Robust Sinusoid Identification with Structured and Unstructured Measurement Uncertainties

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Abstract

In this note a globally stable methodology is proposed to estimate the frequency, phase, and amplitude of a sinusoidal signal affected by additive structured and bounded unstructured disturbances. The structured disturbances belong to the class of time-polynomial signals incorporating both bias and drifts phenomena. Stability and robustness results are given by resorting to Input-to-State stability arguments. Simulation comparative results show the effectiveness of the proposed technique.

I. Introduction

This note deals with the problem of estimating the amplitude, frequency and phase (AFP) of a sinusoidal signal by processing a measurement signal corrupted by bias, drifts and bounded unstructured disturbances. The development of algorithms which are capable of extracting in real-time the parameters of a sinusoid from uncertain measurements turns out to be a very active area of research and many important papers can be found in the literature also with impact on specific application domains like health monitoring, power quality assessment, vibration control, periodic disturbance rejection, noise cancellation, etc..

The robustness properties in the presence of external measurement perturbations (both structured and unstructured), on one hand, and the estimation accuracy, on the other, are the most important features of AFP algorithms towards practical implementation. Beyond some well-known important contributions (see, for example, [1], [2], [3], [4], and the references cited therein), the robust AFP problem has recently received renewed attention (see, for instance, the recent contributions [5], [6], [7], [8] and [9]).

Many different approaches have been proposed in the literature to address the AFP problem that are based on Kalman and Extended Kalman filtering, adaptive notch filtering, and Phase-Locked-Loop (PLL) estimators (a literature review of these methods is out of the scope of the present short note). In this respect, it is worth noting that PLL nonlinear techniques have been recently proposed to obtain robust estimates in presence of noise (see [10], [11], [12], [13] and the references therein). However, the stability results available for the PLL nonlinear AFP algorithms provide, in most cases, only local stability guarantees, or, when averaging analysis is used, global results are valid only for small adaptation gains (see [14] and [15]). Moreover, PLL

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schemes typically require unbiased sinusoidal signals to ensure the convergence of the frequency estimates toward the true value. In order to remove this limitation, the basic PLL algorithm has been suitably modified in [8] to provide globally convergent estimates in presence of bias. An alternative globally convergent nonlinear AFP method for estimation of a biased sinusoid has been proposed in [7]. In the latter approach, a switching strategy is used to reduce the influence of high-frequency measurement noise on the estimates.

The present note deals with a novel AFP method characterized by stability guarantees in the presence of a large class of structured perturbations parametrized in the family of time-polynomial functions. The proposed AFP method is devised in a continuous-time setting which is useful in terms of a possible analog implementation in electronics and power engineering application contexts. The structured measurement disturbances have a practical interest because they may incorporate bias and measurement drifts up to any given order. Moreover, in the spirit of the previous work by the authors on the estimation of unbiased harmonic signals (see [16]), the robustness of the method against bounded unstructured perturbations (noise or additive exogenous signals having limited amplitude) is characterized thanks to Input-to-State-Stability (ISS) analysis. The ISS-Lyapunov tool is also used to assess the transient performance of the frequency-estimator and the practical convergence of the estimates toward a neighborhood of the true values in presence of non-fading perturbations.

II. Problem Statement and Estimation Algorithm

In the following, given an i-times differentiable vector of signals $\mathbf{u}(t) \in \mathbb{R}^n, \forall t \in \mathbb{R}_{\geq 0}$, we denote by $\mathbf{u}^{(i)}$ the vector of the i-th order time-derivative signals. Consider the nominal sinusoidal signal

$$s(t) = A\cos[\vartheta(t)]\,, \quad \text{where} \quad \vartheta^{(1)}(t) = \omega^*\,, \quad t \in \mathbb{R}_{\geq 0} \tag{1}$$

with the initial condition $\vartheta(0) = \vartheta_0$. In this note, we address the task of detecting the frequency $\omega^* \in \mathbb{R}_{>0}$, the phase $\vartheta(t) \in \mathbb{R}$ and the amplitude $A \in \mathbb{R}_{>0}$ on the basis of the perturbed measurement

$$\hat{y}(t) = y(t) + d(t)\,, \text{ with } y(t) = s(t) + \sum_{k=1}^{n_d} b_k t^{k-1}\,, \quad t \in \mathbb{R}_{\geq 0}\,, \tag{2}$$

where, for a given positive and known integer n_d, the term $\sum_{k=1}^{n_d} b_k t^{k-1}$ represents a time-polynomial structured exogenous measurement perturbation[1], with b_k unknown for any $k \in \{1, \ldots, n_d\}$, and where $d(t) \in \mathscr{L}_\infty^1$ is a bounded additive unstructured disturbance with $\|d\|_\infty \leq \overline{d}$, $\overline{d} \in \mathbb{R}_{\geq 0}$ (referred to as *measurement noise* in the sequel).

The proposed AFP methodology exploits the state variable filtering (SVF) tool to compute the unavailable time-derivatives of $y(t)$ (see [17], [18]) that are needed to remove the effect of structured perturbations from the AFP estimates (see also [6]). Moreover, the use of the SVF technique will be instrumental to design the adaptive estimation law in Section III.

Let us consider a simplified setting in which no measurement noise is present, that is, assume for now that $d(t) = 0$, $\forall t \geq 0$ in (2). The SVF paradigm is based on the computation of auxiliary filtered signals $x_1(t)$,

[1] In several application domains structured disturbances are affecting the measurements and are caused by the sensing device. For example, in the context of micro-grids, bias/drift short-time phenomena arise due to power electronics loads. Physical transducers and A/D converters are indeed often affected by offsets, that correspond to $n_d = 1$. In electrical systems variable offsets induced by power electronics are present. Indeed, several sensing devices are influenced by temperature variations that cause drift phenomena ($n_d = 2$).

$x_2(t)$,..., $x_k(t)$,..., $x_{3+n_d}(t)$, obtained as follows:

$$\begin{aligned} x_1^{(1)}(t) &= \lambda\left[\beta y(t) - x_1(t)\right] \\ x_k^{(1)}(t) &= \lambda\left[\beta x_{k-1}(t) - x_k(t)\right], \quad k \in \{2, \ldots, 3+n_d\} \end{aligned} \tag{3}$$

with $x_k(0) = x_{k_0}$, $k \in \{1, \ldots, 3+n_d\}$ and where $\lambda, \beta \in \mathbb{R}_{>0}$ are tunable design parameters. Letting $\mathbf{x}(t) \triangleq [x_1(t), \ldots, x_{3+n_d}(t)]^\top$, we consider the following state-space realization of the filter yielding the signal $x_{3+n_d}(t)$:

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \mathbf{A}_{\lambda,\beta}\mathbf{x}(t) + \boldsymbol{b}_{\lambda,\beta}\, y(t)\,, \\ x_{3+n_d}(t) &= \mathbf{c}^\top \mathbf{x}(t)\,, \end{aligned} \tag{4}$$

for any initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^{3+n_d}$ and where

$$\mathbf{A}_{\lambda,\beta} = \begin{bmatrix} -\lambda & 0 & \cdots & \cdots & 0 \\ \beta\lambda & -\lambda & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta\lambda & -\lambda \end{bmatrix}, \quad \mathbf{b}_{\lambda,\beta} = \begin{bmatrix} \beta\lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{c}^\top = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In view of the proposed filter's structure, it follows that $\mathbf{c}^\top\, \mathbf{A}_{\lambda,\beta}^k\, \mathbf{b}_{\lambda,\beta} = 0,\ k \in \{1, \ldots, 1+n_d\}$. Then, it follows that

$$\begin{aligned} x_{3+n_d}^{(k)}(t) &= \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^k \mathbf{x}(t),\ k \in \{1, \ldots, 2+n_d\} \\ x_{3+n_d}^{(3+n_d)}(t) &= \mathbf{c}^\top \mathbf{A}_{\lambda,\beta}^{2+n_d}\left(\mathbf{A}_{\lambda,\beta}\mathbf{x}(t) + \mathbf{b}_{\lambda,\beta} y(t)\right). \end{aligned} \tag{5}$$

The Laplace transform of the measured signal is given by

$$\mathscr{L}[y](s) = A\frac{s\cos(\vartheta_0) - \omega^*\sin(\vartheta_0)}{s^2 + \omega^{*2}} + \sum_{k=1}^{n_d} b_k \frac{(k-1)!}{s^k}\,.$$

Then, by neglecting the initial conditions of the internal filter's states and by defining $H_k(s) \triangleq \beta^k\lambda^k/(\lambda+s)^k$, the Laplace transform of the n_d-th time-derivative of x_{3+n_d} is

$$\mathscr{L}[x_{3+n_d}^{(n_d)}](s) = H_{3+n_d}(s) A \frac{s\cos(\vartheta_0) - \omega^*\sin(\vartheta_0)}{s^2 + \omega^{*2}} s^{n_d} +\ H_{3+n_d}(s) \sum_{k=1}^{n_d} b_k (k-1)! s^{n_d - k}$$

that gives, in the time-domain, the following asymptotic sinusoidal steady-state time-behaviour $\overline{x}_{3+n_d}^{(n_d)}(t)$ of $x_{3+n_d}^{(n_d)}(t)$:

$$\overline{x}_{3+n_d}^{(n_d)}(t) = A_z \cos[\vartheta_z(t)]\,, \tag{6}$$

where

$$A_z \triangleq A\,\omega^{*n_d}\,|H_{3+n_d}(j\omega^*)|, \quad \vartheta_z(t) \triangleq \vartheta(t) + \angle H_{3+n_d}(j\omega^*) + \frac{\pi}{2} n_d\,. \tag{7}$$

Consider the vector of auxiliary derivatives

$$\mathbf{z}(t) = [\,z_0(t), z_1(t), z_2(t), z_3(t)\,]^\top \triangleq \left[x_{3+n_d}^{(n_d)}(t), -x_{3+n_d}^{(n_d+1)}(t), -x_{3+n_d}^{(n_d+2)}(t), x_{3+n_d}^{(n_d+3)}(t)\right]^\top. \tag{8}$$

The asymptotic result (6) implies that $\mathbf{z}(t)$ tends asymptotically to a sinusoidal stationary equilibrium

$$\overline{\mathbf{z}}(t) = [\,\overline{z}_0(t), \overline{z}_1(t), \overline{z}_2(t), \overline{z}_3(t)\,]^\top \triangleq A_z \left[\cos[\vartheta_z(t)], \omega^* \sin[\vartheta_z(t)], \omega^{*2}\cos[\vartheta_z(t)], \omega^{*3}\sin[\vartheta_z(t)]\right]^\top . \quad (9)$$

Let us now assume that the auxiliary derivative vector $\mathbf{z}(t)$ has reached the stationary sinusoidal equilibrium regime $\overline{\mathbf{z}}(t)$. At any time instant t, the squared frequency $\Omega^* = \omega^{*2}$ can be computed by the following two possible algebraic relations:

$$\Omega^* = \frac{\overline{z}_2(t)}{\overline{z}_0(t)}, \text{ if } t:\ \vartheta_z(t) \neq \pi/2 + i\pi \quad \text{or} \quad \Omega^* = \frac{\overline{z}_3(t)}{\overline{z}_1(t)}, \text{ if } t:\ \vartheta_z(t) \neq i\pi,\ i \in \mathbb{Z}\,.$$

As the actual phase $\vartheta_z(t)$ is not known, it is not possible to choose a priori which of the two expression has to be used. We propose to minimize the following mixed objective:

$$\Omega^* = \arg\min_{\Omega \in \mathbb{R}_{>0}} (\Omega\overline{z}_0(t) - \overline{z}_2(t))^2 + (\Omega\overline{z}_1(t) - \overline{z}_3(t))^2 = \frac{\overline{z}_0(t)\overline{z}_2(t) + \overline{z}_1(t)\overline{z}_3(t)}{[\overline{z}_0(t)]^2 + [\overline{z}_1(t)]^2} \quad (10)$$

Given Ω^*, to avoid the sign dichotomy in the determination of ω^*, we use the positive-sign convention and pick $\omega^* = \sqrt{\Omega^*}$. Note that the previous expression holds for any t at the sinusoidal equilibrium, due to the orthogonality of $\overline{z}_0(t)$ and $\overline{z}_1(t)$. Moreover, from (9) we obtain $\omega^{*2}[\overline{z}_0(t)]^2 + [\overline{z}_1(t)]^2 = A_z^2\omega^{*2}$ which yields

$$A_z = \sqrt{[\Omega^*(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2]/\Omega^*}\,, \quad \vartheta_z(t) = \angle\left[\omega^*\overline{z}_0(t) + j\,\overline{z}_1(t)\right]\,.$$

From (7), a simple algebra finally gives the original amplitude and phase parameters:

$$A = \frac{A_z}{\omega^{*n_d}}\left(\frac{\sqrt{\lambda^2 + \omega^{*2}}}{\beta\lambda}\right)^{3+n_d},\ \vartheta(t) = \vartheta_z(t) + (3 + n_d)\,\mathrm{atan}\left(\frac{\omega^*}{\lambda}\right) - n_d\frac{\pi}{2}\,. \quad (11)$$

To sum up, the equilibrium trajectory of an AFP estimator in the presence of structured perturbations has been addressed by introducing auxiliary filtered signals and by solving a scalar algebraic equation in the squared-frequency Ω^* thus obtaining frequency ω^*. Through (11) amplitude and phase can be computed as well. In the next section, the convergence of an adaptive estimator to the above sinusoidal equilibrium trajectory will be addressed taking into account that during transient the stationary signals $\overline{z}_0(t), \overline{z}_1(t), \overline{z}_2(t), \overline{z}_3(t)$ are of course not available and that noise may affect the measurements.

III. Input-to-State Stable Frequency Estimation

The vector of auxiliary derivatives $\mathbf{z}(t)$ can be expressed in compact form (see (5)) as

$$\mathbf{z}(t) = \mathbf{\Lambda}\begin{bmatrix} y(t) \\ \mathbf{x}(t) \end{bmatrix}, \quad \text{with} \quad \mathbf{\Lambda} \triangleq \begin{bmatrix} 0 & \mathbf{c}^\top\mathbf{A}_{\lambda,\beta}^{n_d} \\ 0 & -\mathbf{c}^\top\mathbf{A}_{\lambda,\beta}^{1+n_d} \\ 0 & -\mathbf{c}^\top\mathbf{A}_{\lambda,\beta}^{2+n_d} \\ \mathbf{c}^\top\mathbf{A}_{\lambda,\beta}^{2+n_d}\mathbf{b}_{\lambda,\beta} & \mathbf{c}^\top\mathbf{A}_{\lambda,\beta}^{3+n_d} \end{bmatrix}. \quad (12)$$

First, observe that there exists an (unknown) initial filter's state $\mathbf{x}(0) = \overline{\mathbf{x}}_0$ giving rise to a filtered state trajectory $\overline{\mathbf{x}}(t)$ whose projection on the subspace containing $\mathbf{z}(t)$ matches the stationary sinusoidal behavior since the initial time-instant $t = 0$, that is:

$$\overline{\mathbf{x}}(t), t \in \mathbb{R}_{\geq 0}:\ \ \mathbf{z}(t) = \mathbf{\Lambda}\,[y(t)\,,\,\overline{\mathbf{x}}(t)]^\top \equiv \overline{\mathbf{z}}(t),\ \forall t \in \mathbb{R}_{\geq 0}. \quad (13)$$

Now, let us consider the measurement signal $\hat{y}(t)$ given by (2). Moreover, let us denote by $\hat{\mathbf{x}}(t)$ the state vector of the filter evolving from an arbitrary initial state $\hat{\mathbf{x}}_0$ according to

$$\hat{\mathbf{x}}^{(1)} = \mathbf{A}_{\lambda,\beta}\hat{\mathbf{x}}(t) + \boldsymbol{b}_{\lambda,\beta}\hat{y}(t)\,, \quad \hat{x}_{3+n_d}(t) = \mathbf{c}^\top \hat{\mathbf{x}}(t)\,, \tag{14}$$

and let $\hat{\mathbf{z}}(t) \triangleq [\,\hat{z}_0(t),\, \hat{z}_1(t), \hat{z}_2(t),\, \hat{z}_3(t)\,]^\top$ be the vector of the computable perturbed derivative

$$\hat{\mathbf{z}}(t) = \mathbf{\Lambda}\,[\hat{y}(t)\,,\, \hat{\mathbf{x}}(t)]^\top\,. \tag{15}$$

During the transient behavior and also because of the presence of measurement noise, the orthogonality of $\hat{z}_0(t)$ and $\hat{z}_1(t)$ cannot be guaranteed and, of course, in general, $\hat{\mathbf{x}}_0 \neq \overline{\mathbf{x}}_0$. Therefore, (10) cannot be directly used. Instead, we resort to a singularity-free dynamic optimization scheme with guaranteed asymptotic convergence properties. The following frequency adaptation law using the perturbed auxiliary filtered signals is thus proposed ($\mu \in \mathbb{R}_{>0}$ is a suitable tunable parameter):

$$\Omega^{(1)}(t) = -\mu\,\{[\hat{z}_0(t)\hat{z}_2(t) + \hat{z}_1(t)\hat{z}_3(t)]\,[\Omega(t)\hat{z}_0(t) - \hat{z}_2(t)]\,\hat{z}_0(t) + \left[(\hat{z}_0(t))^2 + (\hat{z}_1(t))^2\right]\,[\Omega(t)\hat{z}_1(t) - \hat{z}_3(t)]\,\hat{z}_1(t)\Big\}\,. \tag{16}$$

In order to characterize the stability properties of the frequency estimation system (14), (15), and (16), let us first analyze the stability of the filter dynamics. Introducing the error vector with respect to $\overline{\mathbf{x}}(t)$ (see (13)) $\tilde{\mathbf{x}}(t) \triangleq \hat{\mathbf{x}}(t) - \overline{\mathbf{x}}(t)$ and defining $d(t) = \hat{y}(t) - y(t)$, the dynamics of $\tilde{\mathbf{x}}(t)$ can be written as

$$\tilde{\mathbf{x}}^{(1)}(t) = \mathbf{A}_{\lambda,\beta}\tilde{\mathbf{x}}(t) + \mathbf{b}_{\lambda,\beta}\,d(t)\,, \tag{17}$$

where $\tilde{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 - \overline{\mathbf{x}}_0$. As the matrix $\mathbf{A}_{\lambda,\beta}$ is Hurwitz, there exists a positive definite matrix $\mathbf{P}$ that solves the linear Lyapunov's equation: $\mathbf{P}\mathbf{A}_{\lambda,\beta} + \mathbf{A}_{\lambda,\beta}^\top\mathbf{P} = -\mathbf{I}$. Let $W(\tilde{\mathbf{x}}) \triangleq \tilde{\mathbf{x}}^\top\mathbf{P}\tilde{\mathbf{x}}$; then there exist two positive scalars $a_1, a_2 \in \mathbb{R}_{>0}$ such that $a_1|\tilde{\mathbf{x}}|^2 \leq W(\tilde{\mathbf{x}}) \leq a_2|\tilde{\mathbf{x}}|^2\,, \quad \forall\tilde{\mathbf{x}}$. The derivative of W along the system's state trajectory satisfies $\dfrac{\partial W}{\partial \tilde{\mathbf{x}}}\,(\mathbf{A}_{\lambda,\beta}\tilde{\mathbf{x}} + \mathbf{b}_{\lambda,\beta}d) \leq -|\tilde{\mathbf{x}}|^2 + 2\,\|\mathbf{P}\|\,|\mathbf{b}_{\lambda,\beta}|\,|d|\,|\tilde{\mathbf{x}}|$. For any $0 < \epsilon < 1$, let $\mathcal{X}(s) \triangleq \dfrac{2\,\|\mathbf{P}\|\,|\mathbf{b}_{\lambda,\beta}|}{1-\epsilon}s$. with $s \in \mathbb{R}_{\geq 0}$. It is easy to show that $|\tilde{\mathbf{x}}| \geq \mathcal{X}(|d|) \Rightarrow \dfrac{\partial W}{\partial \tilde{\mathbf{x}}}\,(\mathbf{A}_{\lambda,\beta}\tilde{\mathbf{x}} + \mathbf{b}_{\lambda,\beta}d) \leq -|\tilde{\mathbf{x}}|^2$, and that the system is ISS with asymptotic gain $\gamma_x(s) = a_1^{-1}a_2\,\mathcal{X}(s)$. In view of the just shown ISS property of the linear auxiliary filter (17), for any arbitrary $\nu \in \mathbb{R}_{>0}$ and for any finite-norm initial error $\tilde{\mathbf{x}}_0$, the error vector $\tilde{\mathbf{x}}(t)$ will enter in a closed ball of radius $\gamma_x(\|d\|_\infty) + \nu \leq \gamma_x(\overline{d}) + \nu$ in a finite time $T_{\tilde{\mathbf{x}}_0,\nu}$. In view of (15), the vector $\tilde{\mathbf{z}}(t) \triangleq \hat{\mathbf{z}}(t) - \overline{\mathbf{z}}(t)$ will enter in finite-time $T_\delta = T_{\tilde{\mathbf{x}}_0,\nu}$ (depending on initial conditions) in a closed ball of radius $\gamma_z(\overline{d}) + \delta$ centered at the origin, with

$$\delta = \overline{\lambda}\nu\,, \quad \gamma_z(s) = \overline{\lambda}\,(\gamma_x(s) + s), \forall s \in \mathbb{R}_{\geq 0}, \tag{18}$$

where $\overline{\lambda} = \|\Lambda\|$. Let us now write the adaptation law in terms of $\tilde{\mathbf{z}}$:

$$\begin{aligned}\Omega^{(1)}(t) = -\mu\Big\{&\Big[(\overline{z}_0(t) + \tilde{z}_0(t))(\overline{z}_2(t) + \tilde{z}_2(t)) + (\overline{z}_1(t) + \tilde{z}_1(t))(\overline{z}_3(t) + \tilde{z}_3(t))\Big]\\ &\times \Big[\Omega(t)(\overline{z}_0(t) + \tilde{z}_0(t)) - (\overline{z}_2(t) + \tilde{z}_2(t))\Big](\overline{z}_0(t) + \tilde{z}_0(t))\\ &+ \Big[(\overline{z}_0(t) + \tilde{z}_0(t))^2 + (\overline{z}_1(t) + \tilde{z}_1(t))^2\Big]\Big[\Omega(t)(\overline{z}_1(t) + \tilde{z}_1(t)) - (\overline{z}_3(t) + \tilde{z}_3(t))\Big](\overline{z}_1(t) + \tilde{z}_1(t))\Big\}\end{aligned}$$

and hence, after a little algebra,

$$\Omega^{(1)}(t) = -\mu\Big\{[\overline{z}_0(t)\overline{z}_2(t) + \overline{z}_1(t)\overline{z}_3(t)][\Omega(t)\overline{z}_0(t) - \overline{z}_2(t)]\overline{z}_0(t) + [(\overline{z}_0(t))^2 + (\overline{z}_1(t))^2][\Omega(t)\overline{z}_1(t) - \overline{z}_3(t)]\overline{z}_1(t)\Big\} + \mu\tilde{f}_z(t,\tilde{\mathbf{z}}) + \mu\tilde{f}_\Omega(t,\tilde{\mathbf{z}})\Omega(t) \quad (19)$$

where

$$\begin{aligned}\tilde{f}_z(t,\tilde{\mathbf{z}}) \triangleq -\Big[&(\overline{z}_0(t)+\tilde{z}_0(t))(\overline{z}_2(t)+\tilde{z}_2(t)) + (\overline{z}_1(t)+\tilde{z}_1(t))(\overline{z}_3(t)+\tilde{z}_3(t))\Big]\left[-(\overline{z}_2(t)+\tilde{z}_2(t))\right]\tilde{z}_0(t)\\ &+ \Big[(\overline{z}_0(t)+\tilde{z}_0(t))^2 + (\overline{z}_1(t)+\tilde{z}_1(t))^2\Big]\left[-(\overline{z}_3(t)+\tilde{z}_3(t))\right]\tilde{z}_1(t)\\ &+ \Big[\tilde{z}_0(t)\overline{z}_2(t) + \tilde{z}_0(t)\tilde{z}_2(t) + \tilde{z}_2(t)z_0(t) + \tilde{z}_1(t)\overline{z}_3(t) + \tilde{z}_1(t)\tilde{z}_3(t) + \tilde{z}_3(t)z_1(t)\Big]\Big[-(\overline{z}_2(t)+\tilde{z}_2(t))\Big]z_0(t)\end{aligned} \quad (20)$$

and

$$\begin{aligned}\tilde{f}_\Omega(t,\tilde{\mathbf{z}}) \triangleq -\Big[&(\overline{z}_0(t)+\tilde{z}_0(t))(\overline{z}_2(t)+\tilde{z}_2(t)) + (\overline{z}_1(t)+\tilde{z}_1(t))(\overline{z}_3(t)+\tilde{z}_3(t))\Big]\left[\overline{z}_0(t)+\tilde{z}_0(t)\right]\tilde{z}_0(t)\\ &+ \Big[(\overline{z}_0(t)+\tilde{z}_0(t))^2 + (\overline{z}_1(t)+\tilde{z}_1(t))^2\Big]\left[\overline{z}_1(t)+\tilde{z}_1(t)\right]\tilde{z}_1(t)\\ &+ \Big[\tilde{z}_0(t)\overline{z}_2(t) + \tilde{z}_0(t)\tilde{z}_2(t) + \tilde{z}_2(t)z_0(t) + \tilde{z}_1(t)\overline{z}_3(t) + \tilde{z}_1(t)\tilde{z}_3(t) + \tilde{z}_3(t)z_1(t)\Big]\left(\overline{z}_0(t)+\tilde{z}_0(t)\right)z_0(t)\\ &+ \Big[(2\overline{z}_0(t)+\tilde{z}_0(t))\tilde{z}_0(t) + (2\overline{z}_1(t)+\tilde{z}_1(t))\tilde{z}_1(t)\Big]\left[\overline{z}_1(t)+\tilde{z}_1(t)\right]z_1(t).\end{aligned} \quad (21)$$

The adaptation law (16), rewritten in terms of the elements of the disturbance-free vector $\mathbf{z}(t)$ and of $\tilde{\mathbf{z}}(t)$, is described by (19). Note that the functions $\tilde{f}_z(t,\tilde{\mathbf{z}})$ and $\tilde{f}_\Omega(t,\tilde{\mathbf{z}})$ introduced in (19) and defined in (20) and (21), verify $\tilde{f}_z(t,0) = 0$, $\tilde{f}_\Omega(t,0) = 0$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, being the vector $\mathbf{z}(t)$ bounded (this can be deduced from (8), since the filtered derivatives $x^{(n_d)}(t), \ldots, x^{(n_d+3)(t)}$ are bounded for time-polynomial structured uncertainties of order n_d) and owing to the boundedness of $\overline{z}_0(t), \ldots, \overline{z}_3(t)$, there exist two $\mathcal{K}_\infty$-functions $\sigma_z(\cdot)$ and $\sigma_\Omega(\cdot)$ such that

$$|\tilde{f}_z(t,\tilde{\mathbf{z}}(t))| \leq \sigma_z(|\tilde{\mathbf{z}}(t)|)\,, \quad |\tilde{f}_\Omega(t,\tilde{\mathbf{z}}(t))| \leq \sigma_\Omega(|\tilde{\mathbf{z}}(t)|)\,. \quad (22)$$

Now, for a given a squared-frequency estimate Ω, let us consider the following function of Ω and $\overline{\mathbf{z}}(t)$:

$$J(\Omega,\overline{\mathbf{z}}(t)) \triangleq \frac{\overline{z}_0(t)\overline{z}_2(t) + \overline{z}_1(t)\overline{z}_3(t)}{[\overline{z}_0(t)]^2 + [\overline{z}_1(t)]^2}\left[\Omega\overline{z}_0(t) - \overline{z}_2(t)\right]^2 + \left[\Omega\overline{z}_1(t) - \overline{z}_3(t)\right]^2\,. \quad (23)$$

After some algebra, the function (23) can be rewritten as

$$J(\Omega,t) = \Omega^* A_z^2\left(\Omega - \Omega^*\right)^2 \quad (24)$$

which clearly is a positive-definite function depending only on the frequency-estimation error $\tilde{\Omega} \triangleq \Omega - \Omega^*$. Now, with some abuse of notation, letting $V(\tilde{\Omega}) \triangleq J(\Omega,\overline{\mathbf{z}}(t))$ be a candidate ISS-Lyapunov function for the estimation error's dynamics, we obtain

$$\begin{aligned}\frac{\partial V}{\partial\tilde{\Omega}}\Omega^{(1)} &= -2\mu\left[(\overline{z}_0(t))^2+(\overline{z}_1(t))^2\right][\Omega^*(\Omega(t)\overline{z}_0(t)-\overline{z}_2(t))\overline{z}_0(t)+(\Omega(t)\overline{z}_1(t)-\overline{z}_3(t))\overline{z}_1(t)]^2\\ &\quad+2\mu(\tilde{f}_z(t,\tilde{\mathbf{z}})+\tilde{f}_\Omega(t,\tilde{\mathbf{z}})\Omega(t))[\Omega^*(\Omega(t)\overline{z}_0(t)-\overline{z}_2(t))\overline{z}_0(t)+(\Omega(t)\overline{z}_1(t)-\overline{z}_3(t))\overline{z}_1(t)]\\ &\le -2\mu A_z^6\min\{1,\Omega^*\}\Omega^{*2}\left(\Omega(t)-\Omega^*\right)^2+2\mu A_z^2\Omega^*\left|\Omega(t)-\Omega^*\right|\left(\sigma_z(|\tilde{\mathbf{z}}(t)|)+\sigma_\Omega(|\tilde{\mathbf{z}}(t)|)\Omega(t)\right)\\ &\le -\mu(\alpha^*-\sigma_2(|\tilde{\mathbf{z}}(t)|))|\tilde{\Omega}(t)|^2+\mu\sigma_1(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)|\,,\end{aligned}\quad(25)$$

where

$$\begin{aligned}&\alpha^*\triangleq 2A_z^6\min\{1,\Omega^*\}\Omega^{*2},\\ &\sigma_1(|\tilde{\mathbf{z}}(t)|)\triangleq 2A_z^2\Omega^*(\sigma_z(|\tilde{\mathbf{z}}(t)|)+\Omega^*\sigma_\Omega(|\tilde{\mathbf{z}}(t)|)),\;\sigma_2(|\tilde{\mathbf{z}}(t)|)\triangleq 2A_z^2\Omega^*\,\sigma_\Omega(|\tilde{\mathbf{z}}(t)|)\,.\end{aligned}\quad(26)$$

The ISS stability properties of the frequency estimator are characterized in the following result.

Theorem 3.1 (ISS of the adaptive frequency identifier): Given the sinusoidal signal $s(t)$ generated by (1) and the perturbed measurement model (2), the frequency estimation system given by (14), (15) and (16) is ISS with respect to any additive disturbance signal $d(t)\in\mathscr{L}^1_\infty$ such that

$$\|d\|_\infty<\overline{d}<\gamma_z^{-1}\left(\sigma_2^{-1}(\alpha^*)\right)\quad(27)$$

where α^* and σ_2 are given by (26) and γ_z is given by (18). □

Proof: Due to the ISS property of the auxiliary filter (see (18)), for any positive $\delta\in\mathbb{R}_{>0}$ there exists a finite time-instant T_δ such that $|\tilde{\mathbf{z}}(t)|\le\gamma_z(\overline{d})+\delta,\forall t\ge T_\delta$, which implies

$$\sigma_2(|\tilde{\mathbf{z}}(t)|)\le\sigma_2(\gamma_z(\overline{d})+\delta),\;\;\forall t\ge T_\delta.\quad(28)$$

If the bound on disturbances $\overline{d}$ verifies

$$\alpha^*-\sigma_2(\gamma_z(\overline{d})+\delta)>0,\quad(29)$$

for some $\delta\in\mathbb{R}_{>0}$, then, for any $t>T_\delta$, the following bound on the derivative of V can be established

$$\begin{aligned}\frac{\partial V}{\partial\tilde{\Omega}}\Omega^{(1)}(t) &\le -\mu\Big(\alpha^*-\sigma_2(\gamma_z(\overline{d})+\delta)\Big)|\tilde{\Omega}(t)|^2+\sigma_1(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)|\\ &\le -c\,|\tilde{\Omega}(t)|^2+\mu\sigma_1(|\tilde{\mathbf{z}}(t)|)|\tilde{\Omega}(t)|\,,\quad t\ge T_\delta\end{aligned}\quad(30)$$

where $c\triangleq\mu\left[\alpha^*-\sigma_2(\gamma_z(\overline{d})+\delta)\right]$ is a positive constant. Finally, for any $0<\epsilon<1$, let

$$\mathcal{X}_\Omega(s)=\frac{1}{c(1-\epsilon)}\mu\sigma_1(s)\,.\quad(31)$$

It is easy to prove that

$$|\tilde{\Omega}(t)|\ge\mathcal{X}_\Omega(|\tilde{\mathbf{z}}(t)|)\;\Rightarrow\;\frac{\partial V}{\partial\tilde{\Omega}}\Omega^{(1)}(t)\le -c|\tilde{\Omega}(t)|^2,\;\;\forall t\ge T_\delta.\quad(32)$$

Considering that, for any finite initial condition Ω_0, the derivative $\Omega^{(1)}(t)$ is bounded in the interval $[0,T_\delta]$, then $\Omega(T_\delta)$ is finite and $\tilde{\Omega}(T_\delta)$ is, in turn, finite. Hence, thanks to (30) and (32), for any disturbance signal

$d(t)$ bounded by (27), V is an ISS-Lyapunov function for the frequency estimator dynamics with respect to the $\tilde{\mathbf{z}}(t)$ input. The dynamics of $\tilde{\mathbf{z}}$ being ISS with respect to the disturbance $d(t)$, it follows that the frequency estimation system is in turn ISS with respect to $d(t)$, that is, there exist a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathcal{K}$-function $\gamma_\Omega(\cdot)$ such that $|\tilde{\Omega}(t)| \le \beta(\tilde{\Omega}(T_\delta), t - T_\delta) + \gamma_\Omega(\|d\|_\infty)$. In particular, the asymptotic ISS gain is given by $\gamma_\Omega(s) = \mathcal{X}_\Omega(\gamma_z(s))\,,\ \ s \in [0, \overline{d})\,.$

■

It is worth noting that the auxiliary filtered signals provided by the pre-filtering components are combined nonlinearly to obtain a dynamic adaptation law for the squared-frequency which allows to conclude the ISS of the estimation system with respect to additive measurement perturbations.

Remark 3.1 (Practical Stability and Accuracy): The ISS stability analysis can be used to provide some tuning guidelines for the parameters of the proposed AFP scheme. The inequality (29) establishes a bound on the unstructured perturbations that must be fulfilled to ensure practical stability. While α^* depends on the true signal's parameters, the function $\gamma_z(\cdot)$ can be shaped arbitrarily by tuning the parameter λ of the pre-filter, thus allowing to weaken the noise bound depending on the application. In this respect, it is worth noting that the parameter μ has no effect on the practical-stability noise bound. Conversely, assuming that the practical stability condition (29) is met by a proper choice of λ, then the adaptation parameter μ can be decreased to reduce the asymptotic ISS gain (see (31)), which corresponds to an increased frequency-estimation accuracy in case of non-fading unstructured perturbations. Note that a smaller μ also leads to a slower decrease of the estimation error in the adaptation transient, due to the well-known trade-off between asymptotic accuracy and convergence speed. Finally, the filter gain-parameter β adds a useful degree of freedom in designing the SVF.

Remark 3.2 (Estimated frequency bias due to digital implementation): From a practical perspective, one of the issues that deserve investigation is the steady-state bias in the frequency estimate caused by the digital implementation of the proposed continuous-time AFP methodology. Without loss of generality and for the sake of simplicity, let us address the simple case in which no structured exogenous perturbation affects the measurement equation (2) (this simpler case has been addressed in [16]). Considering a Euler discretization with sampling-time T, from the filter equations (3), we immediately get $X_k(z) = \frac{(\lambda\beta T)^k}{(z-1+\lambda T)^k} Y(z),\ k = 1,2,3\,.$ After discretization and some simple algebra, the $\mathcal{Z}$-transforms of the auxiliary derivatives z_0, z_1, z_2 (see (8)) are given by $Z_0 = \frac{(\lambda\beta T)^3}{(z-1+\lambda T)^3} Y(z)\,,\ Z_1 = -\lambda^3\beta^3 T^2 \frac{z-1}{(z-1+\lambda T)^3} Y(z)\,,\ Z_2 = -\lambda^3\beta^3 T \frac{(z-1)^2}{(z-1+\lambda T)^3} Y(z)\,,$ and $Z_3 = \lambda^3\beta^3 \frac{(z-1)^3}{(z-1+\lambda T)^3} Y(z)\,.$ Now, the squared frequency after discretization is given (in the discrete-time domain) by (see (10)) $\Omega_{\text{discr}} \triangleq \frac{z_{0d} z_{2d} + z_{1d} z_{3d}}{z_{0d}^2 + z_{1d}^2}$, where $z_{0d}, z_{1d}, z_{2d}, z_{3d}$ denote the discrete-time sequences corresponding to the auxiliary derivatives. After some lengthy algebra, we get $\Omega_{\text{discr}} = -\frac{(z-1)^2}{T^2}\,.$ Then, for a given frequency ω^* the discrete-time measurements are $y(k) = A\cos(\omega^* T k)$ which gives $\Omega_{\text{discr}} = -\frac{\mathrm{Re}[(e^{j\omega^* T})^2]}{T^2} = \frac{2\cos(\omega^* T)(1-\cos(\omega^* T))}{T^2}$ and hence the steady-state value of the frequency after discretization is

$$\omega_{\text{discr}} = \sqrt{2\cos(\omega^* T)(1 - \cos(\omega^* T))}/T\,. \tag{33}$$

IV. Simulation results

In this section, the digital implementation of the proposed AFP algorithm is addressed and some comparisons with recently proposed techniques are given in the context of two different simulation scenarios.

In the first scenario, we consider a sinusoidal signal $v(t) = \sigma(t) + 5\sin[\omega(t)t + \pi/4] + d(t)$, where $d(t)$ is a $\mathscr{L}_\infty^1$ random noise with uniform distribution in the interval $[-0.5, 0.5]$. The adaptation laws of the AFP technique are discretized by the Euler forward method with sampling period $T_s = 1 \times 10^{-3}s$. Time-varying bias and frequency scenarios are considered, namely: $\omega(t) = 3$, for $0 \le t < 15$s and $\omega(t) = 5$, for $t \ge 15$s; $\sigma(t) = 1$, for $0 \le t < 25$s and $\sigma(t) = 3$, for $t \ge 25$s.

For the sake of comparison, the AFP algorithms presented in [7] and [8] are considered. These two methods are compared with the one proposed in the present note by considering the same initial condition $\hat{\omega}(0) = 1$ for the three AFP algorithms and by choosing the respective tuning parameters in such a way that each algorithm shows the best performances for the given scenario with a comparable initial transient behavior. More specifically, for method [7], we set $\lambda = 3$, $k = 1.2$ whereas, for method [8], we set $K_s = 1, \lambda = 1, \omega_s = 2.7$. Finally, the parameters of the proposed algorithm are given by $\lambda = 2, \beta = 0.7, \mu = 5$.

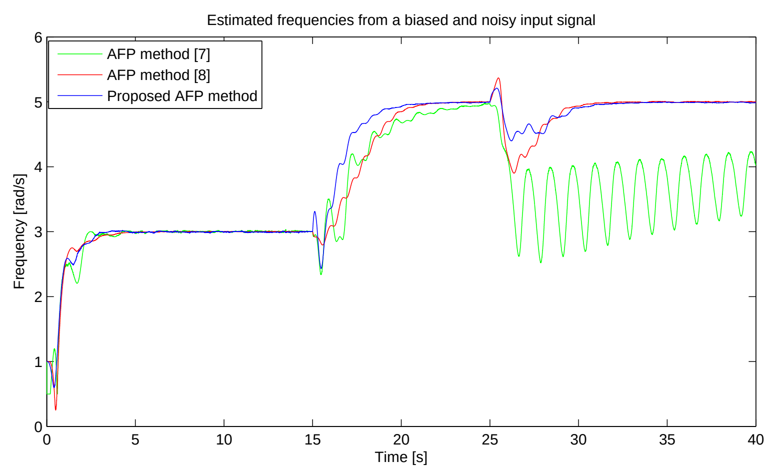

Fig. 1. Time-behavior of the estimated frequency by using the proposed AFP method (blue line) compared with the time behaviors of the estimated frequency by the AFP methods [8] (red line) and [7] (green line), respectively.

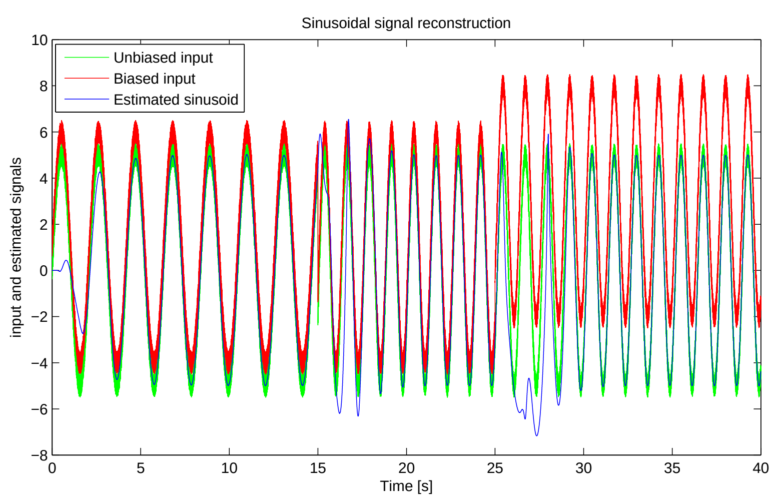

Fig. 2. Estimated sinusoidal signal by the proposed AFP method (blue line). To appreciate the time-behavior of the estimated signal, the the biased noisy input is depicted (red line), as well as the same signal without the time-varying bias term (green line).

It is worth noting from Fig. 1 that all the three AFP estimators succeeded in tracking sudden changes of frequency and bias in presence of noise. However, AFP method proposed in [7] is a little more sensitive to a frequency changes and requires quite a long response time to a bias variation. Fig. 2 shows the time-behavior of the reconstructed sinusoidal signal by the proposed AFP technique. As can be noticed, the sinusoidal signal is estimated successfully even in the presence of noise and time-varying true frequency and bias. For the sake of completeness, in Fig. 3, the time-behaviors of the estimated frequency concerning the previously considered three AFP methods are shown in two cases in which higher levels of noise affect the input. Note that the robustness of AFP method [7] can be improved by exploiting the switching procedure proposed by the authors consisting in adapting the values of the tuning parameters so as to attenuate the effects of the noise at the expense of increasing the transient modes of behavior. Moreover, although method [8] is capable to provide slightly better steady state behaviour, the proposed AFP approach offers better transient behavior during bias and frequency variations.

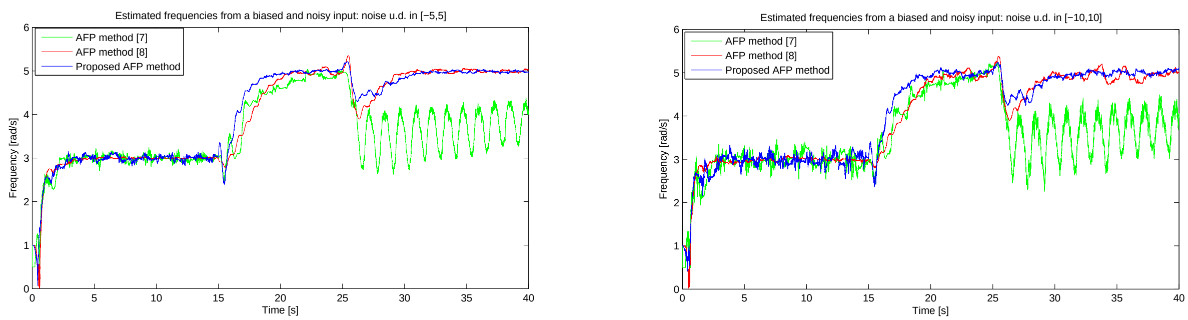

Fig. 3. Time-behavior of the estimated frequency by using the proposed AFP method (blue line) compared with the time behaviors of the estimated frequency by the AFP methods [8] (red line) and [7] (green line), respectively. a) $d(t)$ random noise with uniform distribution in the interval $[-5, 5]$; b) $d(t)$ random noise with uniform distribution in the interval $[-10, 10]$.

The tuning parameters λ and β have a significant influence on the AFP performances. To gain more insight into this important practical aspect, let us refer to Fig. 4. As shown in Fig. 4, the product $\lambda\beta$ strongly influences the noise rejection performances: smaller values of $\lambda\beta$ give rise to better noise rejection. Instead, for a given value of $\lambda\beta$, choosing smaller values of β leads to better transient performances performances.

In the second scenario, a sinusoidal signal incorporating a time-polynomial structured perturbation (drift) is considered, namely $v(t) = 5\sin[3t + \pi/4] + 1 + 0.5t + d(t)$, where $d(t)$ is a random disturbance with the same characteristics as in the previous example. The initial condition is $\hat{\omega}(0) = 1$ and the tuning parameters for the AFP proposed method are now set as $\lambda = 3, \beta = 0.5, \mu = 10$. The results of the simulation are shown in Fig. 5, where the successful detection of the frequency, amplitude and sinusoidal components can be observed even in presence of noise and of the drift term.

V. Concluding remarks

In this note, a new algorithm is proposed for the robust estimation of the frequency, the phase and the amplitude of a sinusoidal signal, in presence of structured uncertainty and bounded additive disturbances. Constant measurement bias and both linear and higher-order polynomial drifts can be handled by the proposed methodology. The convergence of the frequency-identification system is analyzed by an Input-to-State stability

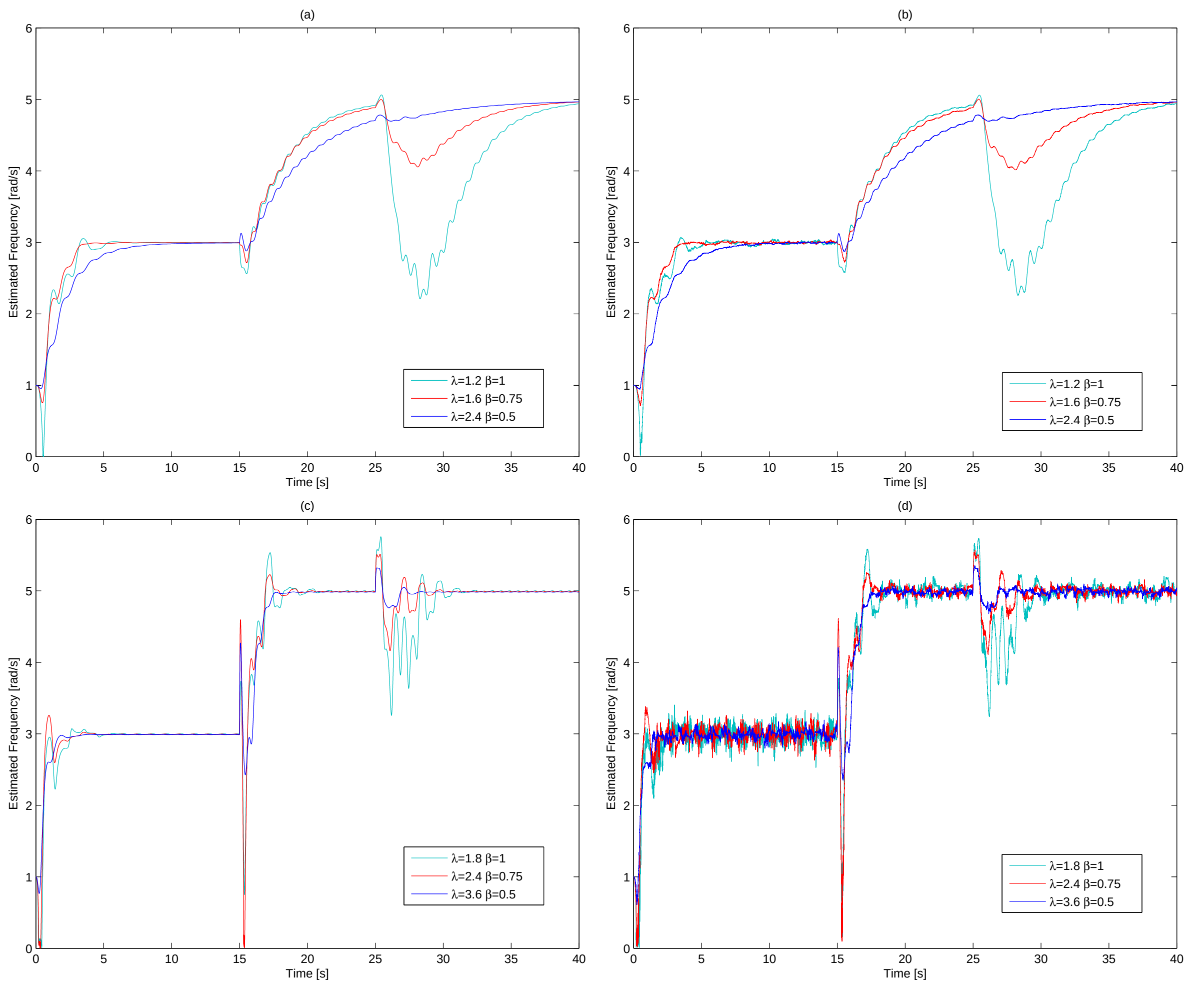

Fig. 4. (a) Estimated frequencies without unstructured noise and (b) with random noise $d(t)$ uniformly distributed in the interval $[-2.5, 2.5]$ using different values of λ and β such that $\lambda\beta = 1.2$. (c) and (d) Same as for (a) and (b) with $\lambda\beta = 1.8$.

analysis, that also provides tuning guidelines for the parameters of the proposed estimation algorithm, depending on the assumed noise level and on the required asymptotic accuracy. Simulation results have shown the effectiveness of the estimation technique.

Future research efforts will be devoted to extend the methodology to the case of multiple-frequencies estimation and to a larger class of structured measurement uncertainties.

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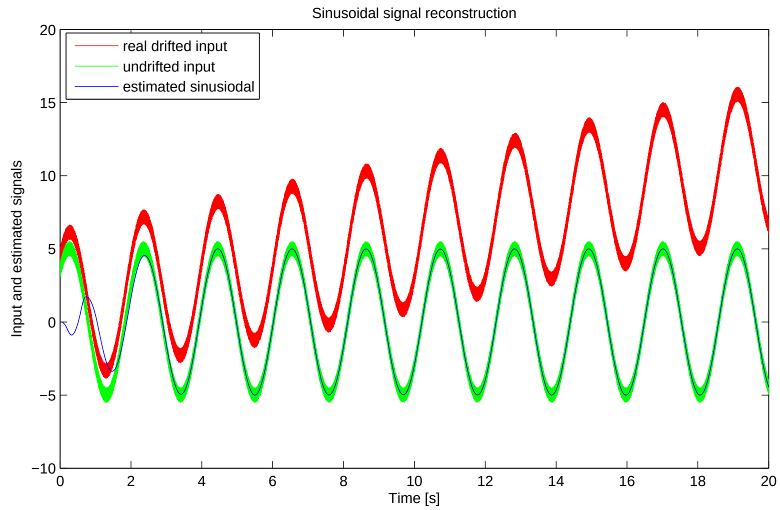

Fig. 5. Estimated sinusoidal signal by the proposed AFP method.

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