q_n^T = (0, 0, 1, 0) \in (\mathbf{R}, \mathbf{R}^{n-1}, \mathbf{R}, \mathbf{R}^{n-1}) = \mathbf{R}^{2n}$.

Using these results, the adaptive system with measurement noise is described by

$$\begin{aligned} \dot{e}(t) &= A_m e(t) + b_m \phi^T(t)\, w_m(t) + b_m \phi^T(t)\, Q\, e(t) + b_m\, \phi^T(t)\, q_n\, n(t) + b_n\, n(t) \\ \dot{\phi}(t) &= -g\, c_m^T e(t)\, w_m(t) - g\, c_m^T e(t)\, Q\, e(t) - g\, c_m^T e(t)\, q_n\, n(t) \\ &\quad - g\, n(t)\, w_m(t) - g\, n(t)\, Q\, e(t) - g\, n^2(t)\, q_n \end{aligned} \tag{5.2.25}$$

which, with the definition of x in (5.2.17), and the definition of f in (5.2.15)-(5.2.16) can be written

$$\dot{x} = f(t, x) + p_1(t) + P_2(t)\, x(t) \tag{5.2.26}$$

where $p_1(t) \in \mathbf{R}^{5n-2}$, and $P_2(t) \in \mathbf{R}^{5n-2 \times 5n-2}$, are given by

$$p_1(t) = \begin{bmatrix} b_n\, n(t) \\ -g\, n(t)\, w_m(t) - g\, n^2(t)\, q_n \end{bmatrix}$$

$$P_2(t) = \begin{bmatrix} 0 & b_m\, n(t)\, q_n^T \\ -g\, n(t)\, q_n\, c_m^T - g\, n(t)\, Q & 0 \end{bmatrix} \tag{5.2.27}$$

Note that if $n \in L_\infty$, then p_1 and $P_2 \in L_\infty$. Therefore, the perturbed system (5.2.26) is a special form of system (5.2.1), where u contains the components of p_1 and P_2. Although $p_1(t)$ depends quadratically on n, given a bound on n, there exists $k_n \geq 0$ such that

$$\| p_1 \|_\infty + \| P_2 \|_\infty \leq k_n \| n \|_\infty \tag{5.2.28}$$

From these derivations, we deduce the following theorem.

Theorem 5.2.2 Robustness to Disturbances

Consider the output error direct adaptive control scheme of section 3.2.2, assuming that the relative degree of the plant is 1. Assume that the measured output y_p of the plant is given by (5.2.18), where $n \in L_\infty$. Let $h > 0$.

If w_m is PE

Then there exists $\gamma_n, c_n > 0$ and $m \geq 1$, such that $\| n \|_\infty \leq c_n$ and $|x(0)| < h/m$ implies that $x(t)$ converges to a B_δ ball of radius $\delta = \gamma_n \| n \|_\infty$, and $|x(t)| \leq m\, |x_0| < h$ for all $t \geq 0$.

Proof of Theorem 5.3.2

Since w_m is PE, the unperturbed system (5.2.15) (i.e. (5.2.16)) is exponentially stable in any B_h by theorem 3.8.2. The perturbed system (5.2.25) (i.e. (5.2.26) is a special case of the general form (5.2.1), so that theorem 5.2.1 can be applied with u containing the components of $p_1(t), P_2(t)$. The results on $p_1(t), P_2(t)$ can be translated into similar results involving $n(t)$, using (5.2.28).

Comments

a) A specific bound c_n on $\| n \|_\infty$ can be obtained such that, within this bound, and provided the initial error is sufficiently small, *the stability of the adaptive system will be preserved*. For this reason, c_n is called a robustness margin of the adaptive system to output disturbances.

b) The deviations from equilibrium are locally *at most proportional* to the disturbances (in terms of L_∞ norms), and their bounds can be made arbitrarily small by reducing the bounds on the disturbances.

c) The L_∞ gain from the disturbances to the deviations from equilibrium can be reduced by *increasing the rate of exponential convergence of the unperturbed system* (provided that other constants remain identical).

d) Rohrs example (R3) of instability of an adaptive scheme with output disturbances on a non persistently excited system, is an example of instability when the persistency of excitation condition of the nominal system is not satisfied.

Robustness to Unmodeled Dynamics

The approach adopted here is similar to that used by Doyle & Stein (1981) to study the robustness of non adaptive control systems. We assume again that there exists a nominal plant $\hat{P}(s)$, satisfying the assumptions on which the adaptive control scheme is based, and we define the *output of the nominal plant* to be

$$y_p^* = \hat{P}(u) \tag{5.2.29}$$

The actual output is modeled as the output of the nominal plant, plus some additive uncertainty represented by a bounded operator H_a

$$y_p(t) = y_p^*(t) + H_a(u)(t) \tag{5.2.30}$$

The operator H_a represents the difference between the real plant, and the idealized plant $\hat{P}(s)$. In the terminology of Doyle & Stein (1981), we refer to it as an *additive unstructured uncertainty*, and it constitutes all the uncertainty, since it is the purpose of the adaptive scheme to reduce to zero the *structured* or *parametric* uncertainty.

We assume that $H_a : L_{\infty e} \to L_{\infty e}$ is a causal operator satisfying

$$\| H_a(u)_t \|_\infty \leqslant \gamma_a \| u_t \|_\infty + \beta_a \tag{5.2.31}$$

for all $t \geq 0$. β_a may include the effect of initial conditions in the unmodeled dynamics, and the possible presence of bounded output disturbances.

The following theorem guarantees the stability of the adaptive system in the presence of unmodeled dynamics satisfying (5.2.28).

Theorem 5.2.3 Robustness to Unmodeled Dynamics

Consider the output error direct adaptive control scheme of section 3.3.2, assuming that the relative degree of the plant is 1. Assume that the nominal plant output and actual measured plant output satisfy (5.2.29)-(5.2.30), where $\hat{P}$ satisfies the assumptions of section 3.3.2, and H_a satisfies (5.2.31).

If w_m is PE

Then for x_0, γ_a, β_a sufficiently small, the states trajectories of the adaptive system remain bounded.

Proof of Theorem 5.2.3

Let $n = H_a(u)$, so that, by assumption

$$\| n_t \|_\infty \leq \gamma_a \| u_t \|_\infty + \beta_a \tag{5.2.32}$$

for all $t \geq 0$. On the other hand, by theorem 5.2.2, there exists $\gamma_n, \beta_n \geq 0$ such that

$$\| x_t \|_\infty \leq \gamma_n \| n_t \|_\infty + \beta_n \leq h \tag{5.2.33}$$

for all $t \geq 0$, provided that $\| n_t \|_\infty \leq c_n$ (so that $x \in B_h$).

The input u is given by

$$\begin{aligned} u &= \theta^T w = \theta^{*T} w + \phi^T w \\ &= \theta^{*T} w_m + \theta^{*T} Q\, e + \theta^{*T} q_n\, n + \phi^T w_m + \phi^T Q\, e + + \phi^T q_n\, n \end{aligned} \tag{5.2.34}$$

where we used (5.2.24). Define $u^* := \theta^{*T} w_m$. Assuming that $x \in B_h$, there exists $k \geq 0$ such that

$$\| u_t \|_\infty \leq \| u_t^* \|_\infty + k \left(\| x_t \|_\infty + \| n_t \|_\infty \right) \tag{5.2.35}$$

for all $t \geq 0$, so that, with (5.2.33)

$$\| u_t \| \leq \gamma_u \| n_t \|_\infty + \beta_u + \| u_t^* \|_\infty \tag{5.2.36}$$

for some γ_u, $\beta_u \geqslant 0$, and for all $t \geqslant 0$.

Applying the small gain theorem (lemma 3.6.6), we find that all signals are bounded if

$$\gamma_a \cdot \gamma_u < 1 \tag{5.2.37}$$

and, to guarantee that $\| n \|_\infty \leqslant c_n$ (so that $x \in B_h$)

$$\frac{\beta_a + \gamma_a(\beta_u + \| u^* \|_\infty)}{1 - \gamma_a \gamma_u} \leqslant c_n \tag{5.2.38}$$

Although the proof in its form appears circular, since we assume that $x \in B_h$ to establish the inequalities used to prove the result, this can be resolved by imposing conditions (5.2.37)-(5.2.38), then show that x must remain in B_h for all $t \geqslant 0$ by a contradiction argument. □

Comments

Condition (5.2.24) is very general, since it includes possible nonlinearities, unmodeled dynamics, etc. provided that they can be represented by additive, bounded-input bounded-output operators.

If the operator H_a is linear time invariant, the stability condition is a condition on the L_∞ gain of H_a. One can use

$$\gamma_a = \| h_a \|_1 = \int_0^\infty | h_a(\tau) | \, d\tau \tag{5.2.39}$$

where $h_a(\tau)$ is the impulse response of $\hat{H}_a$. The constant β_a depends on the initial conditions in the unmodeled dynamics.

The proof of theorem 5.2.3 gives some margins of unmodeled dynamics that can be tolerated without loss of stability of the adaptive system. Given γ_a, β_a it is actually possible to compute these values. The most difficult parameter to determine is possibly the rate of convergence of the unperturbed system, but we saw in chapter 4 how some estimate could be obtained, under the conditions of averaging. Needless to say the expression of these robustness margins depends in a complex way from known parameters, and it is likely that the estimates would be conservative. The importance of the result is to show that if the unperturbed system is persistently excited, it will tolerate *some* amount of disturbance, or conversely that an arbitrary small disturbance *cannot* destabilize the system, such as in example (R3).

5.3 Analysis of the Rohrs Examples

By considering the overall adaptive system, including the plant states, observer states, and the adaptive parameters, we showed in section 5.2 the importance of the exponential convergence to guarantee some robustness of the adaptive system. This convergence depends especially on the *parameter* convergence, and therefore on conditions on the input signal $r(t)$.

A heuristic analysis of the Rohrs examples gives additional insight into the mechanisms leading to instability, and suggest practical methods to improve robustness. Such an analysis can be found in Astrom (1983), and its success relies mainly on the separation of time scales between the evolution of the plant/ observer states, and the evolution of the adaptive parameters. This separation of time scales is especially suited for the application of averaging methods (cf. chapter 4).

Following Astrom (1983), we will show that instability in the Rohrs examples are due to one or more of the following factors

(a) the lack of PE signals to

- allow for parameter convergence in the nominal system,
- prevent the drift of the parameters due to unmodeled dynamics or output disturbances.

(b) the presence of significant excitation at high frequencies, originating either from the reference input, or from output disturbances. These signals cause the adaptive loop to try to get the plant loop to match the model at high frequencies, where it results in a closed-loop unstable plant.

(c) a large reference input with a non-normalized identification algorithm and unmodeled dynamics, resulting in the instability of the identification algorithm.

Analysis

Consider now the mechanisms of instability corresponding to these three cases.

(a) Consider first the case when no unmodeled dynamics or output disturbances are present.

In the nominal case, the output error tends to zero. When the PE condition is not satisfied, the controller parameter does not necessarily converge to its nominal value, but to a value such that the closed-loop transfer function matches the model transfer function at the frequencies of the reference input. Consider for example Rohrs example, without unmodeled dynamics. The closed-loop transfer function from $r \to y_p$, assuming that c_0 and d_0 are fixed is

$$\frac{\hat{y}_p}{\hat{r}} = \frac{2c_0}{s+1-2d_0} \tag{5.3.1}$$

If a constant reference input is used, only the DC gain of this transfer function must be matched with the DC gain of the reference model. This implies the condition that

$$\frac{2c_0}{1-2d_0} = 1 \tag{5.3.2}$$

Any value of c_0, d_0 satisfying (5.3.2) will lead to $y_p - y_m \to 0$ as $t \to \infty$ for a constant reference input. Conversely, when $e \to 0$, so do $\dot{c}_0$, and $\dot{d}_0$, so that the assumption that c_0, d_0 are fixed is justified.

If an output disturbance $n(t)$ enters the adaptive system, it can cause the parameters c_0, d_0 to move along the line (more generally the surface) defined by (5.3.2), leaving $e_0 = y_p - y_m$ at zero. In particular, note that when output disturbances are present, the actual update law for d_0 is not (5.1.6) anymore, but

$$\dot{d}_0 = -g\, y_p^* (y_p^* - y_m) - g\, y_m\, n - g\, n^2 \tag{5.3.3}$$

where we find the presence of the term $-gn^2$, which will tend to make d_0 slowly drift towards the negative direction.

In example (R3), unmodeled dynamics are present, so that the transfer function from $r \to y_p$ is in fact given by

$$\frac{\hat{y}_p}{\hat{r}} = \frac{458c_0}{(s+1)(s^2+30s+229)-458d_0} \tag{5.3.4}$$

which is identical to (5.3.1) for DC signals, but which is unstable for $d_0 \geqslant 1/2$ and $d_0 \leqslant -17.03$.

The result is observed in figures 5.6 and 5.7, where d_0 slowly drifts in the negative direction, until it reaches the limit of stability of the closed-loop plant with unmodeled dynamics.

This instability is called the *slow drift instability*. The error converges to a neighborhood of zero, and the signal available for parameter update is very small and unreliable, since it is indistinguishable fron the output noise $n(t)$. It is the accumulation of updates based on incorrect information that leads to parameter drift, and eventually to instability.

In terms of the discussion of section 5.2, we see that the constant disturbance $-gn^2$ is not counteracted by any restoring force, as would be the case if the original system was exponentially stable. For example, consider the case where $n = 0.1 \sin 16.1 t$. Figure 5.8 shows the evolution of the parameter d_0 in a simulation where $r(t) = 2$ and where $r(t) = 2 \sin t$. In the first case, the parameter slowly drifts, leading eventually to instability. When $r(t) = 2 \sin t$, so that PE conditions are satisfied, the parameter d_0 deviates from d_0^*, but remains close to the nominal value.

(b) Consider now the case when the reference input, or the output disturbance, contain a large component at a frequency where unmodeled dynamics are significant.

Let us return to Rohrs example, with a sinusoidal reference input $r(t) = r_0 \sin(\omega_0 t)$. With unmodeled dynamics, there are still *unique* values of c_0, d_0 such that the transfer function from $r \to y_p$ matches $\hat{M}$ at the frequency of the reference input ω_0. Without unmodeled dynamics, these would be the nominal c_0^*, d_0^*, but now they are the values c_0^+, d_0^+, which are usually called the *tuned values*, such that

$$\left. \frac{458 c_0^+}{(s+1)(s^2+30s+229) - 458 d_0^+} \right|_{j\omega_0} = \left. \frac{3}{s+3} \right|_{j\omega_0} \tag{5.3.5}$$

where ω_0 is the frequency of the reference input. Note that the tuned values depend on $\hat{M}$, $\hat{P}$, the unmodeled dynamics, and also *on the reference input* r.

On the other hand, it may be verified through simulations, that the output error tends to zero, and that the controller parameters converge to the following values $c_{o_{ss}}$ and $d_{o_{ss}}$ (cf. Astrom (1983))

ω_0	$c_{o_{ss}}$	$d_{o_{ss}}$
1	1.69	-1.26
2	1.67	-1.44
5	1.53	-2.72
10	1.04	-7.31

It may be verified that these values are identical to the tuned values defined above. Therefore, the adaptive control system updates the parameters, trying to match the closed-loop transfer function - including the unmodeled dynamics - to the model reference transfer function. Note that the parameter $d_{o_{ss}} = d_0^+$ quickly decreases for $\omega_0 > 5$. On the other hand, the closed-loop system is unstable when $d_0 \leqslant -17.03$, and $d_0^+ \leqslant -17.03$, when $\omega_0 \geqslant 16.09$. Therefore, by attempting to match the reference model at a high frequency, the adaptive system leads to an unstable closed-loop system, and thereby to an unstable overall system.

This is the instability observed in example (R2). In contrast, figure 5.9 shows a simulation where $r = 0.3 + 1.85 \sin t$, that is where the sinusoidal component of the input is at a frequency where model matching is possible. Then, the parameters converge to values c_0^+, d_0^+ close to c_0^*, d_0^*, and the adaptive system remains stable, despite the unmodeled dynamics.

Finally, note that instabilities of this type can be obtained even without unmodeled dynamics, and can lead to the so-called *bursting phenomenon* (cf. Anderson (1985)).

(c) Consider finally the mechanism of instability observed in example (R1).

This mechanism will be called the *high-gain identifier instability*. Although we do not have explicitly a high adaptation gain g, we recall that the adaptation law is given by

$$\dot{c}_0 = -g\, r\, e \tag{5.3.6}$$

$$\dot{d}_0 = -g\, y_p\, e \tag{5.3.7}$$

Therefore, multiplying r by 2, means multiplying y_m, y_p and e by 2, and therefore is equivalent to multiplying the adaptation gain by 4.

The instabilities obtained for high values of the adaptation gain are comparable to instabilities caused by high gain feedback in LTI systems with relative degree greater than

2 (cf. Astrom (1983) for a simple root-locus argument). A simple fix to these problems is to replace the identification algorithm by a normalized algorithm.

5.4 Methods to Improve Robustness

From the discussions in the previous sections, we deduce some basic guidelines to improve the robustness of adaptive systems.

Persistency of Excitation

Persistency of excitation should be used to ensure that parameters converge to the neighborhood of their nominal values, and track these values if the plant is slowly varying. PE should be achieved by injecting inputs in the frequency range of interest (i.e. the frequency where model matching is achievable). PE has the advantage of directly increasing the information available to the identification algorithm. The disadvantage is that it may not be practical, since inputs are generally restricted by external constraints. Sometimes, small signals may be added to the reference input with acceptable disturbance of the system, and with sufficient excitation to ensure convergence of the parameters. Another disadvantage of this method is however to lead to a robustness that is not internal (or "structural"), but instead depends on external signals.

Deadzone

This method consists in turning off the adaptation law when the identifier error is below the threshold under which it only consists of measurement noise. The parameters are therefore not updated if the identifier error is sufficiently small. A more complex version of this is to monitor the frequency content of the control input, and to turn-off adaptation when PE conditions are not met.

The use of a deadzone is simple and practical, but it absolutely requires the measurement noise to be bounded. Otherwise, occasional disturbances may cause parameter drift, and eventually instability.

Slow Adaptation

As we saw in the previous section, fast adaptation can lead to instability of the identifier, and is in general nonrobust. Slow adaptation reduces the influence of noise by averaging it. To some extent, this is therefore a method to increase robustness. In this category, we can also include modifications of the update law where some signals are replace by averaged or filtered signals.

Although fast adaptation is not recommended, very slow adaptation is not either. First, slow adaptation goes against a basic performance consideration, which is to track parameter variations. Second, the effect of slow adaptation on drift instabilities is only to delay instabilities, not to prevent them.

Prior Information

Prior information is useful to constrain adaptive parameters to some arbitrary set (with the use of projection in the update law for example). Drift instability can be prevented in this manner. Also, if an approximate value θ_a of the adaptive parameter θ is known, the update law may be replaced by

$$\dot{\theta} = -\sigma(\theta - \theta_a) \ + \ \text{(previous update law)} \tag{5.4.1}$$

This modification includes the σ modification proposed by Ioannou & Kokotovic (1984). It has the advantage of being simple and efficient, but its efficiency depends strongly on the approximate θ_a. Note also that, even without unmodeled dynamics, the output error and the parameter error do not tend to zero unless $\theta_a = \theta^*$.

References

Research work along these lines (and others) can be found in Kreisselmeier and Narendra (1982), Peterson and Narendra (1982), Anderson and Johnstone (1983), Sastry (1984), Riedle, Cyr, and Kokotovic (1984), Kosut and Johnson (1984), Ortega, Praly, and Landau (1985), Kokotovic, Praly, and Landau (1985), Kreisselmeier (1986), Kreisselmeier and Anderson (1986), Narendra and Annaswamy (1986).

5.5 Conclusions

In this chapter, we studied the problem of the robustness of adaptive systems, that is their ability to maintain stability despite modeling errors and measurement noise.

We first reviewed the Rohrs examples, illustrating several mechanisms of instability. Then, we derived a general result relating exponential stability to robustness. We also showed how it could be used to compute robustness margins of an adaptive control scheme in the presence of measurement noise or unmodeled dynamics. The result indicated that the property of exponential stability is robust, although examples show that the BIBS stability property is not (that is, BIBS stable systems can become unstable in the presence of arbitrarily small disturbances). In practice, the amplitude of the disturbances should be checked against robustness margins to determine if stability is guaranteed. The complexity of the relationship between the robustness margins and known parameters, and the dependence of these margins on external signals unfortunately made the result more conceptual than practical.

The mechanisms of instability found in the Rohrs examples were discussed in view of the relationship between exponential stability and robustness. Further explanations of the mechanisms of instability were presented. Finally, various methods to improve robustness were briefly reviewed.

Much research is still needed in the area of robustness. As for traditional control, we confront the problem of the tradeoff between robustness and performance. It is also necessary to develop useful methods of analysis of robustness of adaptive systems, and methods to quantify robustness to allow comparison between different approaches.

Conclusions

Specific Conclusions

In this thesis, we addressed three issues of prime importance to adaptive systems: the stability under ideal conditions, the convergence of the adaptive parameters, and the robustness to modeling errors and to measurement noise. Identification and model reference adaptive control schemes were considered, but the attention was focused on single-input single-output, continuous time, linear time invariant systems.

New results were presented, as well as simplified and unified proofs of existing results. Therefore, connections between different schemes, and apparently different issues were found: for example, between input error and output error schemes, between direct proofs of exponential convergence and proofs using averaging techniques, and between parameter convergence and robustness.

First, some identification algorithms were reviewed, and their stability and parameter convergence properties were established. It was shown that, under general conditions, the identifier parameter was a bounded function of time, and the identifier error converged to zero as time approached infinity. Similar results were found for gradient and least-squares algorithms, and for linear as well as SPR error equations. Parameter convergence followed from an additional persistency of excitation condition.

Three model reference adaptive control schemes were presented. One was the output error adaptive control scheme of Narendra, Lin, and Valavani. Another was a simple indirect adaptive control scheme. The third was a new, input error, direct adaptive control scheme, that was an alternate scheme to the Narendra, Lin, and Valavani algorithm. It did not require a strictly positive real condition on the reference model, and no over-parametrization was needed when the high-frequency gain was unknown.

Useful lemmas were presented, and unified stability proofs were derived for the input and output error schemes, as well as for the indirect adaptive control scheme. The results showed that all three schemes had similar stability properties: the state trajectories were bounded functions of time, and the error between the plant and the reference model converged to zero as time approached infinity. Therefore, stability was not an argument in selecting one scheme instead of the other. In practice however, differences appeared. The input error and the indirect schemes had the advantage of leading to a linear error equation, and of allowing for a useful separation between identification and control. Parameter convergence was also established for the adaptive control schemes, under persistency of excitation conditions on model signals.

The parameter convergence of the adaptive schemes was further analyzed using averaging techniques. For this, we assumed that the reference input possessed some stationarity properties, and that the adaptation gain was sufficiently small. It was shown that the nonautonomous adaptive systems could be approximated by autonomous systems, thereby considerably simplifying the analysis. In particular, estimates of the rates of exponential convergence of the parameters were obtained for the linear identification scheme, as well for a nonlinear adaptive control scheme.

Although the class of inputs under consideration was restricted to stationary inputs, this class was quite large (more general than almost periodic inputs), and resulting expressions in the frequency domain were especially appealing. The assumption of slow adaptation was not really restrictive, and it appeared to simply require that adaptive parameters vary slower than other states and signals in the adaptive system. Practical considerations in chapter 5 suggested that this should be the normal operation of adaptive systems, and that fast adaptation was essentially non robust. It should not be deduced however that very slow adaptation would be desirable for robustness, or required for the applicability of averaging methods.

Finally, the robustness of adaptive algorithms was investigated. The Rohrs examples were first reviewed. A connection between exponential convergence and robustness was established in a general framework. The result was applied to a model reference adaptive control scheme, and stressed the importance of the persistency of excitation condition for robustness. Robustness margins were also obtained. An important parameter of the analysis was the rate of exponential convergence of the adaptive algorithm, which can be obtained - or approximated - using methods described earlier. The mechanisms of instability observed in the Rohrs examples were explained, and methods to improve robustness were briefly investigated.

General Conclusions

Appropriate techniques are needed for the analysis of the nonlinear time varying dynamics of adaptive systems. Among these, averaging methods constitute a very successful, and promising approach. The success of the application of averaging methods to adaptive systems partially relies on the separation between the adaptive parameters and the remaining states of the adaptive system. This is probably due to the fact that we can exploit the linearity of the underlying system for fixed adaptive parameters. In fact, we saw in chapter 4 that, by this mean, we could eventually deal with the nonlinear dynamics without linearization of any type, and even obtain frequency domain results. Using a similar decomposition in chapter 5, we found interesting explanations of the mechanisms of instability observed in Rohrs examples.

In general, it is curious to note that many results, besides those using averaging, were proved by relying on a fixed parameter approximation. This was found in the proofs of exponential convergence in chapter 2, and in the proofs of stability in chapter 3 (swapping lemma). It is therefore likely that successful approaches will keep in mind the separation between the adaptive parameters and the other states of the adaptive system.

Robustness is a very important topic that needs to be better understood to stimulate practical applications. Again, averaging methods are a very promising approach in this direction. Practical solutions are needed to enhance the robustness of adaptive systems, but some methods to quantify robustness and compare different methodologies of control would be desirable. We hope that the basic work of this thesis will help to strengthen the foundations on which such research can be built.

Suggestions for Future Research

The thesis suggests several avenues for future research. As mentioned above, an important area is that of the robustness of adaptive systems to measurement noise and unmodeled dynamics. We need practical methods to improve robustness, tools for the analysis of robustness, and in general a better understanding of what the problem is. The relationship between reference input, nominal plant, unmodeled dynamics, and tuned parameters is a particular topic of interest in that regard.

The development of averaging methods, and their application to specific problems is worth special interest. Among these problems is the robustness of adaptive systems, but also the optimum input design for parameter convergence, and the comparison of adaptive algorithms (gradient vs. least squares, MIT rule vs. others), etc.

We did not address numerical considerations in this thesis. It is clear that the choice of structure, parametrization, and identification algorithms will strongly influence the numerical stability of the algorithms, and research in that area would definitely be beneficial.

Appendix

Proof of Lemma 1.4.2

Let

$$r(t)=\int_0^t a(\tau)x(\tau)d\tau \qquad \text{(A1.4.1)}$$

so that, by assumption

$$\dot{r}(t)=a(t)x(t)\leqslant a(t)r(t)+a(t)u(t) \qquad \text{(A1.4.2)}$$

i.e. for some positive $s(t)$

$$\dot{r}(t)-a(t)r(t)-a(t)u(t)+s(t)=0 \qquad \text{(A1.4.3)}$$

Solving the differential equation with $r(0) = 0$

$$r(t)=\int_0^t e^{\int_\tau^t a(\sigma)d\sigma}(a(\tau)u(\tau)-s(\tau))d\tau \qquad \text{(A1.4.4)}$$

Since exp(.) and s(.) are positive functions

$$r(t)\leqslant\int_0^t e^{\int_\tau^t a(\sigma)d\sigma}a(\tau)u(\tau)d\tau \qquad \text{(A1.4.5)}$$

By assumption $x(t)\leqslant r(t)+u(t)$ so that (1.4.11) follows. Inequality (A1.4.12) is obtained by integrating (A1.4.11) by parts. □

Proof of Lemma 2.5.2

We consider the system

$$\dot{x}(t)=A(t)x(t)$$

$$y(t)=C(t)x(t) \qquad \text{(A2.5.1)}$$

and the system, under output injection

$$\dot{w}(t) = (A(t)+K(t)C(t))w(t)$$

$$z(t) = C(t)w(t) \qquad \text{(A2.5.2)}$$

where $x, w \in \mathbf{R}^n$, $A \in \mathbf{R}^{n \times n}$, $C \in \mathbf{R}^{m \times n}$, $K \in \mathbf{R}^{n \times m}$, and $y, z \in \mathbf{R}^m$.

It is sufficient to derive equations the inequalities giving β_1', β_2', β_3'.

(a) Derivation of β_1'

Consider the trajectories $x(\tau)$ and $w(\tau)$, corresponding to systems (A2.5.1) and (A2.5.2) respectively, with identical initial conditions $x(t_0)=w(t_0)$. Then

$$w(\tau) - x(\tau) = \int_{t_0}^{\tau} \Phi(\tau,\sigma)K(\sigma)C(\sigma)w(\sigma)\,d\sigma \qquad \text{(A2.5.3)}$$

Let $e(\sigma) = K(\sigma)C(\sigma)w(\sigma) / \, |K(\sigma)C(\sigma)w(\sigma)| \in \mathbf{R}^n$, so that

$$|C(\tau)(w(\tau)-x(\tau))|^2 = |\int_{t_0}^{\tau} C(\tau)\Phi(\tau,\sigma)K(\sigma)C(\sigma)w(\sigma)d\sigma|^2$$

$$\leqslant (\int_{t_0}^{\tau} |C(\tau)\Phi(\tau,\sigma)e(\sigma)| \, \|K(\sigma)\| \, |C(\sigma)w(\sigma)|\,d\sigma)^2$$

$$\leqslant \int_{t_0}^{\tau} |C(\nu)w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2 \, \|K(\sigma)\|^2 d\sigma \qquad \text{(A2.5.4)}$$

using the definition of the induced norm, and Schwartz inequality. On the other hand, using the triangular inequality

$$(\int_{t_0}^{t_0+\delta} |C(\tau)w(\tau)|^2 d\tau)^{1/2} \geqslant (\int_{t_0}^{t_0+\delta} |C(\tau)x(\tau)|^2 d\tau)^{1/2}$$

$$- (\int_{t_0}^{t_0+\delta} |C(\tau)(w(\tau)-x(\tau))|^2 d\tau)^{1/2} \qquad \text{(A2.5.5)}$$

so that, using (A2.5.4), and the UCO of the original system

$$(\int_{t_0}^{t_0+\delta} |C(\tau)w(\tau)|^2 d\tau)^{1/2}$$

$$\geqslant \sqrt{\beta_1}\,|w(t_0)| - (\int_{t_0}^{t_0+\delta} \int_{t_0}^{\tau} |C(\nu)w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2 \|K(\sigma)\|^2 \, d\sigma \, d\tau)^{1/2}$$

$$\geqslant \sqrt{\beta_1}\,|w(t_0)|$$

$$-\Big(\int_{t_0}^{t_0+\delta} |C(\nu)w(\nu)|^2 d\nu\Big)^{1/2}\Big(\int_{t_0}^{t_0+\delta}\int_{t_0}^{\tau} \|K(\sigma)\|^2 |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2 \,d\sigma\, d\tau\Big)^{1/2} \quad \text{(A2.5.6)}$$

Changing the order of integration, the integral in the last parenthesis becomes

$$\int_{t_0}^{t_0+\delta} \|K(\sigma)\|^2 \int_{\sigma}^{t_0+\delta} |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2 \,d\tau\, d\sigma \quad \text{(A2.5.7)}$$

Note that $t_0+\delta-\sigma \leqslant \delta$, $|e(\sigma)|=1$, while $\Phi(\tau,\sigma)e(\sigma)$ is the solution of system (A2.5.1) starting at $e(\sigma)$. Therefore, using the UCO property on the original system, and the condition on $K(.)$, (A2.5.7) becomes

$$\int_{t_0}^{t_0+\delta} \|K(\sigma)\|^2 \int_{\sigma}^{t_0+\delta} |C(\tau)\Phi(\tau,\sigma)\,e(\sigma)|^2 \,d\tau\, d\sigma \leqslant k_\delta\,\beta_2 \quad \text{(A2.5.8)}$$

Inequality (2.5.12) follows directly from (A2.5.6) and (A2.5.8).

(b) Derivation of β_2'

We use a similar procedure, using (A2.5.4)

$$|C(\tau)w(\tau)|^2 \leqslant |C(\tau)x(\tau)|^2 + \Big|\int_{t_0}^{\tau} C(\tau)\Phi(\tau,\sigma)K(\sigma)C(\sigma)w(\sigma)d\sigma\Big|^2$$

$$\leqslant |C(\tau)x(\tau)|^2 + \Big(\int_{t_0}^{\tau} |C(\sigma)w(\sigma)|\,|C(\tau)\Phi(\tau,\sigma)e(\sigma)|\,\|K(\sigma)\|\,d\sigma\Big)^2$$

$$\leqslant |C(\tau)x(\tau)|^2$$

$$+ \int_{t_0}^{\tau} |C(\nu)w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2\, \|K(\sigma)\|^2\, d\sigma \quad \text{(A2.5.9)}$$

and, for all $t_0 \leqslant t \leqslant t_0+\delta$

$$\int_{t_0}^{t} |C(\tau)w(\tau)|^2 d\tau \leqslant \int_{t_0}^{t_0+\delta} |C(\tau)x(\tau)|^2 d\tau$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau} |C(\nu)w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2\, \|K(\sigma)\|^2 d\sigma\, d\tau \quad \text{(A2.5.10)}$$

and, using the Bellman-Gronwall lemma (lemma 1.4.2), together with the UCO of the original system

$$\int_{t_0}^{t} |C(\tau)w(\tau)|^2 d\tau$$

$$\leqslant \beta_2 |w(t_0)|^2 \exp\left(\int_{t_0}^{t}\int_{t_0}^{\tau} |C(\tau)\Phi(\tau,\sigma)e(\sigma)|^2 \, \|K(\sigma)\|^2 d\sigma \, d\tau\right) \qquad \text{(A2.5.11)}$$

for all t, and in particular for $t = t_0 + \delta$.

The integral in the exponential can be transformed, by changing the order of integration, as in (A2.5.8). Inequality (2.5.13) follows directly from (A2.5.8) and (A2.5.11).

(c) Derivation of $\beta_3'(.)$

Using (A2.5.3)

$$|w(t)| \leqslant |x(t)| + \int_{t_0}^{t} \|\Phi(t,\sigma)\| \, \|K(\sigma)\| \, |C(\sigma)w(\sigma)| \, d\sigma$$

$$\leqslant \beta_3(|t - t_0|)\,|w(t_0)| + \sup_{\tau \in [0, t - t_0]} \beta_3(|\tau|) \int_{t_0}^{t} \|K(\sigma)\| \, |C(\sigma)w(\sigma)| \, d\sigma$$

$$\leqslant \beta_3(|t - t_0|)\,|w(t_0)| + \sup_{\tau \in [0, t - t_0]} \beta_3(|\tau|)\,(k_{t - t_0} \int_{t_0}^{t} |C(\sigma)w(\sigma)|^2 d\sigma)^{1/2} \qquad \text{(A2.5.12)}$$

Inequality (2.5.14) follows directly from (A2.5.12). □

Proof of Lemma 2.6.5

We wish to prove that for some δ, α_1, $\alpha_2 > 0$, and for all x with $|x| = 1$

$$\alpha_2 \geqslant \int_{t_0}^{t_0+\delta} ((w^T + e^T)x)^2 d\tau \geqslant \alpha_1 \qquad \text{for all } t_0 \geqslant 0 \qquad \text{(A2.6.1)}$$

By assumption, $e \in L_2$, so that $\int_0^\infty (e^T x)^2 d\tau \leqslant m$ for some $m \geqslant 0$. Since w is PE, there exist σ, β_1, $\beta_2 > 0$ such that

$$\beta_2 \geqslant \int_{t_0}^{t_0+\sigma} (w^T x)^2 d\tau \geqslant \beta_1 \qquad \text{for all } t_0 \geqslant 0 \qquad \text{(A2.6.2)}$$

Let $\delta \geqslant \sigma(1+\frac{m}{\beta_1})$, $\alpha_1 = \beta_1$, $\alpha_2 = m + \beta_2(1+\frac{m}{\beta_1})$ so that

$$\int_{t_0}^{t_0+\delta} ((w^T + e^T)x)^2 d\tau \geqslant \int_{t_0}^{t_0+\delta} (w^T x)^2 d\tau - \int_{t_0}^{t_0+\delta} (e^T x)^2 d\tau$$

$$\geqslant \beta_1 (1 + \frac{m}{\beta_1}) - m = \alpha_1 \tag{A2.6.3}$$

and

$$\int_{t_0}^{t_0+\delta} ((w^T + e^T)x)^2 d\tau \leqslant \int_{t_0}^{t_0+\delta} (w^T x)^2 d\tau + \int_{t_0}^{t_0+\delta} (e^T x)^2 d\tau$$

$$\leqslant \beta_2 (1 + \frac{m}{\beta_1}) + m = \alpha_2 \tag{A2.6.4}$$

□

Proof of Lemma 2.6.6

We wish to prove that for some δ, α_1, $\alpha_2 > 0$, and for all x with $|x| = 1$

$$\alpha_2 \geqslant \int_{t_0}^{t_0+\delta} (\hat{H}(w^T)x)^2 d\tau \geqslant \alpha_1 \qquad \text{for all } t_0 \geqslant 0 \tag{A2.6.5}$$

Denote $u = w^T x$, and $y = \hat{H}(u) = \hat{H}(w^T x) = \hat{H}(w^T)x$ (where the last inequality is true because x does not depend on t). We thus wish to show that

$$\alpha_2 \geqslant \int_{t_0}^{t_0+\delta} y^2(\tau) d\tau \geqslant \alpha_1 \qquad \text{for all } t_0 \geqslant 0 \tag{A2.6.6}$$

Since w is PE, there exists σ, β_1, $\beta_2 > 0$ such that

$$\beta_2 \geqslant \int_{t_0}^{t_0+\sigma} u^2(\tau) d\tau \geqslant \beta_1 \qquad \text{for all } t_0 \geqslant 0 \tag{A2.6.7}$$

In this form, the problem appears on the relationship between truncated L_2 norms of the input and output of a stable, minimum phase LTI system. Similar problems are addressed in section 3.6, and we will therefore use results from lemmas in that section.

Let $\delta = m\,\sigma$, where m is an integer to be defined later. Since u is bounded, and $\hat{y} = \hat{H}(u)$, it follows that y is bounded (lemma 3.6.1), and the upper bound in (A2.6.6) is satisfied. The lower bound is obtained now, by inversing $\hat{H}$ in a similar way as is used in the proof of lemma 3.6.2. We let

$$\hat{z}(s) = \frac{a^r}{(s+a)^r} \hat{u}(s) \tag{A2.6.8}$$

where $a > 0$ will be defined later, and r is the relative degree of $\hat{H}(s)$. Thus

$$\hat{y}(s) = \frac{(s+a)^r}{a^r} \hat{H}(s) \hat{z}(s) \tag{A2.6.9}$$

The transfer function from $\hat{z}(s)$ to $\hat{y}(s)$ has relative degree 0. Being minimum phase, it has a proper and stable inverse. By lemma 3.6.1, there exist $k_1, k_2 \geqslant 0$ such that

$$\int_{t_0}^{t_0+\delta} z^2(\tau) d\tau \leqslant k_1 \int_{t_0}^{t_0+\delta} y^2(\tau) d\tau + k_2 \tag{A2.6.10}$$

Since $\dot{u}$ is bounded

$$\int_{t_0}^{t_0+\delta} \dot{u}^2(\tau) d\tau \leqslant k_3 \delta \tag{A2.6.11}$$

for some $k_3 \geqslant 0$. Using the results in the proof of lemma 3.6.2 ((A3.6.14)), we can also show that, with the properties of the transfer function $a^r / (s+a)^r$

$$\int_{t_0}^{t_0+\delta} u^2(\tau) d\tau \leqslant \int_{t_0}^{t_0+\delta} z^2(\tau) d\tau + \frac{r}{a} k_3 \delta + k_4 \tag{A2.6.12}$$

where k_4 is another constant due to initial conditions. It follows that

$$\int_{t_0}^{t_0+\delta} y^2(\tau) d\tau \geqslant \frac{1}{k_1} \left(\int_{t_0}^{t_0+\delta} u^2(\tau) d\tau - \frac{r}{a} k_3 \delta - k_2 - k_4 \right)$$

$$\geqslant \frac{1}{k_1} \left(m \left(\beta_1 - \frac{r}{a} k_3 \sigma\right) - k_2 - k_4 \right) \tag{A2.6.13}$$

Note that r / a is arbitrary, and although k_1 depends on r / a, the constants β_1, k_3, and σ do not. Consequently, we can let r / a sufficiently small that $\beta_1 - (r/a) k_3 \sigma \geqslant \beta_1 / 2$. We can also let m be sufficiently large that $m \beta_1 / 2 - k_2 - k_4 \geqslant \beta_1$. Then the lower bound in (A2.6.6) is satisfied with

$$\alpha_1 = \frac{\beta_1}{k_1} \tag{A2.6.14}$$

□

Proof of Lemma 3.6.2

The proof of lemma 3.6.2 relies on the auxiliary lemma presented herafter.

Auxiliary Lemma

Consider the transfer function

$$\hat{K}(s) = \frac{a^r}{(s+a)^r} \qquad a > 0 \tag{A3.6.1}$$

where r is a positive integer.

Let $k(t)$ be the corresponding impulse response and define

$$g(t-\tau) = \int_{t-\tau}^{\infty} k(\sigma) d\sigma = \int_{-\infty}^{\tau} k(t-\sigma) d\sigma \qquad t-\tau \geqslant 0 \tag{A3.6.2}$$

Then

$$k(t) = \frac{a^r}{(r-1)!} t^{r-1} e^{-at} \qquad t \geqslant 0 \tag{A3.6.3}$$

and $k(t) = 0$ for $t < 0$. It follows that $k(t) \geqslant 0$ for all t, and

$$\| k \|_1 = \int_0^{\infty} k(\sigma) d\sigma = \int_{-\infty}^{t} k(t-\sigma) d\sigma = 1 \tag{A3.6.4}$$

Similarly

$$g(t) = e^{-at} \sum_{k=1}^{r} \frac{t^{r-k}}{(r-k)!} a^{r-k} \qquad t \geqslant 0 \tag{A3.6.5}$$

and $g(t) = 0$ for $t < 0$. It follows that $g(t) \geqslant 0$ for all t, and

$$\| g \|_1 = \int_0^{\infty} g(\sigma) d\sigma = \int_{-\infty}^{t} g(t-\sigma) d\sigma = \frac{r}{a} \tag{A3.6.6}$$

□

We are now ready to prove lemma 3.6.2. Let r be the relative degree of $\hat{H}$, and

$$\hat{z}(s) = \frac{a^r}{(s+a)^r} \hat{u}(s) \tag{A3.6.7}$$

where $a > 0$ is an arbitrary constant to be defined later. Using (A3.6.7)

$$\hat{y}(s) = \frac{(s+a)^r}{a^r} \hat{H}(s)\, \hat{z}(s) \tag{A3.6.8}$$

Since the transfer function from $\hat{z}(s)$ to $\hat{y}(s)$ has relative degree 0, and is minimum phase, it has a proper and stable inverse. By lemma 3.6.1

$$\| z_t \|_p \leq b_1 \| y_t \|_p + b_2 \tag{A3.6.9}$$

We will prove that

$$\| u_t \|_p \leq c_1 \| z_t \|_p + c_2 \tag{A3.6.10}$$

so that the lemma will be verified with $a_1 = c_1 b_1$, $a_2 = c_1 b_2 + c_2$.

Derivation of (A3.6.10)

We have that

$$z(t) = \epsilon(t) + \int_0^t k(t-\tau)\, u(\tau)\, d\tau \tag{A3.6.11}$$

where $\epsilon(t)$ is an exponentially decaying term due to the initial conditions, and $k(t)$ is the impulse response corresponding to the transfer function in (A3.6.7) (derived in the auxiliary lemma). Integrate (A3.6.11) by parts to obtain

$$z(t) = \epsilon(t) + u(t) \int_{-\infty}^{t} k(t-\sigma)\, d\sigma - u(0) \int_{-\infty}^{0} k(t-\sigma)\, d\sigma$$

$$- \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} k(t-\sigma)\, d\sigma \right) \dot{u}(\tau)\, d\tau \tag{A3.6.12}$$

Using the results of the auxiliary lemma

$$z(t) = \epsilon(t) + u(t) - u(0)\, g(t) - \int_0^t g(t-\tau)\, \dot{u}(\tau) d\tau \tag{A3.6.13}$$

Since $g(t)$ is exponentially decaying, $u(0)g(t)$ can be included in $\epsilon(t)$. Also, using again the auxiliary lemma, together with lemma 3.6.1, and then the assumption on $\dot{u}$, it follows that

$$\| u_t \|_p \leq \| z_t \|_p + \| \epsilon_t \|_p + \frac{r}{a} \| \dot{u}_t \|_p$$

$$\leq \| z_t \|_p + \| \epsilon_t \|_p + \frac{r}{a} k_1 \| u_t \|_p + \frac{r}{a} k_2 \tag{A3.6.14}$$

Since a is arbitrary, let it be sufficiently large that $\frac{r}{a} k_1 < 1$. Consequently

$$\|u_t\|_p \leqslant \frac{1}{1-\frac{r}{a}k_1} \|z_t\|_p + \frac{\|\epsilon\|_p + \frac{r}{a}k_2}{1-\frac{r}{a}k_1}$$

$$:= c_1 \|z_t\|_p + c_2 \tag{A3.6.15}$$

□

Proof of Corollary 3.6.3

(a) From lemma 3.6.2.

(b) Since $\hat{H}$ is strictly proper, both y and $\dot{y}$ are bounded.

(c) We have that $y = \hat{H}(u)$ and $\dot{y} = \hat{H}(\dot{u})$. Using succesively lemma 3.6.1, the regularity of u, and lemma 3.6.2, it follows that for some constants $k_1,...k_6$

$$\begin{aligned} |\dot{y}| &\leqslant k_1 \|\dot{u}_t\|_\infty + k_2 \\ &\leqslant k_3 \|u_t\|_\infty + k_4 \\ &\leqslant k_5 \|y_t\|_\infty + k_6 \end{aligned} \tag{3.6.16}$$

The proof can easily be extended to the vector case. □

Proof of Lemma 3.6.4

Let

$$\hat{H}(s) = h_0 + \hat{H}_1(s) \tag{A3.6.17}$$

where $\hat{H}_1$ is strictly proper (and stable). Let h_1 be the impulse response corresponding to $\hat{H}_1$. The output $y(t)$ is given by

$$y(t) = \epsilon(t) + h_0 u(t) + \int_0^t h_1(t-\tau)\, u(\tau)\, d\tau \tag{A3.6.18}$$

where $\epsilon(t)$ is due to the initial conditions. Inequality (3.6.9) follows, if we define

$$\gamma_1(t) := |h_0|\, \beta_1(t) + \int_0^t |h_1(t-\tau)|\, \beta_1(\tau)\, d\tau \tag{A3.6.19}$$

and

$$\gamma_2(t) := |\epsilon(t)| + |h_0|\beta_2(t) + \int_0^t |h_1(t-\tau)|\beta_2(\tau)\,d\tau \tag{A3.6.20}$$

Since $\epsilon \in L_2$, and $h_1 \in L_1 \cap L_\infty$, we also have that $|\epsilon| \in L_2$, $|h_1| \in L_1 \cap L_\infty$. Since $\beta_1, \beta_2 \in L_2$, it follows that the last term of (A3.6.19), and similarly the last term of (A3.6.20) belong to $L_2 \cap L_\infty$, and go to zero as $t \to \infty$ (see e.g. Desoer and Vidyasagar (1975), exercise 5, p. 242). The conclusions follow directly from this observation. □

Proof of Lemma 3.6.5

Let $[A, b, c^T, d]$ be a minimal realization of $\hat{H}$, with $A \in \mathbf{R}^{m \times m}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^m$, and $d \in \mathbf{R}$. Let $x : \mathbf{R}_+ \to \mathbf{R}^m$, and $y_1 : \mathbf{R}_+ \to \mathbf{R}$ such that

$$\dot{x} = A\,x + b\,(w^T \phi)$$
$$y_1 = c^T x \tag{A3.6.21}$$

and $W : \mathbf{R}_+ \to \mathbf{R}^{m \times n}$, $y_2 : \mathbf{R}_+ \to \mathbf{R}$ such that

$$\dot{W} = A\,W + b\,w^T$$
$$y_2 = c^T\,W\,\phi \tag{A3.6.22}$$

Thus

$$\hat{H}\,(w^T \phi) = y_1 + d\,(w^T \phi) \qquad \hat{H}\,(w^T)\phi = y_2 + (d\,w^T)\phi \tag{A3.6.23}$$

Since

$$\frac{d}{dt}\,(W\phi) = \dot{W}\,\phi + W\,\dot{\phi} = A\,W\,\phi + b\,w^T\,\phi + W\,\dot{\phi} \tag{A3.6.24}$$

it follows that

$$\frac{d}{dt}\,(x - W\,\phi) = A\,(x - W\,\phi) - W\,\dot{\phi}$$
$$y_1 - y_2 = c^T\,(x - W\,\phi) \tag{A3.6.25}$$

The result then follows since

$$\hat{H}(w^T \phi) - \hat{H}(w^T)\phi = y_1 - y_2 = \hat{H}_c\,(W\dot{\phi}) = \hat{H}_c\,(\hat{H}_b(w^T)\dot{\phi}) \tag{A3.6.26}$$

□

Proof of Lemma 4.2.1

Define

$$w_\epsilon(t,x) = \int_0^t d(\tau,x)e^{-\epsilon(t-\tau)}d\tau \tag{A4.2.1}$$

and

$$w_0(t,x) = \int_0^t d(\tau,x)d\tau \tag{A4.2.2}$$

From the assumptions

$$|w_0(t+t_0,x) - w_0(t_0,x)| \leqslant \gamma(t).t \tag{A4.2.3}$$

for all $t, t_0 \geqslant 0$, $x \in B_h$. Integrating (A4.2.1) by parts

$$w_\epsilon(t,x) = w_0(t,x) - \epsilon\int_0^t e^{-\epsilon(t-\tau)}w_0(\tau,x)\,d\tau \tag{A4.2.4}$$

Using the fact that

$$\epsilon\int_0^t e^{-\epsilon(t-\tau)}w_0(t,x)\,d\tau = w_0(t,x) - w_0(t,x)\,e^{-\epsilon t} \tag{A4.2.5}$$

(A4.2.4) can be rewritten as

$$w_\epsilon(t,x) = w_0(t,x)e^{-\epsilon t} + \epsilon\int_0^t e^{-\epsilon(t-\tau)}(w_0(t,x) - w_0(\tau,x))\,d\tau \tag{A4.2.6}$$

and, using (A4.2.3) and the fact that $w_0(0,x) = 0$

$$|w_\epsilon(t,x)| \leqslant \gamma(t)\,t\,e^{-\epsilon t} + \epsilon\int_0^t e^{-\epsilon(t-\tau)}(t-\tau)\,\gamma(t-\tau)d\tau \tag{A4.2.7}$$

Consequently

$$|\epsilon w_\epsilon(t,x)| \leqslant \sup_{t'\geqslant 0}\gamma\left(\frac{t'}{\epsilon}\right)t'e^{-t'} + \int_0^\infty \gamma\left(\frac{\tau'}{\epsilon}\right)\tau' e^{-\tau'}d\tau' \tag{A4.2.8}$$

Since, for some β, $|d(t,x)| \leqslant \beta$, we also have that $\gamma(t) \leqslant \beta$. Note that, for all $t' \geqslant 0$, $t'e^{-t'} \leqslant e^{-1}$, and $t'e^{-t'} \leqslant t'$, so that

$$|\epsilon w_\epsilon(t,x)| \leqslant \sup_{t'\in[0,\sqrt{\epsilon}]}\left|\gamma\left(\frac{t'}{\epsilon}\right)t'e^{-t'}\right| + \sup_{t'\geqslant\sqrt{\epsilon}}\left|\gamma\left(\frac{t'}{\epsilon}\right)t'e^{-t'}\right|$$

$$+\int_0^{\sqrt{\epsilon}} \gamma(\frac{\tau'}{\epsilon})\tau' e^{-\tau'} d\tau' + \int_{\sqrt{\epsilon}}^{\infty} \gamma(\frac{\tau'}{\epsilon})\tau' e^{-\tau'} d\tau' \quad \text{(A4.2.9)}$$

This, in turn, implies that

$$|\epsilon w_\epsilon(t,x)| \leqslant \beta\sqrt{\epsilon} + \gamma(\frac{1}{\sqrt{\epsilon}})e^{-1} + \beta\frac{\epsilon}{2} + \gamma(\frac{1}{\sqrt{\epsilon}})(1+\sqrt{\epsilon})e^{-\sqrt{\epsilon}}$$

$$:= \xi(\epsilon) \quad \text{(A4.2.10)}$$

From the assumption on γ, it follows that $\xi(\epsilon) \in K$. From (A4.2.1)

$$\frac{\partial w_\epsilon(t,x)}{\partial t} - d(t,x) = -\epsilon\, w_\epsilon(t,x) \quad \text{(A4.2.11)}$$

so that the first part of the lemma is verified.

If $\gamma(T) = a / T^r$, then the right-hand side of (A4.2.8) can be computed explicitly

$$\sup_{t' \geqslant 0} a\,\epsilon^r (t')^{1-r} e^{-t'} = a\,\epsilon^r (1-r)^{1-r} e^{r-1} \leqslant a\,\epsilon^r \quad \text{(A4.2.12)}$$

and, with Γ denoting the standard gamma function

$$\int_0^{\infty} a\,\epsilon^r (\tau')^{1-r} e^{-\tau'} d\tau' = a\,\epsilon^r\,\Gamma(2-r) \leqslant a\,\epsilon^r \quad \text{(A4.2.13)}$$

Defining $\xi(\epsilon) = 2a\,\epsilon^r$, the second part of the lemma is verified. □

Proof of Lemma 4.2.2

Define $w_\epsilon(t,x)$ as in lemma 4.2.1. Consequently,

$$\frac{\partial w_\epsilon(t,x)}{\partial x} = \frac{\partial}{\partial x}\left(\int_0^t d(\tau,x) e^{-\epsilon(t-\tau)} d\tau\right) = \int_0^t \left(\frac{\partial}{\partial x} d(\tau,x)\right) e^{-\epsilon(t-\tau)} d\tau \quad \text{(A4.2.14)}$$

Since $\frac{\partial d(t,x)}{\partial x}$ is zero mean, and is bounded, lemma 4.2.1 can be applied to $\frac{\partial d(t,x)}{\partial x}$, and inequality (4.2.6) of lemma 4.2.1 becomes inequality (4.2.10) of lemma 4.2.2. Note that since $\frac{\partial d(t,x)}{\partial x}$ is bounded, and $d(t,0) = 0$ for all $t \geqslant 0$, $d(t,x)$ is Lipschitz.

Since $d(t,x)$ is zero mean, with convergence function $\gamma(T)|x|$, the proof of lemma 4.2.1 can be extended, with an additional factor $|x|$. This leads directly to (4.2.8) and (4.2.9) (although the function $\xi(\epsilon)$ may be different from that obtained with $\frac{\partial d(t,x)}{\partial x}$.

these functions can be replaced by a single $\xi(\epsilon)$). □

Proof of Lemma 4.2.3

The proof proceeds in two steps.

Step 1: for ϵ sufficiently small, and for t fixed, the transformation is a homeomorphism.

Apply lemma 4.2.2, and let ϵ_1 such that $\xi(\epsilon_1) < 1$. Let $\epsilon \leqslant \epsilon_1$. Given $z \in B_h$, the corresponding x such that

$$x = z + \epsilon\, w_\epsilon(t, z) \tag{A4.2.15}$$

may not belong to B_h. Similarly, given $x \in B_h$, the solution z of (A4.2.15) may not exist in B_h. However, for any x, z satisfying (A4.2.15), inequality (4.2.8) implies (4.2.16) and

$$(1-\xi(\epsilon))\,|z| \leqslant |x| \leqslant (1+\xi(\epsilon))\,|z| \tag{A4.2.16}$$

Define

$$h'(\epsilon) = \min\left(h\,(1-\xi(\epsilon)), \frac{h}{1+\xi(\epsilon)}\right) = h\,(1-\xi(\epsilon)) \tag{A4.2.17}$$

and note that $h'(\epsilon) \to h$ as $\epsilon \to 0$.

We now show that

- for all $z \in B_{h'}$, there exists a unique $x \in B_h$ such that (A4.2.15) is satisfied,
- for all $x \in B_{h'}$, there exists a unique $z \in B_h$ such that (A4.2.15) is satisfied.

In both cases, $|x - z| \leqslant \xi(\epsilon) h$.

The first part follows directly from (A4.2.15), (A4.2.16). The fact that $|x - z| \leqslant \xi(\epsilon) h$ also follows from (A4.2.15), (4.2.8), and implies that, if a solution z exists to (A4.2.15), it must lie in the closed ball U of radius $\xi(\epsilon) h$ around x. It can be checked, using (4.2.10), that the mapping $F_x(z) = x - \epsilon w_\epsilon(t, z)$ is a contraction mapping in U, provided that $\xi(\epsilon) < 1$. Consequently, F has a unique fixed point z in U. This solution is also a solution of (A4.2.15), and since it is unique in U, it is also unique in B_h (and actually in R^n). For $x \in B_h$, but outside $B_{h'}$, there is no guarantee that a solution z exists in B_h, but if it exists, it is again unique in B_h. Consequently, the map $x \to z$ defined by (A4.2.15) is well-defined. From the smoothness of $w_\epsilon(t, z)$ with respect to z, it follows that the map is a homeomorphism.

Step 2: the transformation of variable leads to the differential equation (4.2.17)

Applying (A4.2.15) to the system (4.2.1)

$$(I+\epsilon\frac{\partial w_\epsilon}{\partial z})\dot z=\epsilon f_{av}(z)+\epsilon(f(t,z,0)-f_{av}(z)-\frac{\partial w_\epsilon}{\partial z})$$

$$+\epsilon(f(t,z+\epsilon w_\epsilon,\epsilon)-f(t,z,\epsilon))+\epsilon(f(t,z,\epsilon)-f(t,z,0))$$

$$:=\epsilon f_{av}(z)+\epsilon p'(t,x,z,\epsilon) \qquad \text{(A4.2.18)}$$

where, using the assumptions, and the results of lemma 4.2.2

$$|p'(t,z,\epsilon)|\leqslant\xi(\epsilon)|z|+\xi(\epsilon)l_1|z|+\epsilon l_2|z| \qquad \text{(A4.2.19)}$$

For $\epsilon\leqslant\epsilon_1$, (4.2.10) implies that $(I+\epsilon\frac{\partial w_\epsilon}{\partial z})$ has a bounded inverse for all $t\geqslant 0$, $z\in B_h$. Consequently, z satisfies the differential equation

$$\dot z=\left[I+\epsilon\frac{\partial w_\epsilon}{\partial z}\right]^{-1}(\epsilon f_{av}(z)+\epsilon p'(t,z,\epsilon))$$

$$=\epsilon f_{av}(z)+\epsilon p(t,z,\epsilon) \qquad z(0)=x_0 \qquad \text{(A4.2.20)}$$

where

$$p(t,z,\epsilon)=\left[I+\epsilon\frac{\partial w_\epsilon}{\partial z}\right]^{-1}\left[p'(t,z,\epsilon)-\epsilon\frac{\partial w_\epsilon}{\partial z}f_{av}(z)\right] \qquad \text{(A4.2.21)}$$

and

$$|p(t,z,\epsilon)|\leqslant\frac{1}{1-\xi(\epsilon_1)}(\xi(\epsilon)+\xi(\epsilon)l_1+\epsilon l_2+\xi(\epsilon)l_{av})|z|$$

$$:=\psi(\epsilon)|z| \qquad \text{(A4.2.22)}$$

for all $t\geqslant 0$, $\epsilon\leqslant\epsilon_1$, $z\in B_h$. □

Proof of Lemma 4.4.1

The proof is similar to the proof of lemma 4.2.3. We consider the transformation of variable

$$x = z + \epsilon w_\epsilon(t, z) \tag{A4.4.1}$$

with $\epsilon \leq \epsilon_1$, such that $\xi(\epsilon_1) < 1$. (4.4.1) becomes

$$\dot{z} = (I + \epsilon \frac{\partial w_\epsilon}{\partial z})^{-1} \epsilon \Bigg\{ f_{av}(z) + (f(t, z, 0) - f_{av}(z) - \frac{\partial w_\epsilon}{\partial t})$$

$$+ (f(t, z + \epsilon w_\epsilon, 0) - f(t, z, 0))$$

$$+ (f(t, z + \epsilon w_\epsilon, y) - f(t, z + \epsilon w_\epsilon, 0)) \Bigg\} \tag{A4.4.2}$$

or

$$\dot{z} = \epsilon f_{av}(z) + \epsilon p_1(t, z, \epsilon) + \epsilon p_2(t, z, y, \epsilon) \qquad z(0) = x_0 \tag{A4.4.3}$$

where

$$| p_1(t, z, \epsilon) | \leq \frac{1}{1 - \xi(\epsilon_1)} (\xi(\epsilon) l_{av} + \xi(\epsilon) + \xi(\epsilon) l_1) | z | := \xi(\epsilon) k_1 | z | \tag{A4.4.4}$$

and

$$| p_2(t, z, y, \epsilon) | \leq \frac{1}{1 - \xi(\epsilon_1)} l_2 | y | := k_2 | y | \tag{A4.4.5}$$

□

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Figures

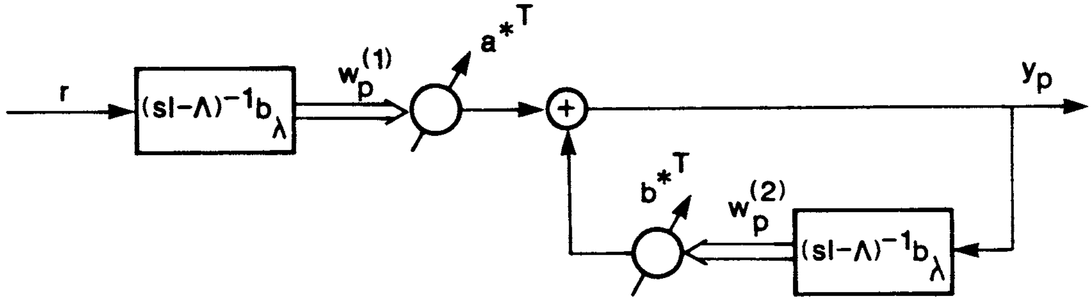


Figure 2.1 Plant Parametrization

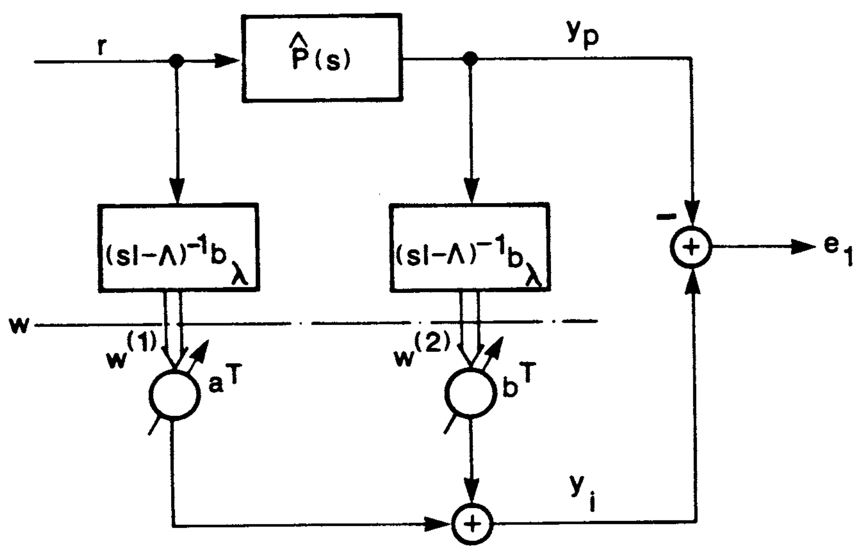


Figure 2.2 Identifier Structure

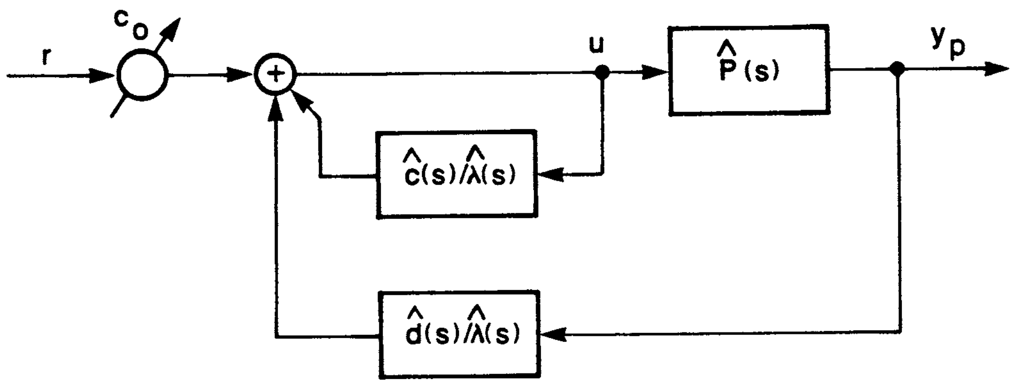


Figure 3.1 Controller Structure

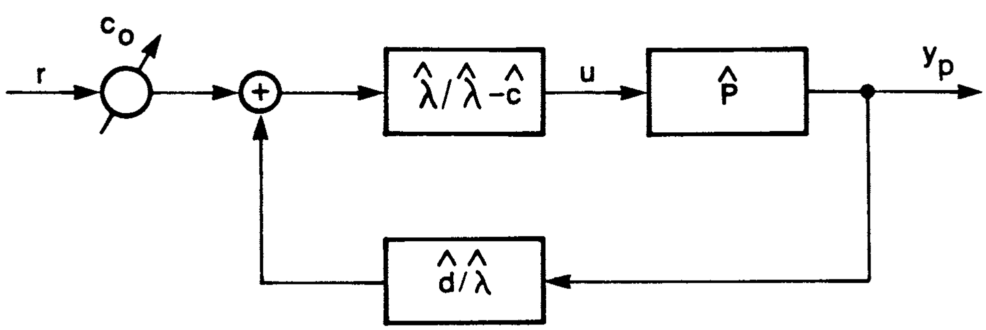


Figure 3.2 Controller Structure - Equivalent Form

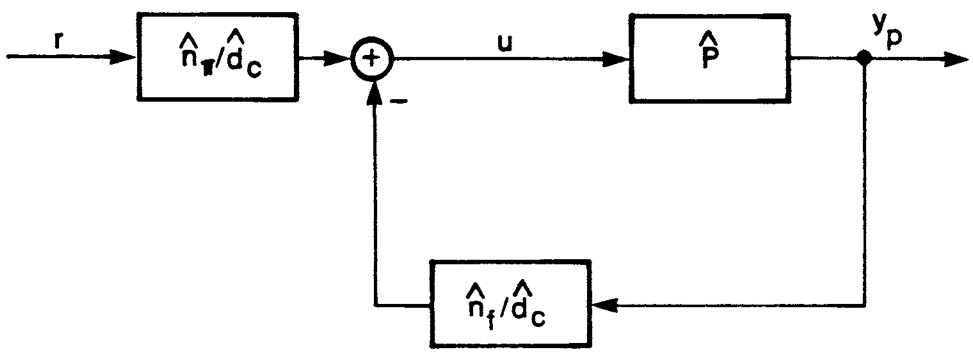


Figure 3.3 Alternate Controller Structure

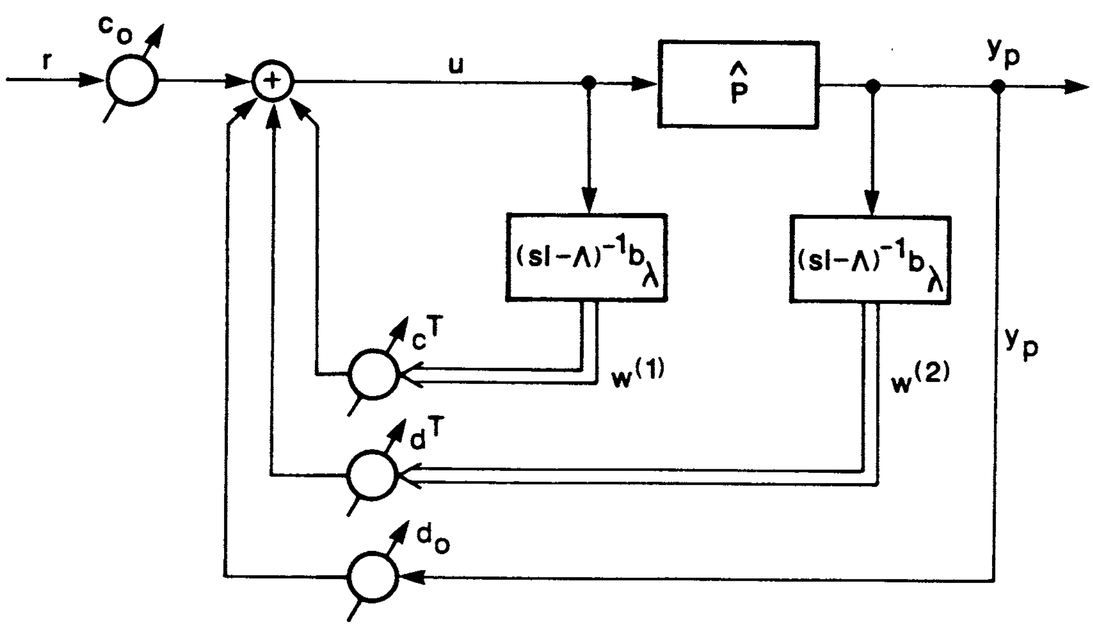


Figure 3.4 Controller Structure - Adaptive Form

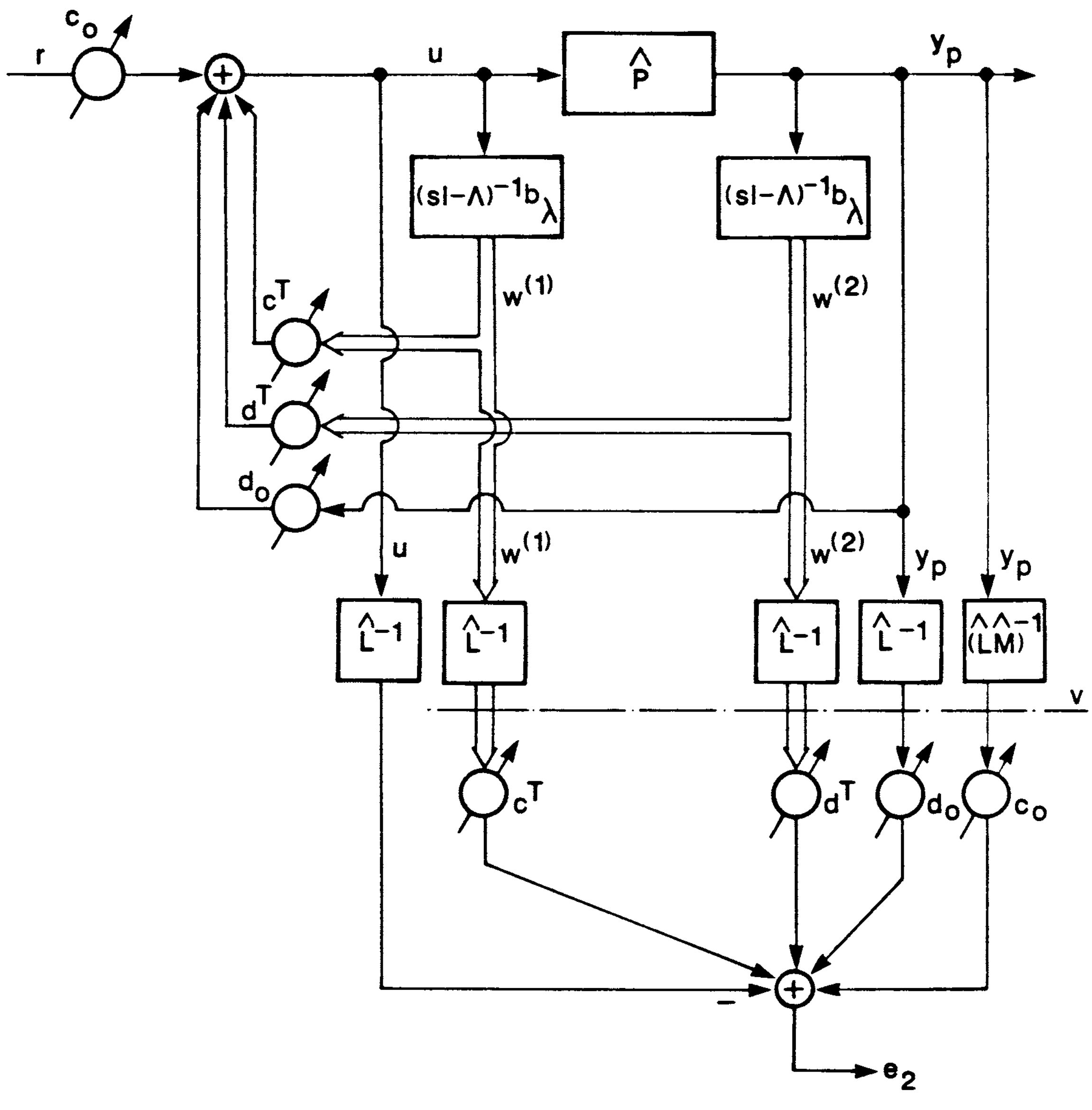


Figure 3.5 Controller and Input Error Identifier Structures

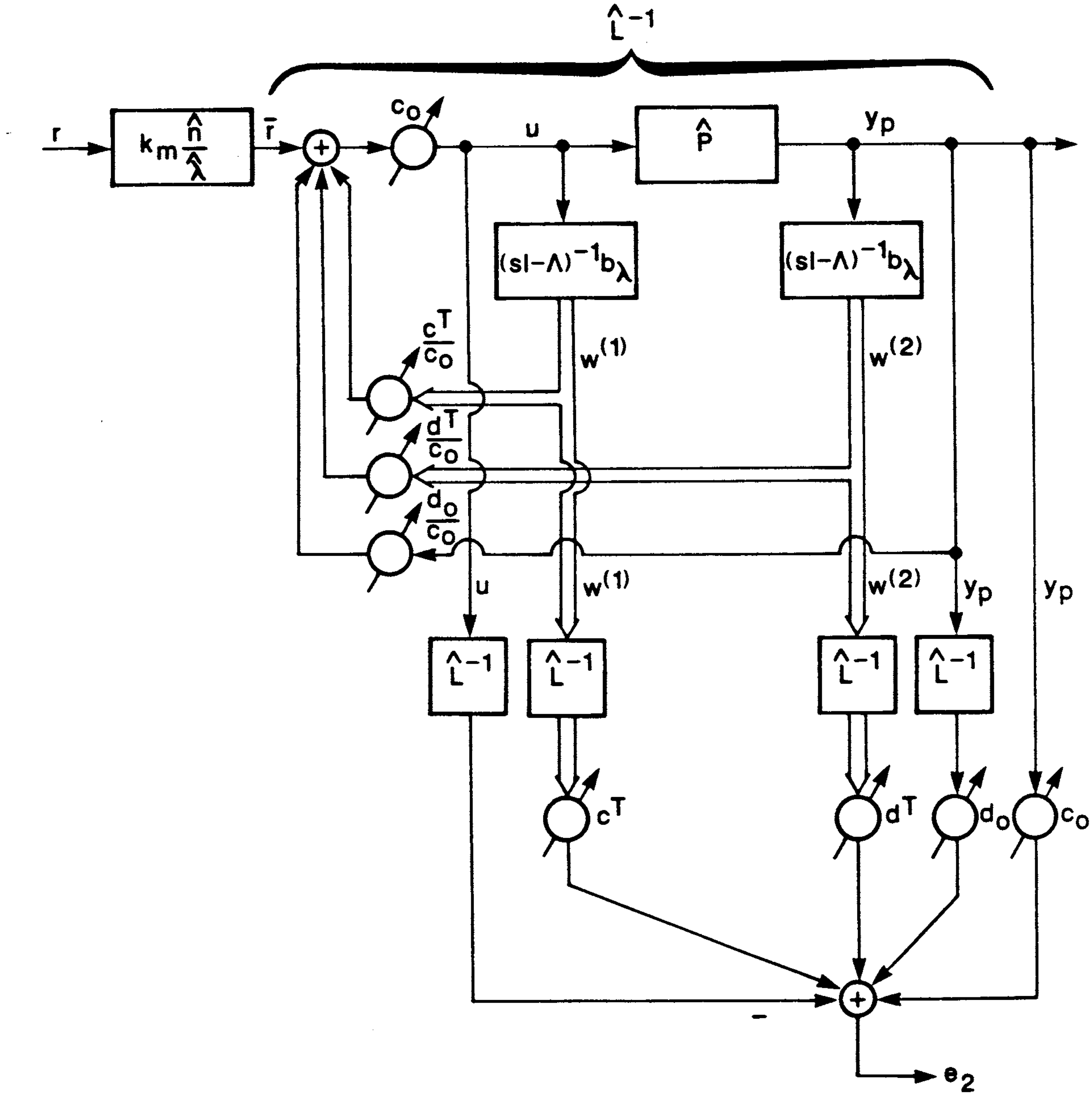


Figure 3.6 Alternate Input Error Scheme

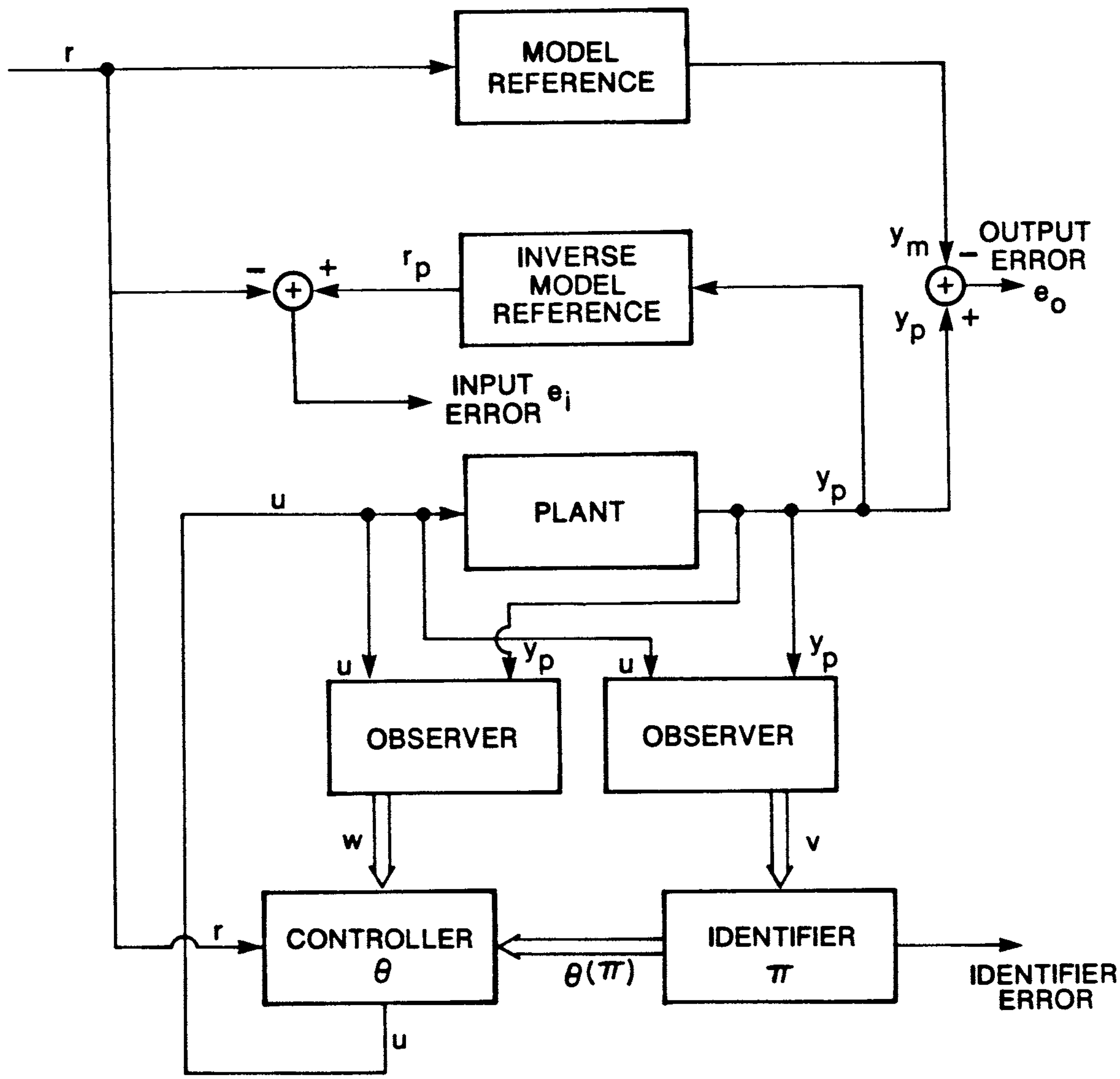


Figure 3.7 Generic Model Reference Adaptive Control System

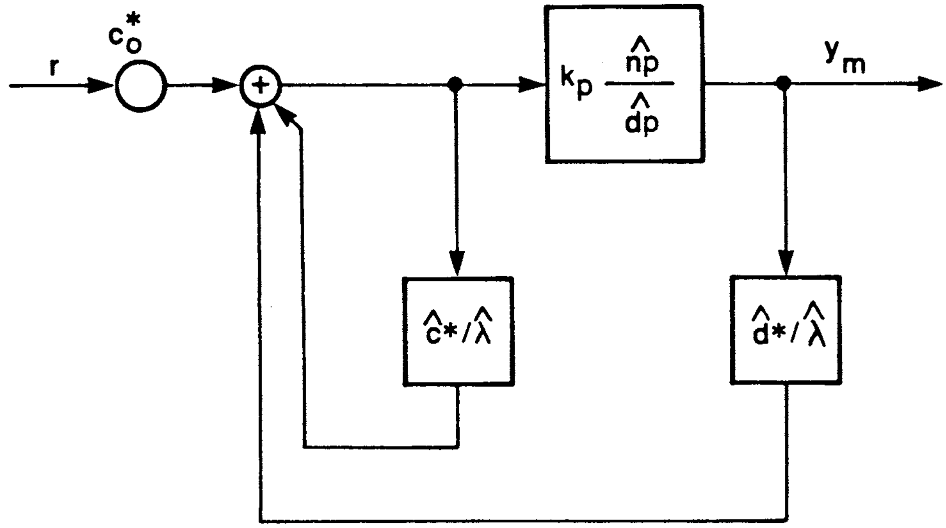


Figure 3.8 Representation of the Reference Model

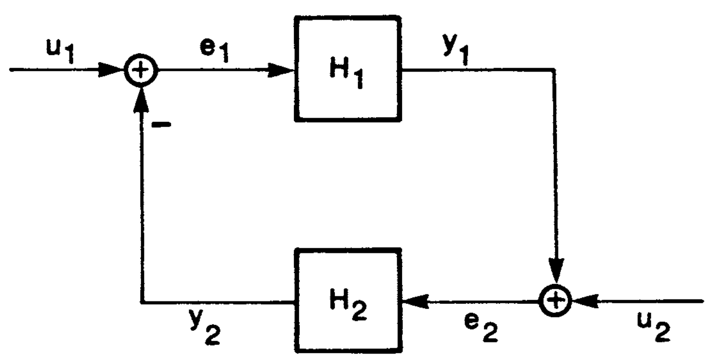


Figure 3.9 Feedback System for the Small Gain Theorem

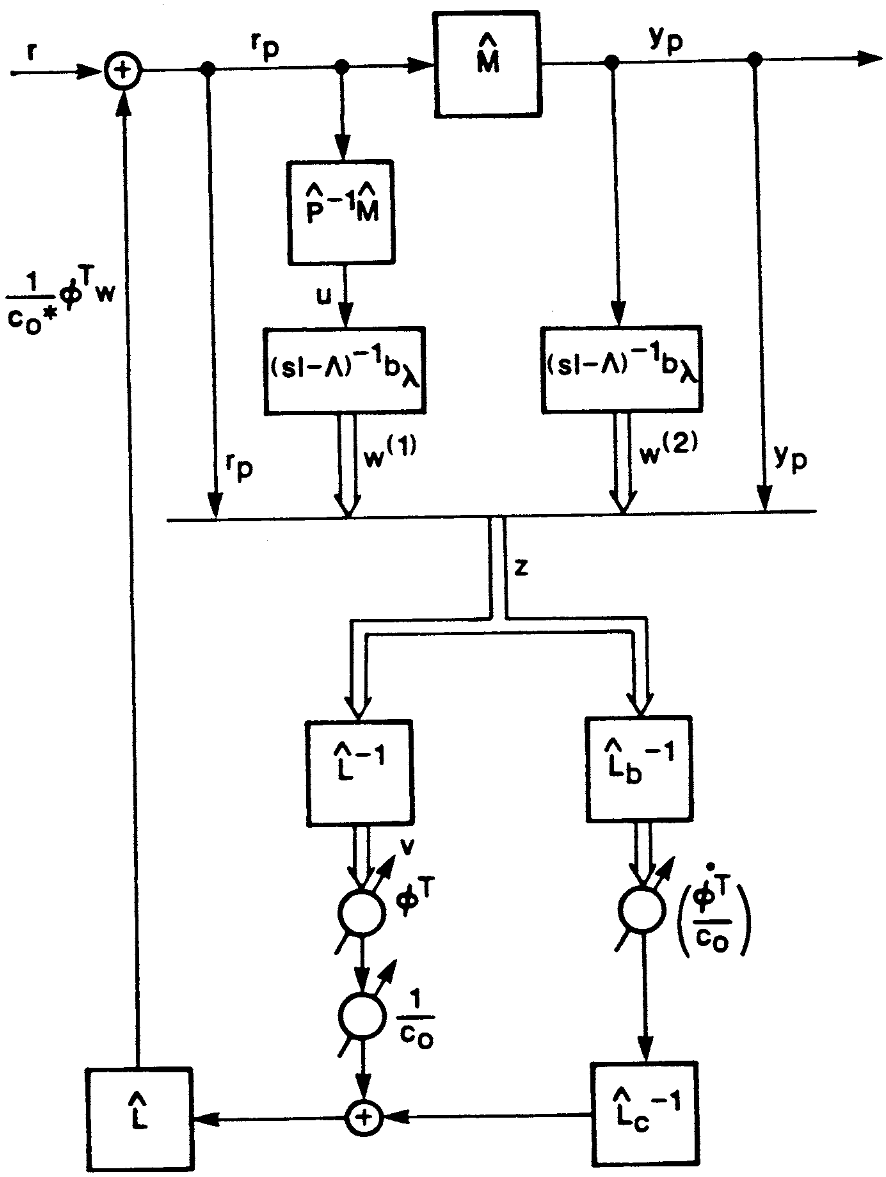


Figure 3.10 Representation of the Plant for the Stability Analysis

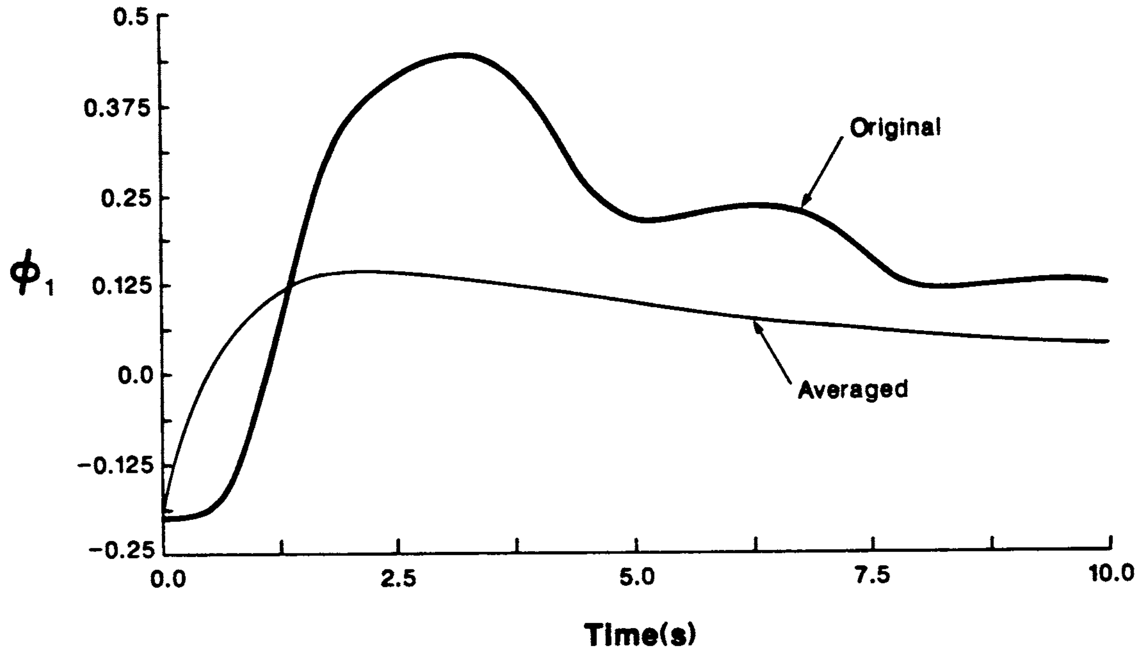


Figure 4.1 Parameter Error ϕ_1 ($\epsilon = 1$)

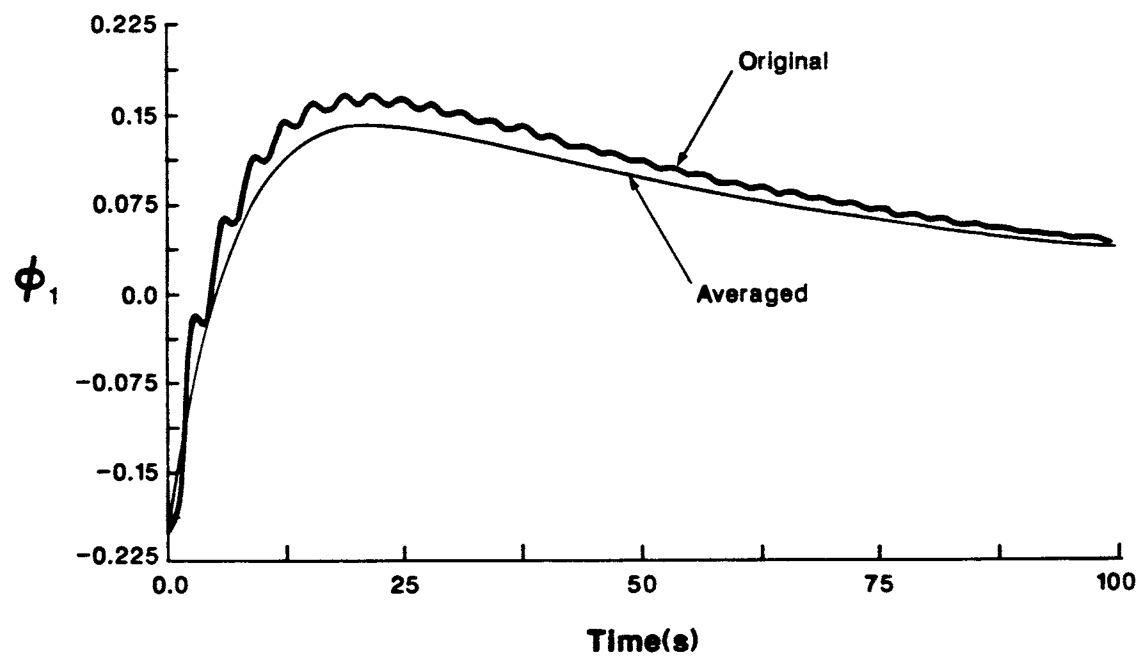


Figure 4.2 Parameter Error ϕ_1 ($\epsilon = 0.1$)

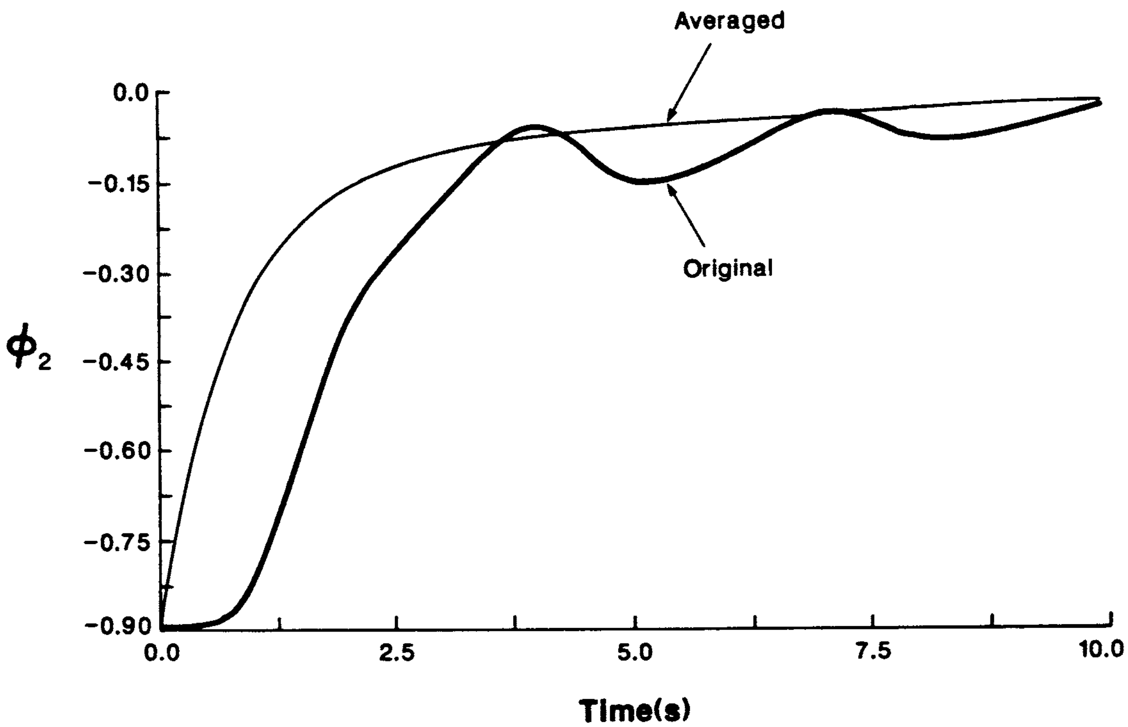
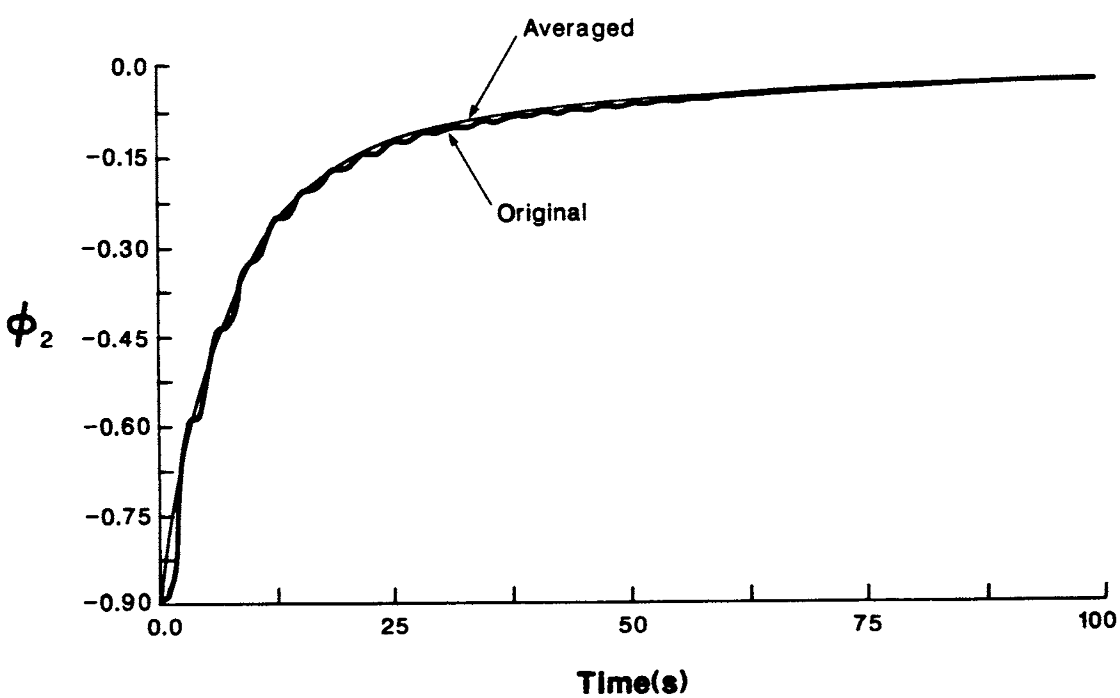


Figure 4.3 Parameter Error ϕ_2 ($\epsilon = 1$)



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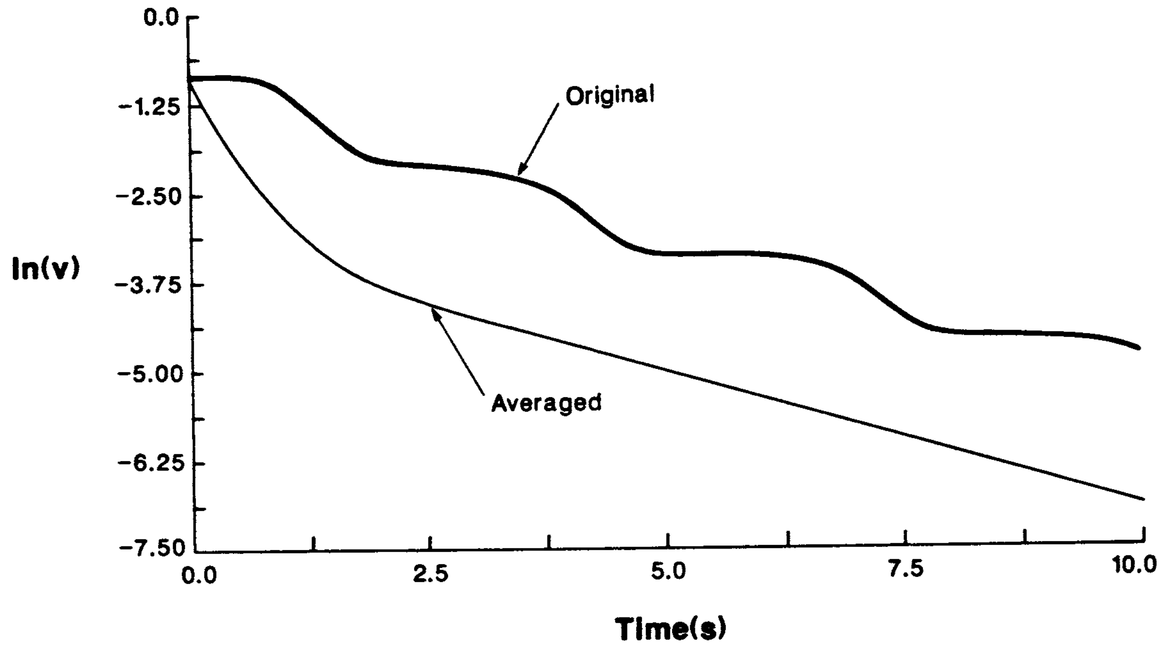

Figure 4.5 Logarithm of the Lyapunov Function ($\epsilon = 1$)

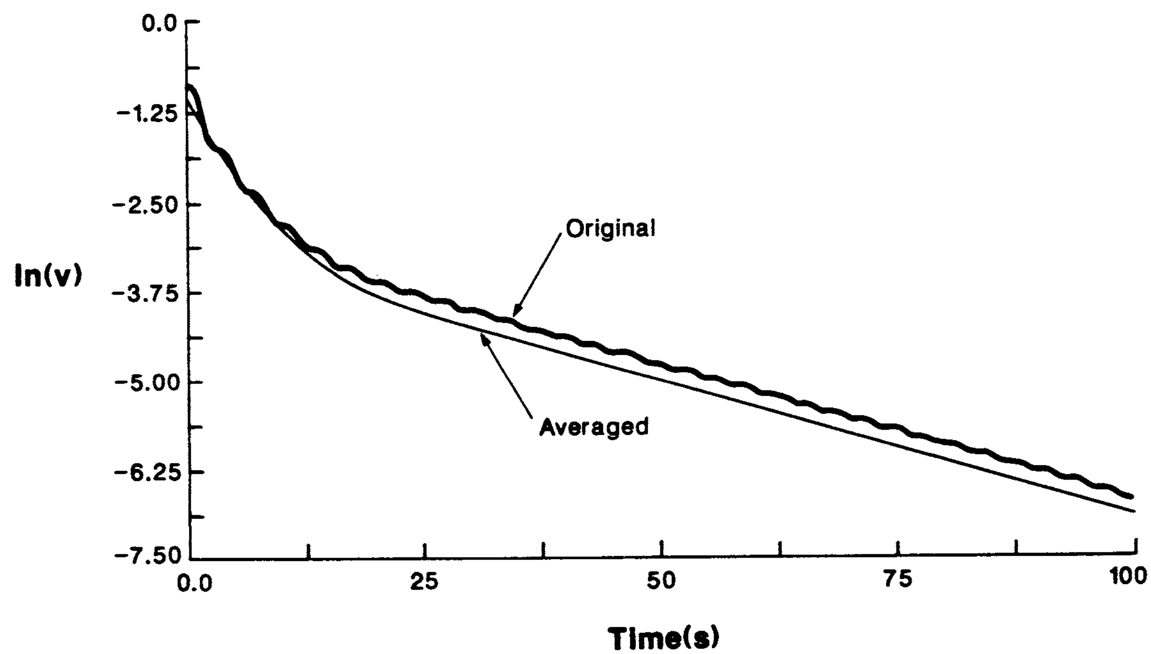

Figure 4.6 Logarithm of the Lyapunov Function ($\epsilon = 0.1$)

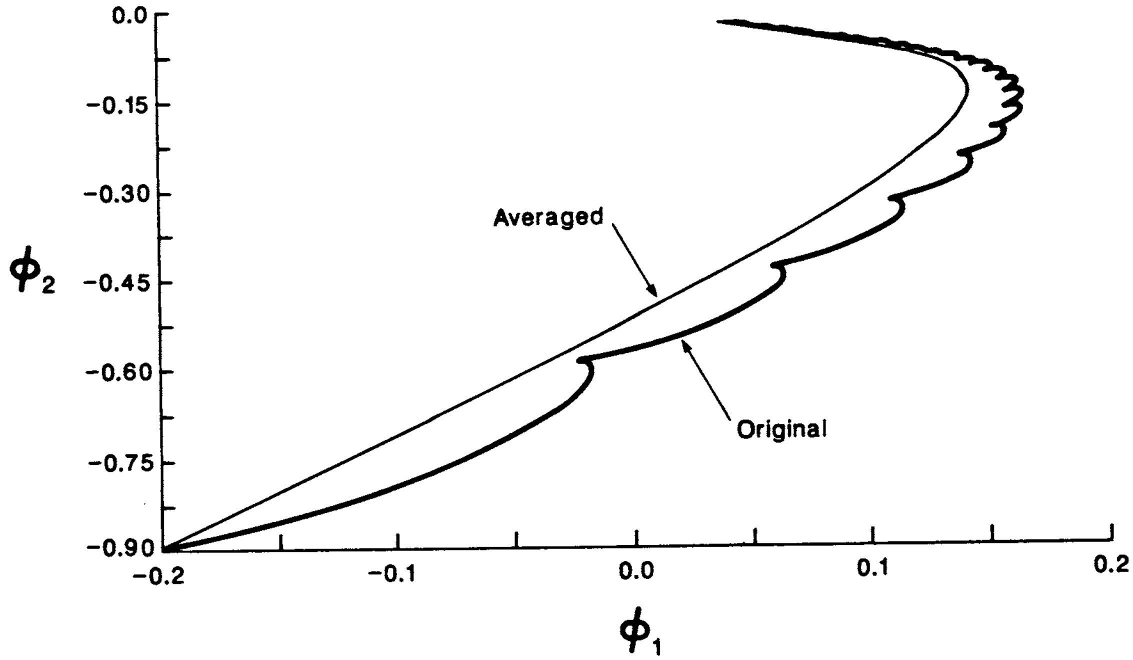

Figure 4.7 Parameter Error $\phi_2(\phi_1)$ ($\epsilon = 0.1$)

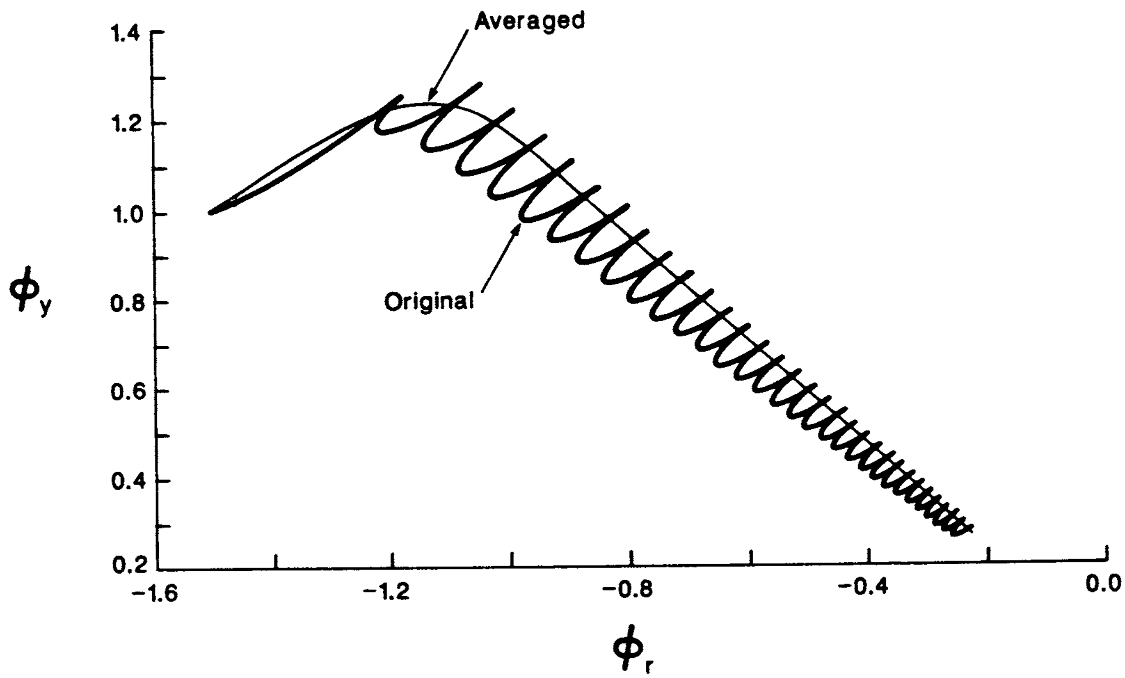

Figure 4.8 Parameter Error $\phi_y(\phi_r)$ ($r = \sin t$)

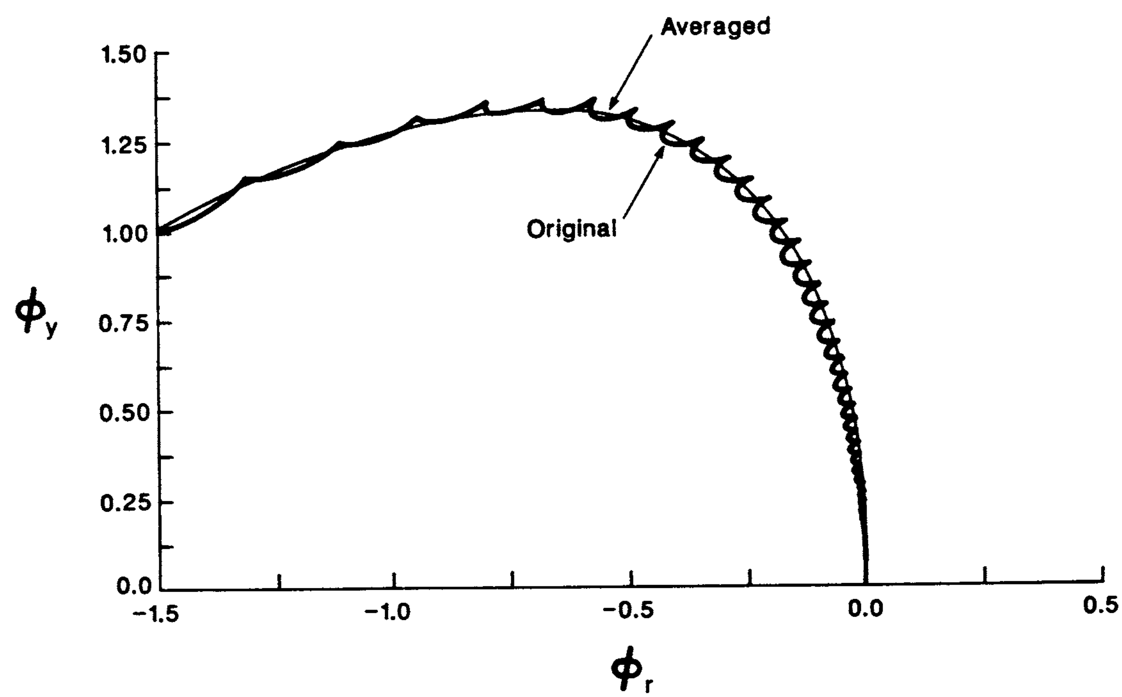

Figure 4.9 Parameter Error $\phi_y(\phi_r)$ ($r = \sin 3t$)

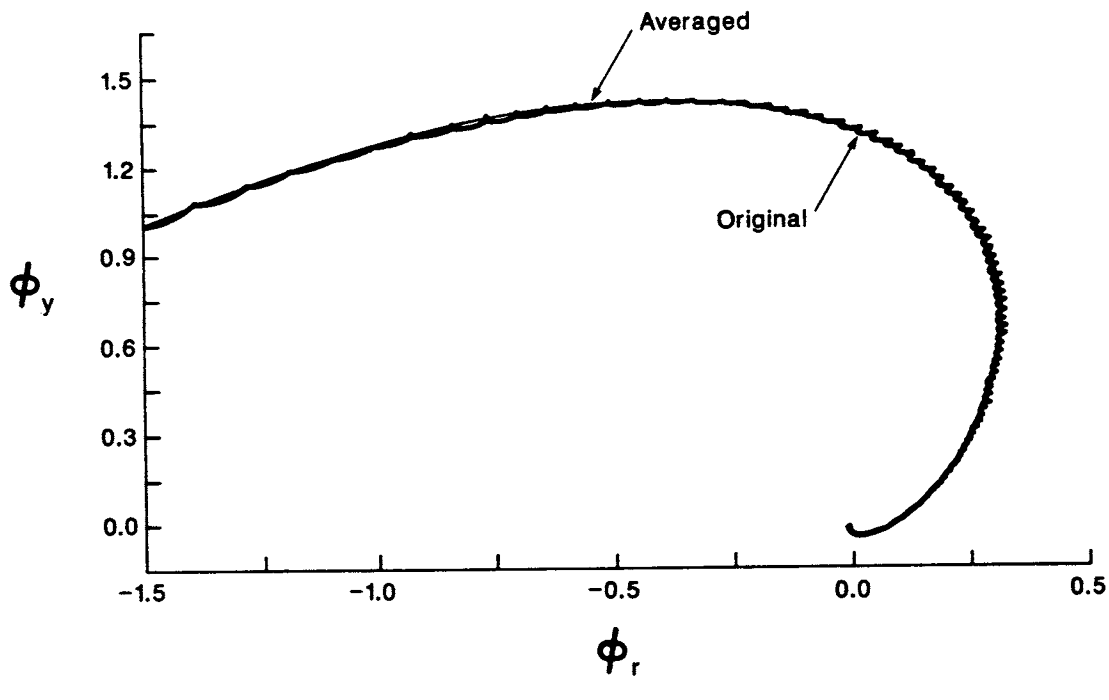

Figure 4.10 Parameter Error $\phi_y\,(\phi_r)$ $(r = \sin 5t)$

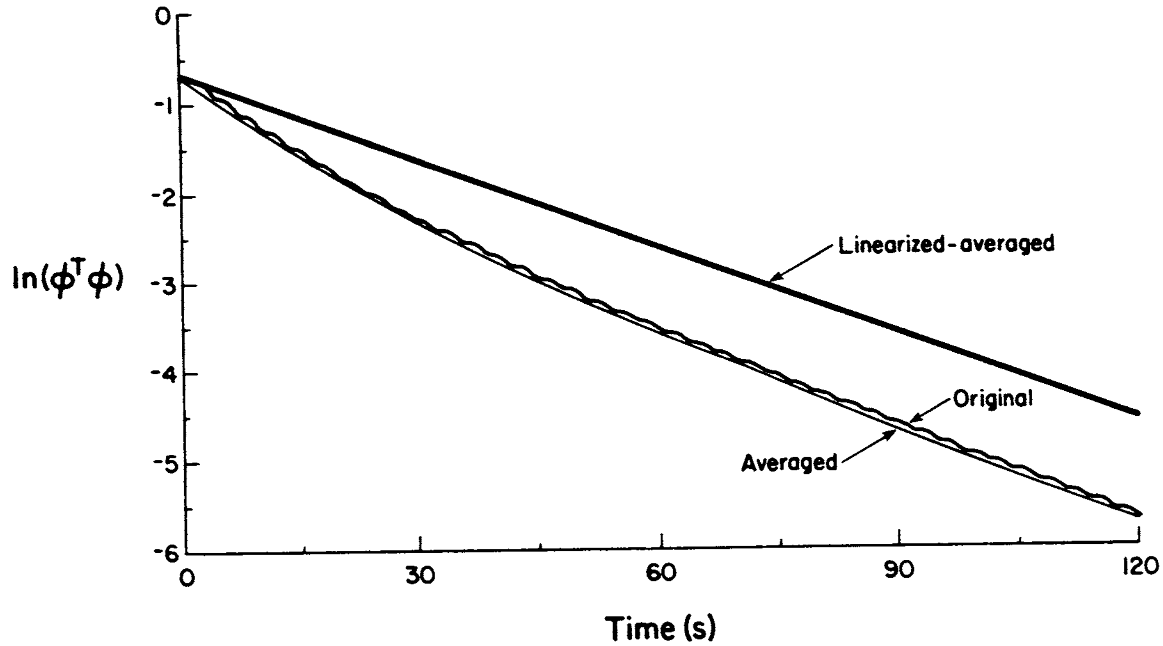

Figure 4.11 Logarithm of the Lyapunov Function

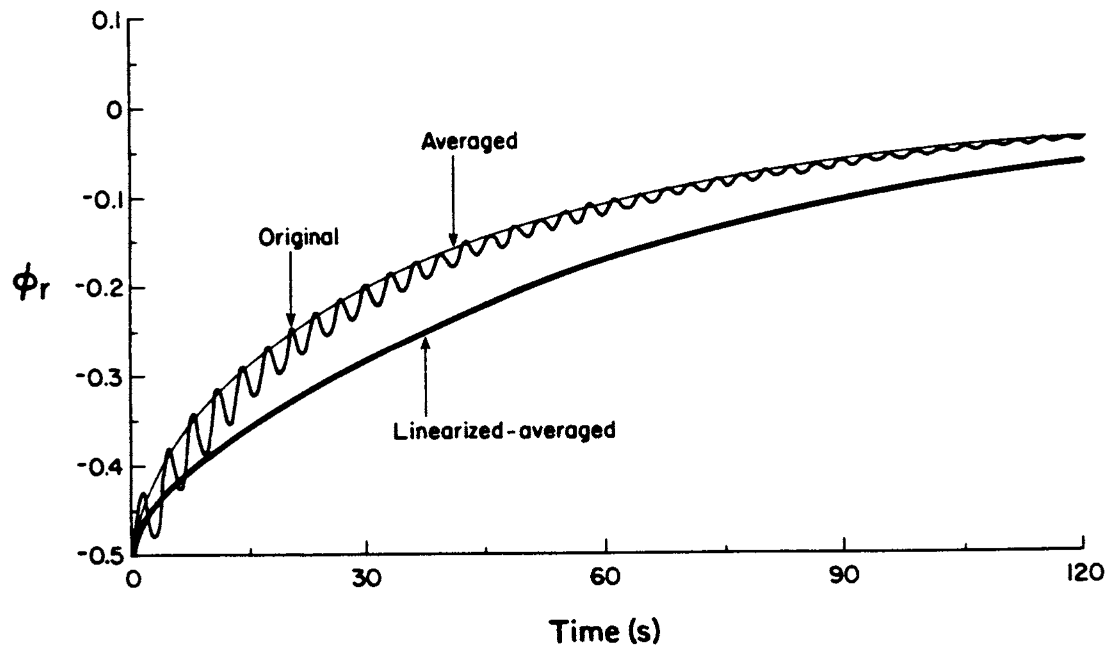

Figure 4.12 Parameter Error ϕ_r

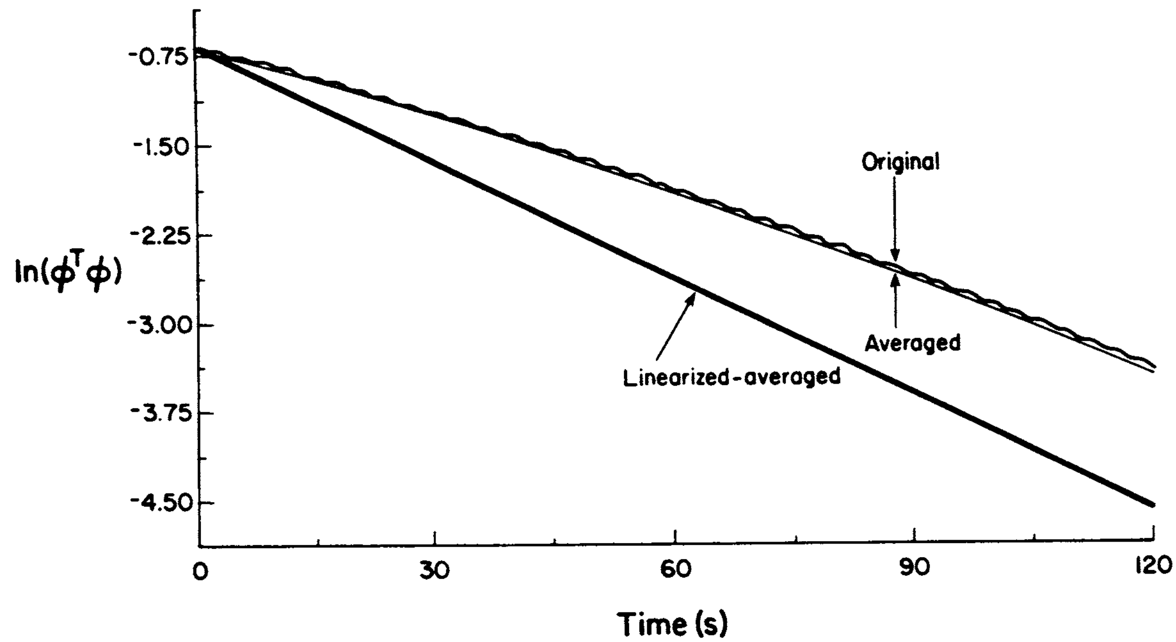

Figure 4.13 Parameter Error ϕ_y

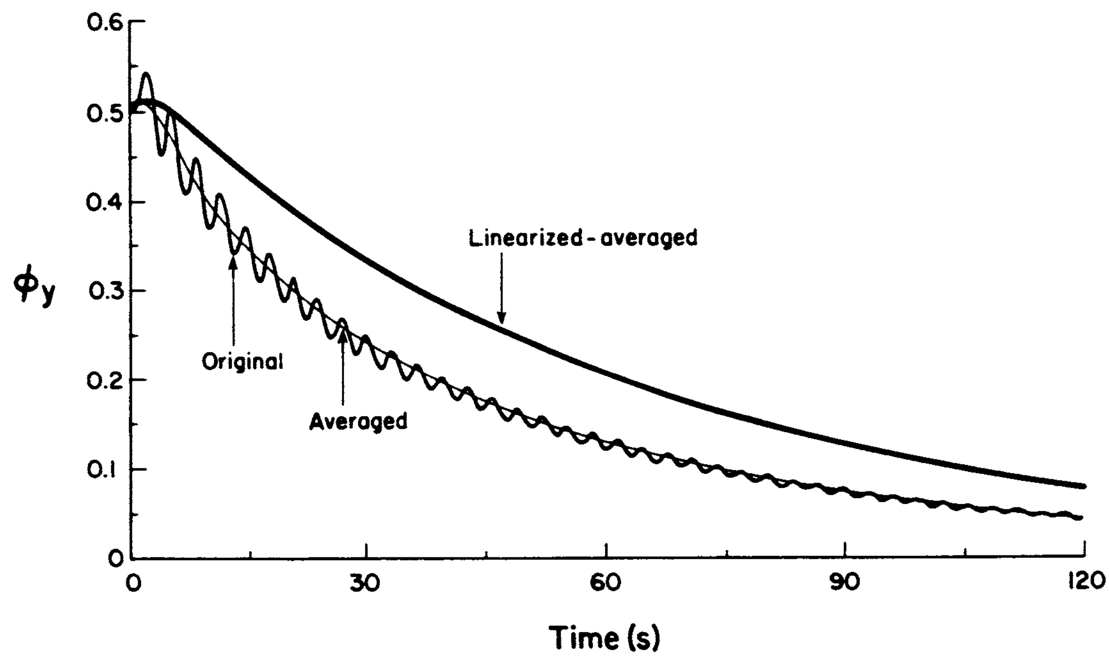

Figure 4.14 Logarithm of the Lyapunov Function

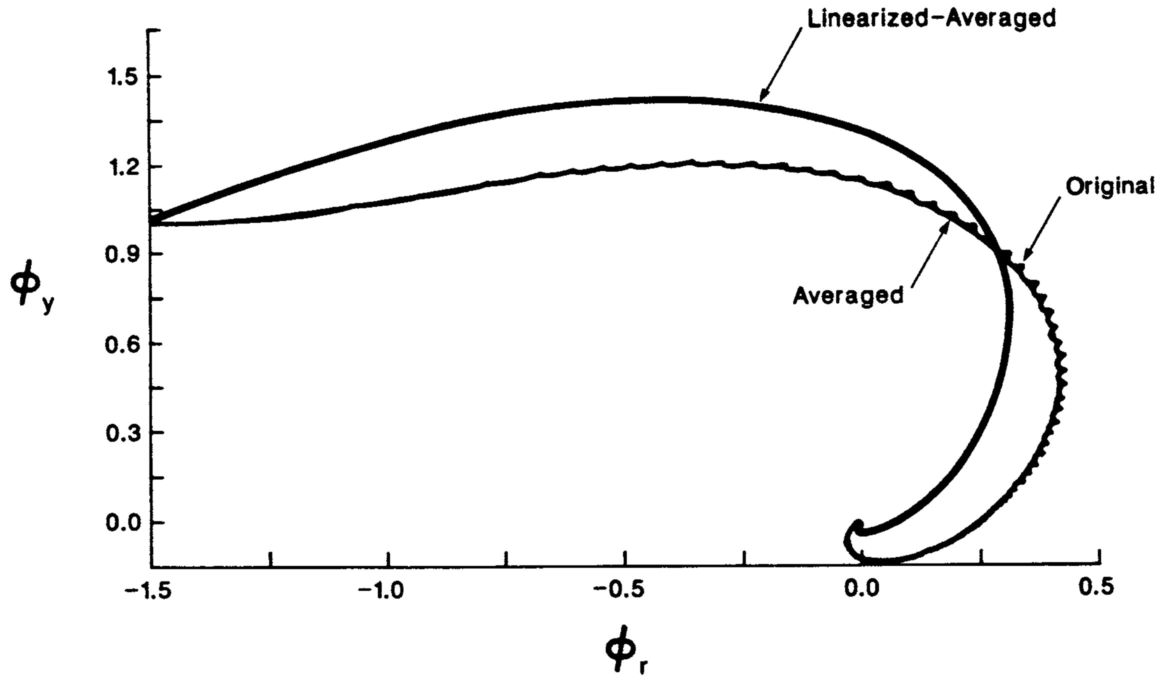

Figure 4.15 Parameter Error $\phi_y(\phi_r)$ ($r = \sin 5t$)

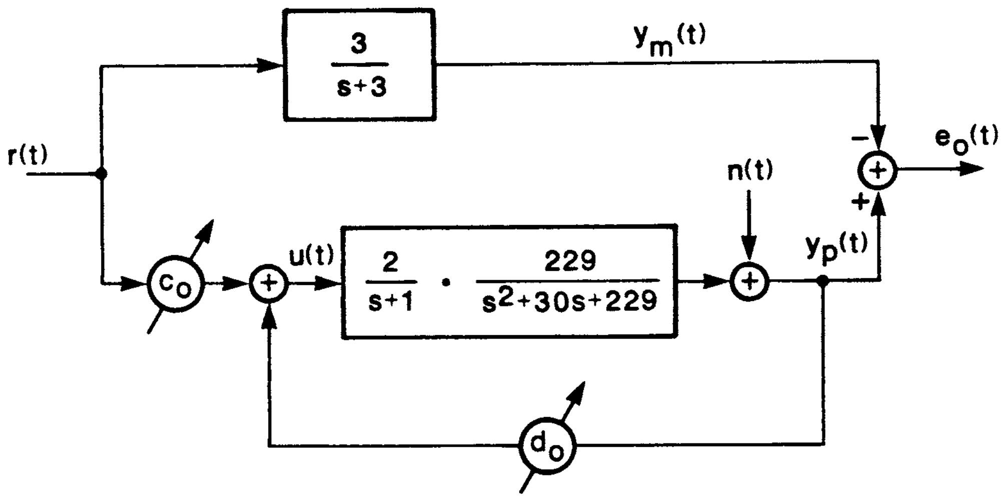

Figure 5.1 Rohrs Example - Plant, Reference Model, and Controller

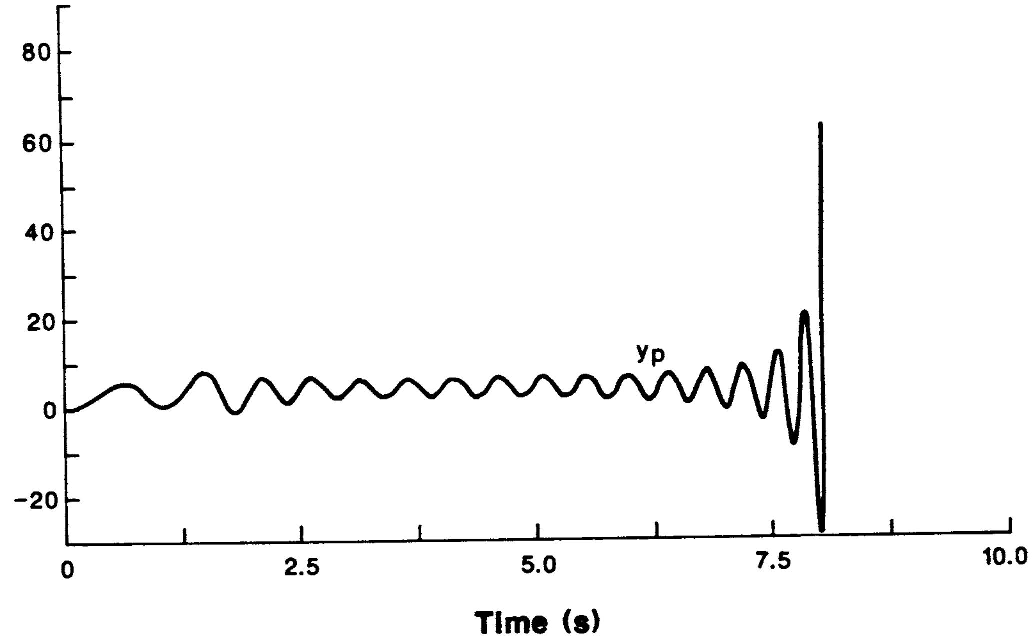

Figure 5.2 Plant Output (r = 4.3, n = 0)

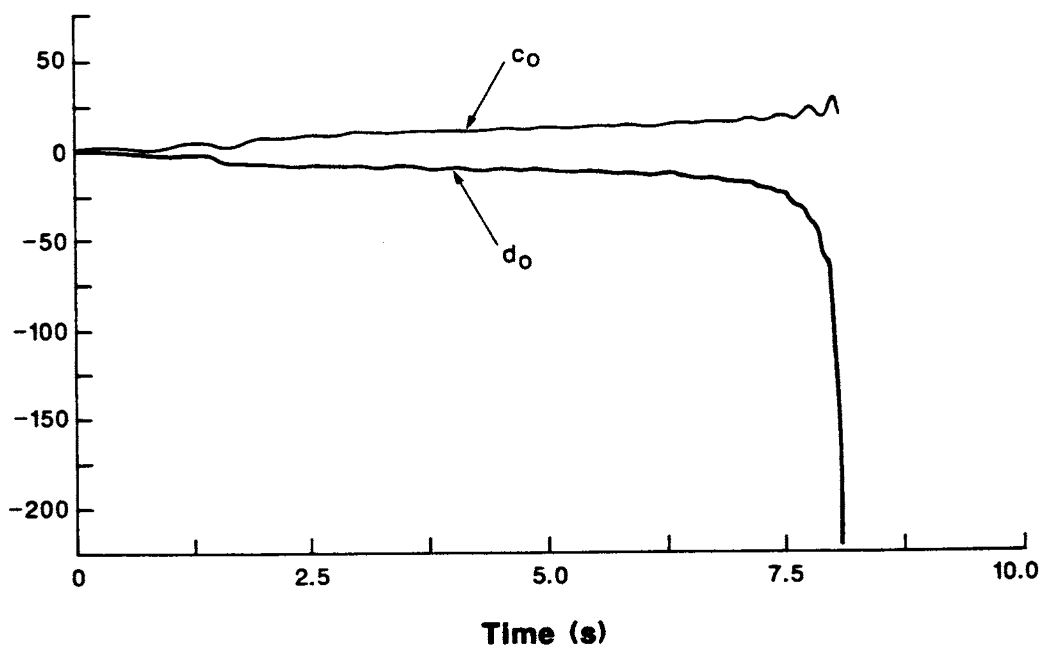

Figure 5.3 Controller Parameters (r = 4.3, n = 0)

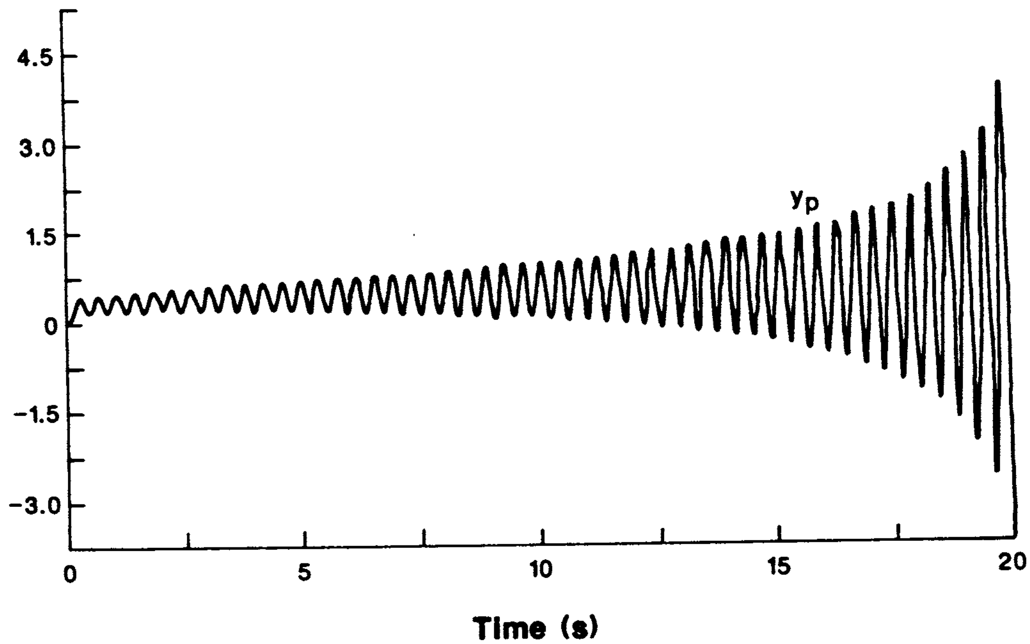

Figure 5.4 Plant Output ($r = 0.3 + 1.85 \sin 16.1\, t$, $n = 0$)

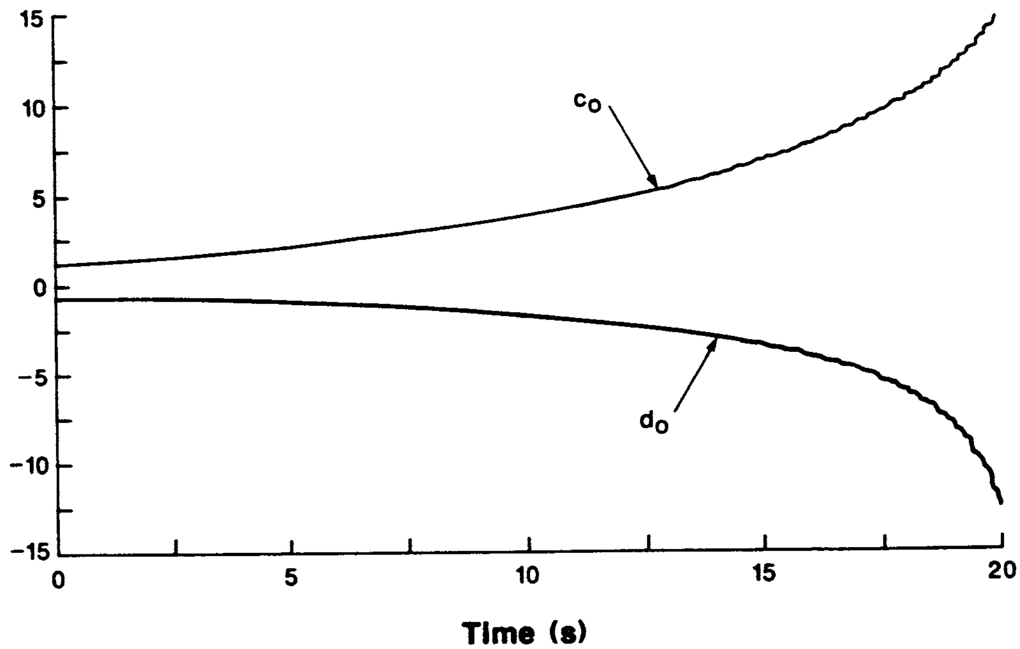

Figure 5.5 Controller Parameters ($r = 0.3 + 1.85 \sin 16.1\, t$, $n = 0$)

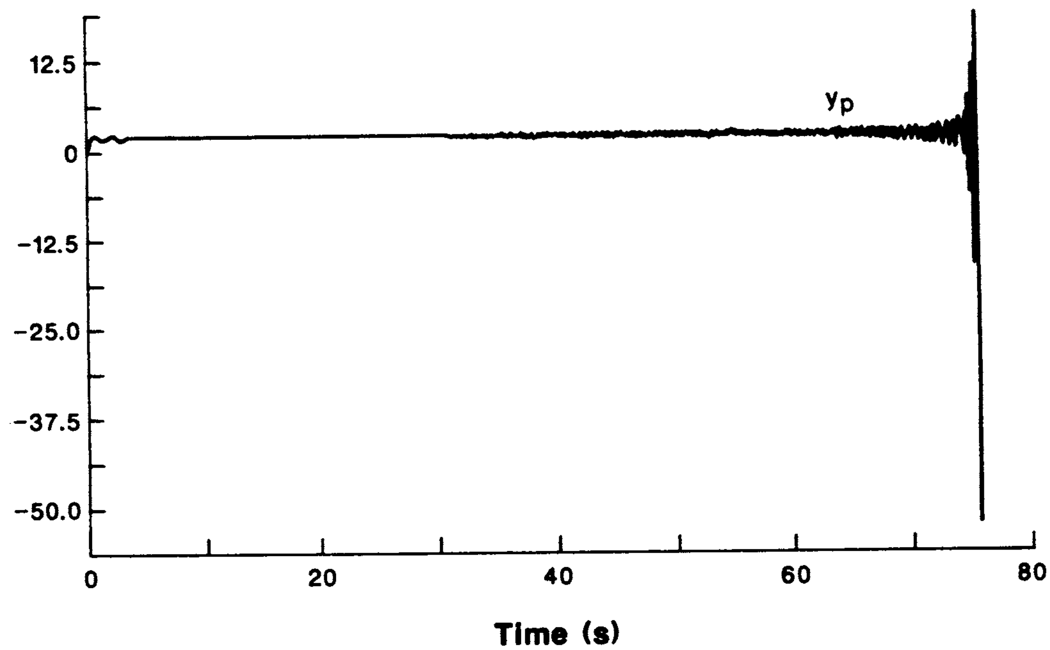

Figure 5.6 Plant Output ($r = 2, n = 0.5 \sin 16.1\, t$)

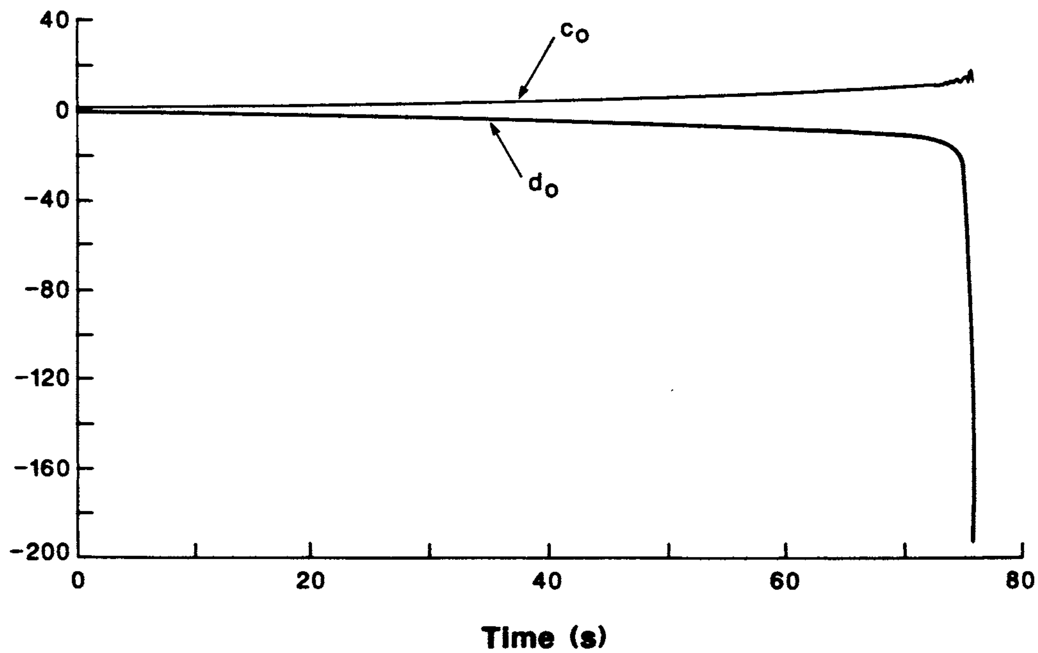

Figure 5.7 Controller Parameters ($r = 2, n = 0.5 \sin 16.1\, t$)

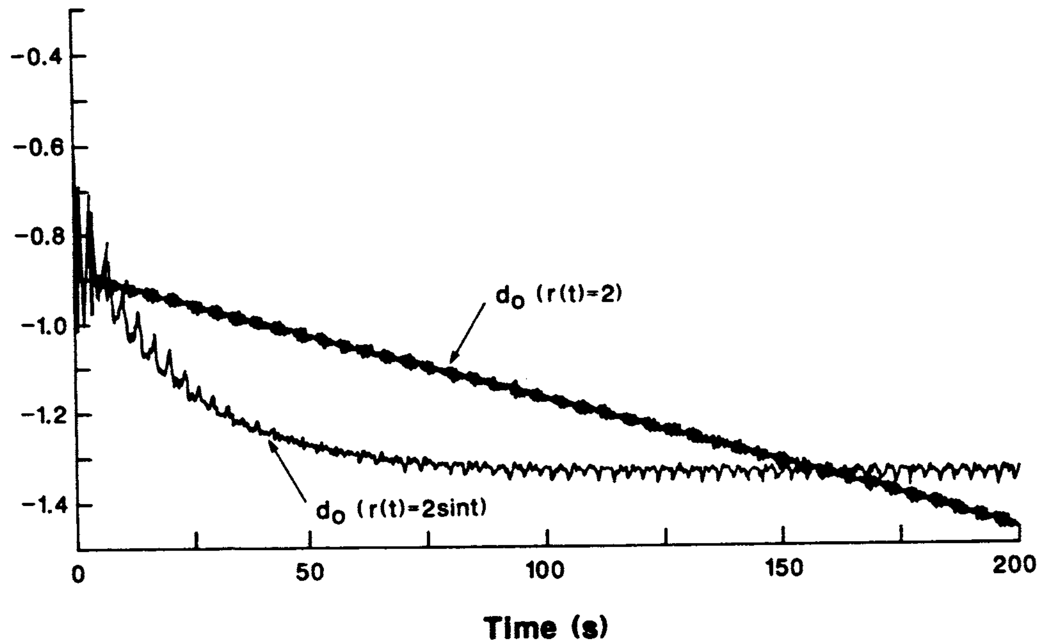

Figure 5.8 Controller Parameter d_o ($n = 0.1 \sin 16.1 t$)

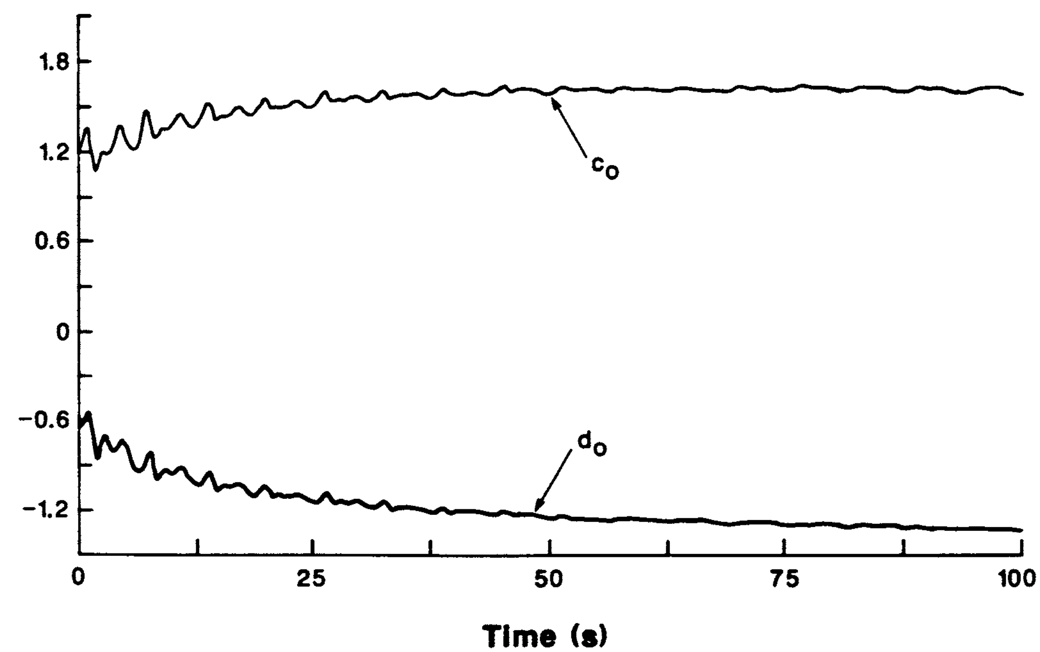

Figure 5.9 Controller Parameters ($r = 0.3 + 1.85 \sin t$, $n = 0$)