Explaining the Routh-Hurwitz criterion

A tutorial presentation

Marc Bodson
bodson@eng.utah.edu
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Routh's treatise [1] was a landmark in the analysis of stability of dynamic systems and became a core foundation of control theory. The remarkable simplicity of the result was in stark contrast with the challenge of the proof. Efforts were devoted by many researchers to extend the result to singular cases, with some of the earlier techniques shown to be inadequate [2]. Together with the extensions to singular cases, shorter proofs were also proposed. Noteworthy is the proof of [3], which followed the root locus arguments of [4]. A key feature of the proof is a continuity argument that had been used in an earlier derivation [5]. In [6], the more conventional approach using Cauchy's principle of the argument is followed. A relatively simple proof is proposed, considering the extension to complex polynomials and to singular cases.

Control textbooks describe the Routh-Hurwitz criterion, but do not explain how the result is obtained. Consequently, the procedure remains mysterious to many students and their teachers. The paper shows that the interpretation of the Routh array is straightforward, and that two proofs of the criterion can be completed shortly. The first proof is based on [3] and the second is inspired from [6], but using the Nyquist criterion instead of Cauchy's principle. The second proof is also similar to the one found in [7]. Small changes are made to the proofs to remove some technical steps and further simplify them. The derivations require only standard knowledge available from textbooks on feedback systems.

Given the computing power available today, the Routh-Hurwitz criterion has lost some of its importance, but it remains valuable in practical problems. The procedure makes it possible to obtain analytic stability conditions for specific applications involving multiple plant and controller parameters. In any case, the Routh-Hurwitz criterion remains a remarkable result of historical significance.

The Routh-Hurwitz criterion

Consider a polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0. \tag{1}$$

The first two rows of the *Routh array* are obtained by copying the coefficients of $p(s)$ using the pattern shown below.

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & \cdots \\
s^{n-2} & x_1 & x_2 & x_3 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

When a_0 is reached in one of the first two rows, blanks are left in the remaining slots, which are equivalent to zeros. The first two rows are labelled s^n and s^{n-1}, respectively. The third row is labelled s^{n-2} and has elements

$$x_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}},\ x_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}},\ \cdots \tag{2}$$

The computation is repeated for subsequent rows until the row labelled s^0 is reached. The case is called *regular* if no coefficient of the first column (also called a leading coefficient) is zero. Otherwise, the case is called *singular* and the algorithm stops prematurely.

If the case is regular, the Routh-Hurwitz criterion states that the number of right half-plane (RHP) roots of the polynomial $p(s)$ is equal to the number of sign changes in the first column of the array. The right half-plane (or left half-plane) is taken to be the part of the plane such that $\mathrm{Re}(s) > 0$ (or $\mathrm{Re}(s) < 0$). It turns out that there can be no root on the imaginary axis (such that $\mathrm{Re}(s) = 0$) in the regular case. Conversely, if the roots are in the left half-plane (LHP), the case must be regular. Therefore, the Routh-Hurwitz criterion implies that the roots of $p(s)$ are in the LHP if and only if all the elements of the first column are nonzero and have the same signs.

Explanation of the Routh array

The first two rows of the array contain the coefficients of the polynomials

$$p_1(s) = a_n s^n + a_{n-2} s^{n-2} + \ldots \tag{3}$$

$$p_2(s) = a_{n-1} s^{n-1} + a_{n-3} s^{n-3} + \ldots \tag{4}$$

where the elements that are zero by construction are omitted from the array. One of the polynomials $p_1(s)$ and $p_2(s)$ is even (that is, only has even powers of s, including s^0), and the other polynomial is odd (only has odd powers of s). A polynomial $p_3(s)$ is defined that is the remainder of the division of polynomial $p_1(s)$ by $p_2(s)$, so that

$$p_1(s) = q_1(s)p_2(s) + p_3(s), \tag{5}$$

where $q_1(s) = a_n\ s/a_{n-1}$ is the quotient. The third row of the array contains the coefficients of the remainder

$$p_3(s) = (a_{n-2} - a_{n-3}\ a_n/a_{n-1})s^{n-2} + (a_{n-4} - a_{n-5}\ a_n/a_{n-1})s^{n-4} + \dots \tag{6}$$

Repeating the procedure, polynomials $p_k(s)$ are constructed so that

$$p_{k+2}(s) = p_k(s) - q_k(s)p_{k+1}(s) \quad \text{for } k = 1, \cdots, n-1. \tag{7}$$

The polynomials $p_k(s)$ are of the form

$$p_k(s) = c_k s^{n-k+1} + \dots \tag{8}$$

where c_k is the leading coefficient of row k, with $c_1 = a_n$ and $c_2 = a_{n-1}$. The quotient polynomials are given by

$$q_k(s) = \frac{c_k}{c_{k+1}}\ s \quad \text{for } k = 1, \cdots, n-1. \tag{9}$$

The polynomials $p_k(s)$ alternate as even and odd polynomials of decreasing order. The Routh array contains the coefficients of these polynomials, omitting the coefficients that are always equal to zero due to the even/odd property. The labels on the left of the array give the highest power of s of the polynomials. If no c_k is equal to zero, the last two polynomials of the sequence are $p_n(s) = c_n s$ and $p_{n+1}(s) = c_{n+1}$.

Together with the polynomials $p_k(s)$, the procedure also produces a sequence of polynomials $p_k(s)+p_{k+1}(s)$, starting from the original polynomial $p(s) = p_1(s)+p_2(s)$. The Routh-Hurwitz criterion originates from a key property that applies to these polynomials at every step of the procedure.

Key property: assuming that $c_1, \cdots c_{k+1} \neq 0$, the number of roots of $p_k(s) + p_{k+1}(s)$ with $\text{Re}(s) < 0$ (or $\text{Re}(s) > 0$) is equal to the number of roots of $(1 + q_k(s))\,(p_{k+1}(s) + p_{k+2}(s))$ with $\text{Re}(s) < 0$ (or $\text{Re}(s) > 0$). The roots with $\text{Re}(s) = 0$ are identical in both polynomials, including their multiplicity.

Note that the last polynomial in the sequence is $p_n(s) + p_{n+1}(s) = c_n s + c_{n+1}$. Given that $1 + q_k(s) = (c_k s + c_{k+1})/c_{k+1}$, the Routh-Hurwitz criterion follows from the key property in a straightforward manner. One can also conclude that:

- a case where $p(s)$ has imaginary roots must be singular. Indeed, $1 + q_k(s)$ and $c_n s + c_{n+1}$ can only have real roots, so that the procedure must stop before the last step if there are imaginary roots.
- a case with $c_{k+1} = 0$ for some k has roots with $\mathrm{Re}(s) \geqslant 0$. Indeed, $c_{k+1} = 0$ if and only if the second coefficient of $p_k(s) + p_{k+1}(s)$ is zero. The second coefficient is the sum of the roots of $p_k(s) + p_{k+1}(s)$, which implies that some roots must be on the imaginary axis or in the right half-plane. The original polynomial must have at least the same number of roots with $\mathrm{Re}(s) \geqslant 0$.
- conversely, a case where $p(s)$ has all roots with $\mathrm{Re}(s) < 0$ must be regular.

First proof of the key property using continuity

The proof relies on the even/odd nature of the polynomials and properties that are straightforward to prove. An even polynomial $p_e(s)$ is such that $p_e(j\omega)$ is purely real. With $p_e(s) = p_e(-s)$, its roots must be pairs of imaginary roots ($s = \pm jb$), pairs of real roots ($s = \pm a$), or quadruples of complex roots ($s = \pm a \pm jb$). An odd polynomial $p_o(s)$ is such that $p_o(j\omega)$ is purely imaginary and $p_o(s) = s\ p_e(s)$, where $p_e(s)$ is an even polynomial. Its roots must include a root at $s = 0$, plus the same types of roots as an even polynomial. The sum of two even/odd polynomials is even/odd. The product of two even or two odd polynomials is even, and the product of an even polynomial with an odd polynomial is odd.

The proof presented here is mostly the same as to the one found in [3], with a small simplification obtained by considering a different polynomial in the analysis. The polynomial is

$$
\begin{aligned}
d_{k,g}(s) &= p_k(s) + p_{k+1}(s) + g\ q_k(s)\ p_{k+2}(s) && (10)\\
&= p_{k+2}(s) + q_k(s)p_{k+1}(s) + p_{k+1}(s) + g\ q_k(s)\ p_{k+2}(s). && (11)
\end{aligned}
$$

where $g \in [0, 1]$. For $g = 0$, $d_{k,0}(s) = p_k(s) + p_{k+1}(s)$, while for $g = 1$

$$
d_{k,1}(s) = (1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s)). \tag{12}
$$

The polynomial $d_{k,g}(s)$ in (10) is the sum of $p_k(s)$ and two polynomials of lower degree. Therefore, $d_{k,g}(s)$ has degree $n - k + 1$ for all $g \in [0, 1]$, and continuous branches connect the roots of $d_{k,0}(s)$ to the roots of $d_{k,1}(s)$.

Next, note that a root of $d_{k,g}(s)$ belongs to the imaginary axis if and only if, for some ω_0,

$$
p_{k+2}(j\omega_0) + q_k(j\omega_0)p_{k+1}(j\omega_0) + p_{k+1}(j\omega_0) + g\ q_k(j\omega_0)\ p_{k+2}(j\omega_0) = 0. \tag{13}
$$

Due to the even/odd alternation of the polynomials $p_k(s)$ and with $q_k(s)$ being an odd polynomial, the equation can be split into real and imaginary parts to give

$$
\begin{aligned}
p_{k+2}(j\omega_0) + q_k(j\omega_0)p_{k+1}(j\omega_0) &= 0 && (14)\\
p_{k+1}(j\omega_0) + g\ q_k(j\omega_0)\ p_{k+2}(j\omega_0) &= 0. && (15)
\end{aligned}
$$

It follows that

$$\left(1 - g\ q_k^2(j\omega_0)\right) p_{k+2}(j\omega_0) = 0. \tag{16}$$

With $1 - g\ q_k^2(j\omega_0) = 1 + g\left(c_k\ \omega_0/c_{k+1}\right)^2 \geqslant 1$, $p_{k+2}(j\omega_0) = 0$, and $p_{k+1}(j\omega_0) = 0$ as well. This result is true for all $g \in [0, 1]$, so that any root of $d_{k,g}(s)$ on the imaginary axis for some g is a root of $p_k(s) + p_{k+1}(s)$, a root of $p_{k+1}(s) + p_{k+2}(s)$, and a root of $d_{k,g}(s)$ for *all* g. Imaginary roots remain at their location, and no root of $d_{k,g}(s)$ can move from the right half-plane or the left half-plane to the imaginary axis. Therefore, no root can also move from the right half-plane to the left half-plane and vice-versa. The key property follows.

Second proof of the key property using the Nyquist criterion

The key property can also be proved by using the Nyquist criterion, and we assume that $p_k(s) + p_{k+1}(s)$ and $p_{k+1}(s) + p_{k+2}(s)$ have no roots on the imaginary axis to keep the proof simple. Consider the open-loop transfer function

$$G_k(s) = \frac{-q_k(s)p_{k+2}(s)}{(1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s))}. \tag{17}$$

The poles of this transfer function are the roots of

$$p_{ol}(s) = (1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s)), \tag{18}$$

while the poles of the closed-loop transfer function $G_k(s)/(1 + G_k(s))$ are the roots of

$$
\begin{aligned}
p_{cl}(s) &= (1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s)) - q_k(s)p_{k+2}(s) && (19)\\
&= p_k(s) + p_{k+1}(s). && (20)
\end{aligned}
$$

The Nyquist criterion specifies that the number of RHP roots of $p_{cl}(s)$ is equal to the number of RHP roots of $p_{ol}(s)$ plus the number of clockwise encirclements of (-1, 0) by the curve $G_k(s)$ computed along the Nyquist contour. Because $G_k(s)$ has more poles than zeros, $\lim_{\omega\to\infty} G_k(j\omega) = \lim_{\omega\to-\infty} G_k(j\omega) = 0$. Also, $G_k(0) = 0$ because $q_k(s)$ has a zero at $s = 0$. With no pole on the imaginary axis, the Nyquist curve is a bounded and closed curve that reaches the origin for $\omega = 0$ and for $\omega \to \pm\infty$. Note that, with $c_{k+1} \neq 0$,

$$\left|\frac{q_k(j\omega)}{1 + q_k(j\omega)}\right| = \left|\frac{jc_k\omega}{c_{k+1} + jc_k\omega}\right| < 1 \ \text{ for all } \omega. \tag{21}$$

Similarly, $p_{k+1}(j\omega)$ is real and $p_{k+2}(j\omega)$ is imaginary, or vice-versa, so that

$$\left|\frac{p_{k+2}(j\omega)}{p_{k+1}(j\omega)+p_{k+2}(j\omega)}\right| \leqslant 1 \quad \text{for all } \omega. \tag{22}$$

It follows that $|G_k(j\omega)| < 1$ for all ω, including as $\omega \to \pm\infty$. As a result, there can be no encirclements of (-1,0) by the Nyquist curve and the key property follows.

Singular cases

The regular procedure stops when the leading coefficient $c_{k+1} = 0$. Two singular cases can be defined:

- **Singular case #1**: the leading coefficient is zero, but the row is not identically zero. Polynomial division could proceed, but would produce an odd polynomial $q_k(s)$ of degree 3 (or higher if the next coefficient is also zero). The sum of the roots of $1 + q_k(s)$ would be equal to zero, so that some roots would not be in the left half-plane.
- **Singular case #2**: the row of the Routh array is identically zero, so that $p_{k+2}(s) = 0$ and $p_{k+1}(s) + p_{k+2}(s) = p_{k+1}(s)$, which is either even or odd. Some roots of $p_{k+1}(s) + p_{k+2}(s)$ would not be in the left half-plane.

The two cases confirm that the polynomial $p(s)$ cannot have all roots with $\mathrm{Re}(s) < 0$ if some leading coefficient of the array is equal to zero. To continue counting the roots in the singular case, an alternate procedure is needed. In the most recent work, the preferred approach has consisted in replacing $p_{k+1}(s) + p_{k+2}(s)$ by a polynomial to which the regular procedure can be applied and to which the root locations can be related. [3] gives an approach for singular cases based on [8] and even provides a short Matlab code to count the roots in the right half-plane, left half-plane, and on the imaginary axis. However, the main justification for counting the roots in the singular case is to determine whether a system is marginally stable. So, one needs to know whether any root on the imaginary axis is repeated. [9] and [10] propose Routh-like procedures for singular cases to determine whether any imaginary root is repeated. Still, the usefulness of procedures for singular cases is limited from a practical perspective, since a system is known to be bounded-input bounded-output unstable as soon as a zero leading coefficient is encountered in the Routh array.

Invariant roots

The key property implies that imaginary roots remain invariant at every step of the procedure. Interestingly, other roots are invariant as well. In [11], it was observed that the roots of the polynomial $p_{k+1}(s)$ in singular case #2 must appear in the original polynomial $p(s)$. This property follows from the recursion

$$p_k(s) = q_k(s) p_{k+1}(s) + p_{k+2}(s). \tag{23}$$

With $p_{k+2}(s) = 0$, $p_k(s)$ must be a multiple of $p_{k+1}(s)$. Similarly, $p_{k-1}(s)$ must be a multiple of $p_{k+1}(s)$, as well as every $p_j(s)$ for $j < k$. It follows that $p(s)$ must be a multiple of the last nonzero polynomial $p_{k+1}(s)$.

Conversely, suppose that we started from a polynomial $p(s) = p_a(s)\ p_m(s)$, where $p_m(s)$ is an even polynomial. Letting $p_a(s) = p_e(s) + p_o(s)$ where $p_e(s)$ is even and $p_o(s)$ is odd, $p(s)$ is the sum of the even polynomial $p_e(s)p_m(s)$ and the odd polynomial $p_o(s)p_m(s)$. $p_1(s)$ and $p_2(s)$ are equal to these two polynomials, and are therefore multiples of $p_m(s)$. The same result is true if $p_m(s)$ is an odd polynomial. From (23), every $p_k(s)$ is a multiple of $p_m(s)$ until the procedure stops.

The conclusion is that, if $p(s)$ is the multiple of an even or odd polynomial, every polynomial $p_k(s) + p_{k+1}(s)$ is a multiple of that polynomial. As a result, not only are purely imaginary roots invariant in the procedure, but also *any* pair of roots that are symmetric with respect to the imaginary axis. The presence of such roots in the polynomial $p(s)$ implies that the case must be singular.

Examples

Example 1 - Using the Routh-Hurwitz criterion to find stability conditions

Consider the control system of Fig. 1. The plant is an electric motor with an inner torque control loop, resulting in the equation

$$\theta = \frac{1}{Js^2} \tau_{COM}, \tag{24}$$

where θ is the angular position of the motor (in rad), J is the inertia of the motor and load (in kg-m^2), and τ_{COM} is the torque command (in N-m). The controller is a proportional integral derivative (PID) control law

$$\tau_{COM} = \left(k_P + \frac{k_I}{s} \right) (\theta_{REF} - \theta) - k_D \frac{a_F\ s}{s + a_F} \theta, \tag{25}$$

where θ_{REF} is the reference input for the position, and k_P, k_I, and k_D are the PID gains. The derivative term is filtered by a first-order system with a pole at $s = -a_F$ to reduce the high-frequency noise originating from the differentiation of the position measurement. The derivative action is not applied to the reference input to avoid large transients when step inputs are applied. The objective is to find conditions on the PID gains so that the closed-loop system is stable. J and a_F are positive parameters.

The closed-loop polynomial is

$$p(s) = Js^4 + Ja_F s^3 + (k_P + k_D a_F)\, s^2 + (k_P a_F + k_I)\, s + k_I a_F, \tag{26}$$

so that the Routh array is given by

$$\begin{array}{c|ccc} s^4 & J & k_P + k_D a_F & k_I a_F \\ s^3 & Ja_F & k_P a_F + k_I & \\ s^2 & x_1 & k_I a_F & \\ s^1 & y_1 & & \\ s^0 & k_I a_F & & \end{array}$$

where

$$x_1 = k_D a_F - k_I/a_F, \;\; y_1 = k_P a_F + k_I - \frac{J k_I a_F^2}{x_1}. \tag{27}$$

It follows that the conditions that the PID gains must satisfy for stability are

$$k_I > 0, \;\; k_D > \frac{k_I}{a_F^2}, \;\; k_P > \frac{J k_I a_F^2}{k_D a_F^2 - k_I} - \frac{k_I}{a_F}. \tag{28}$$

Example 2 - Root locus in a regular case

Consider the polynomial $p(s) = s^6 + 4s^5 + 8s^4 + 6s^3 + s^2 + 10s + 50$, with the Routh array

$$\begin{array}{c|cccc} s^6 & 1 & 8 & 1 & 50 \\ s^5 & 4 & 6 & 10 & \\ s^4 & 6.5 & -1.5 & 50 & \\ s^3 & 6.92 & -20.77 & & \\ s^2 & 18 & 50 & & \\ s^1 & -40 & & & \\ s^0 & 50 & & & \end{array}$$

Fig. 2 shows the root locus obtained through the procedure of the Routh-Hurwitz criterion. The locus is a sequence of root loci truncated to $g \in [0, 1]$, rather than a single conventional root locus with $g \in [0, \infty)$. The locations of the roots at each step are marked by red dots. The roots of $p_1(s) + p_2(s)$ are marked with the green label 1. For $k > 1$, the roots of $(1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s))$ are identified by the number $k + 1$, with the label for the root of $1 + q_k(s)$ placed in a box. Such a root marks the end of a branch. The procedure is repeated at every step with a decreasing number of roots. All roots end their journey on the real axis, and on the same side of the imaginary axis as the side from which they started.

Example 3 - Root locus in a singular case with imaginary roots

Consider the polynomial $p(s) = s^6 + 2s^5 + 3s^4 + 26s^3 + 26s^2 + 72s + 720$. The polynomial has a pair of imaginary roots, so that the Routh array stops before the end.

s^6	1	3	26	720
s^5	2	26	72	
s^4	-10	-10	720	
s^3	24	216		
s^2	80	720		
s^1	0			

The example corresponds to singular case #2, with the row s^1 equal to zero. The root locus is shown on Fig. 3. One finds that the imaginary roots do not move throughout the procedure. The other roots find their way to the real axis, and the algorithm stops when the two imaginary roots remain alone. The roots of $p_5(s)+p_6(s) = 80s^2+720$ are the same as the original imaginary roots at $s = \pm j3$.

Example 4 - Root locus in a singular case without imaginary roots

Consider the polynomial $p(s) = s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$, with the Routh array

s^5	1	3	3
s^4	2	2	2
s^3	2	2	0
s^2	0	2	0

The procedure ends prematurely after two steps, even though there are no imaginary roots. The example corresponds to singular case #1, with the leading coefficient of row s^2 equal to zero. The root locus is shown on Fig. 4. The last polynomial is $p_3(s) + p_4(s) = 2s^3 + 2s + 2$ and has roots at $0.3412 \pm 1.1615j$, and -0.6823. The sum of the roots is equal to zero. These roots are marked with the label 3 (without the box) on the figure

Example 5 - Nyquist diagram

Consider the polynomial $p(s) = s^3 + 3s^2 + 3s + (1 + g_0)$, with the Routh array

s^3	1	3
s^2	3	$(1 + g_0)$
s^1	$(8 - g_0)/3$	
s^0	$1 + g_0$	

The Routh-Hurwitz criterion implies that no roots of $p(s)$ lie in the RHP if $-1 < g_0 < 8$. For $g_0 > 8$, there are two sign changes and therefore two roots in the RHP. Fig. 5 shows the Nyquist plots of $G_k(s)$ for $k = 1$ and $k = 2$, and for $g_0 = 1$ and $g_0 = 20$. A third curve shows the

Nyquist plot for $k = 2$ and $g_0 = 10$ (the $k = 1$, $g_0 = 10$ curve is omitted to avoid overloading the plot). The positive and negative frequency curves for $k = 2$ happen to overlap exactly in this example.

There are no encirclements of (-1,0) by any curve because $|G_k(j\omega)| < 1$ for all k and for all ω. The intersection with the real axis becomes closer and closer to (-1,0) for $k = 2$ as g_0 reaches 8, but the intersection remains to the right of (-1,0) for any $g_0 > 0$ different from 8. The number of encirclements does not change regardless of the stability of the system, because the Nyquist criterion is not used to count the number of RHP roots of the original polynomial, but to compare two polynomials with the same number of RHP roots.

Conclusions

The paper gave an explanation and two short proofs of the Routh-Hurwitz criterion. The proofs were based on results presented in the literature after the original work of Routh. The author hopes that this tutorial presentation will be valuable in satisfying the curiosity of motivated students and their teachers, while providing interesting examples of application of root locus plots and of the Nyquist criterion.

References

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Figures

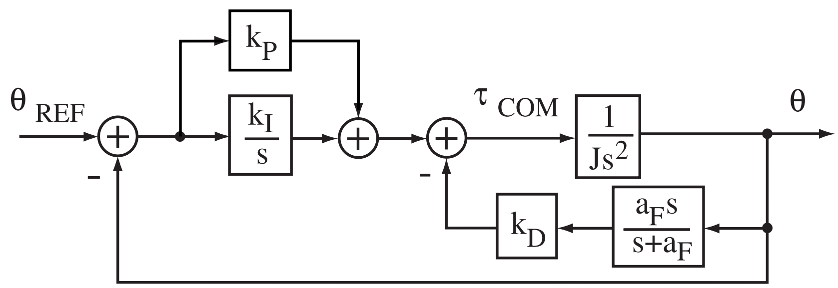

Figure 1. Proportional integral derivative control scheme for an electric motor. θ_{REF} is the reference position, τ_{COM} is the torque command, and θ is the angular position of the motor. A first-order filter is integrated with the derivative term.

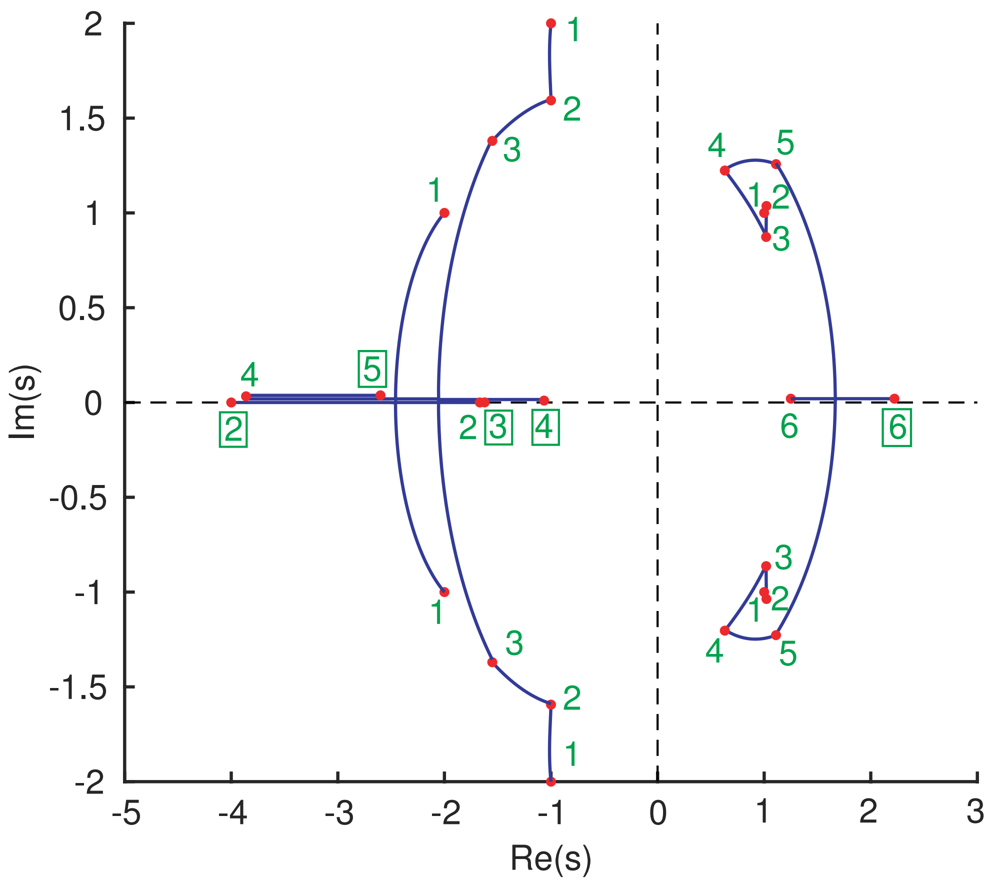

Figure 2. Root locus plot for a regular case. The roots move at every step but remain on the same side of the imaginary axis. Roots in a box are roots of $1 + q_k(s)$ and mark the end of a branch.

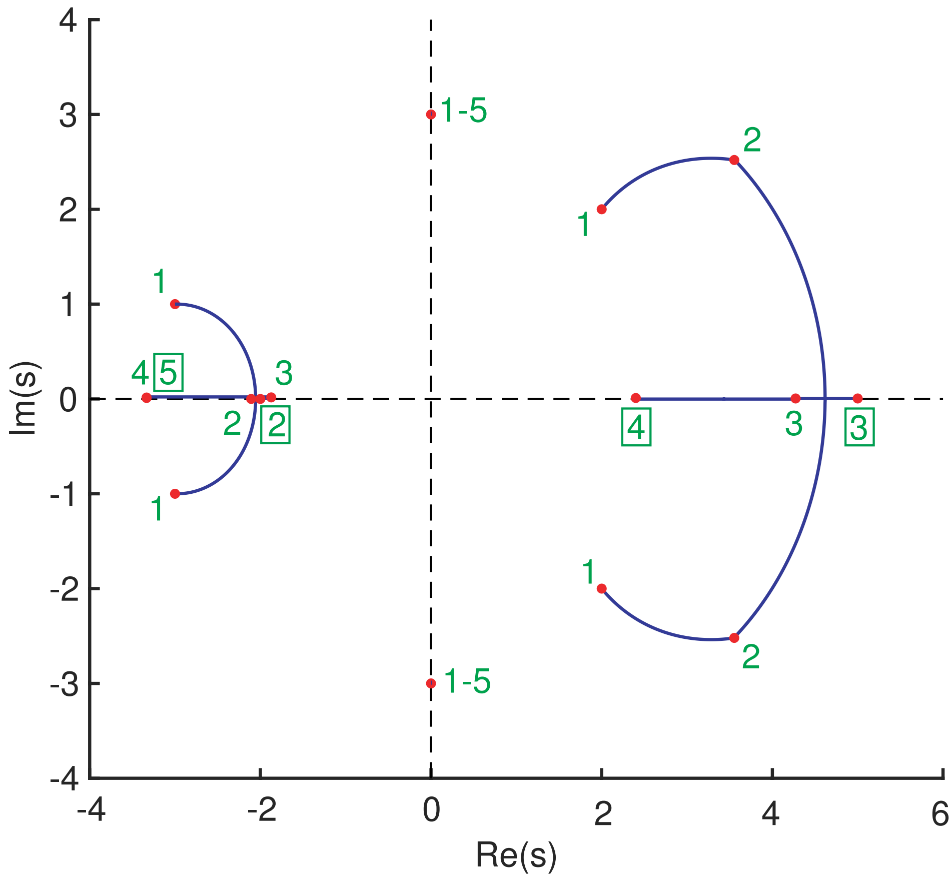

Figure 3. Root locus plot for a singular case with imaginary roots. The roots with nonzero real parts remain on the same side of the imaginary axis. The imaginary roots stay at the same place, eventually causing the procedure to stop with a zero leading coefficient in the Routh array.

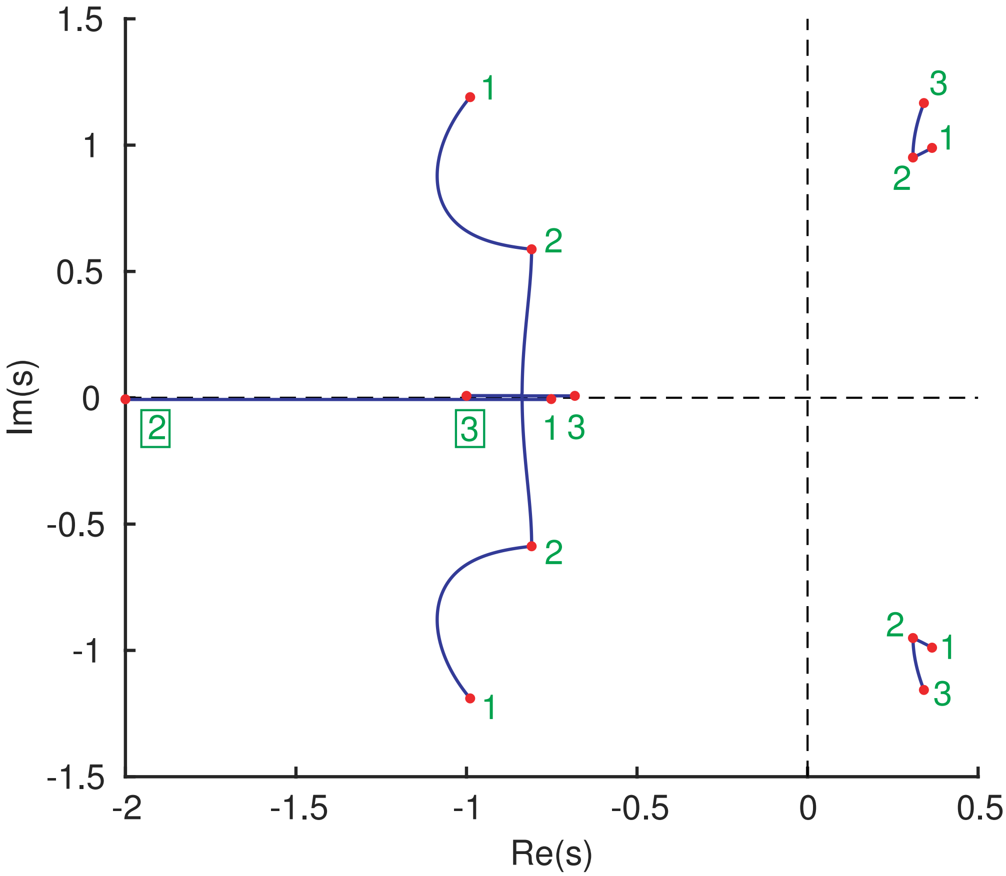

Figure 4. Root locus plot for a singular case without imaginary roots. The procedure stops because the sum of the three roots labelled 3 (without the box) is equal to zero, causing a leading coefficient of the Routh array to be equal to zero.

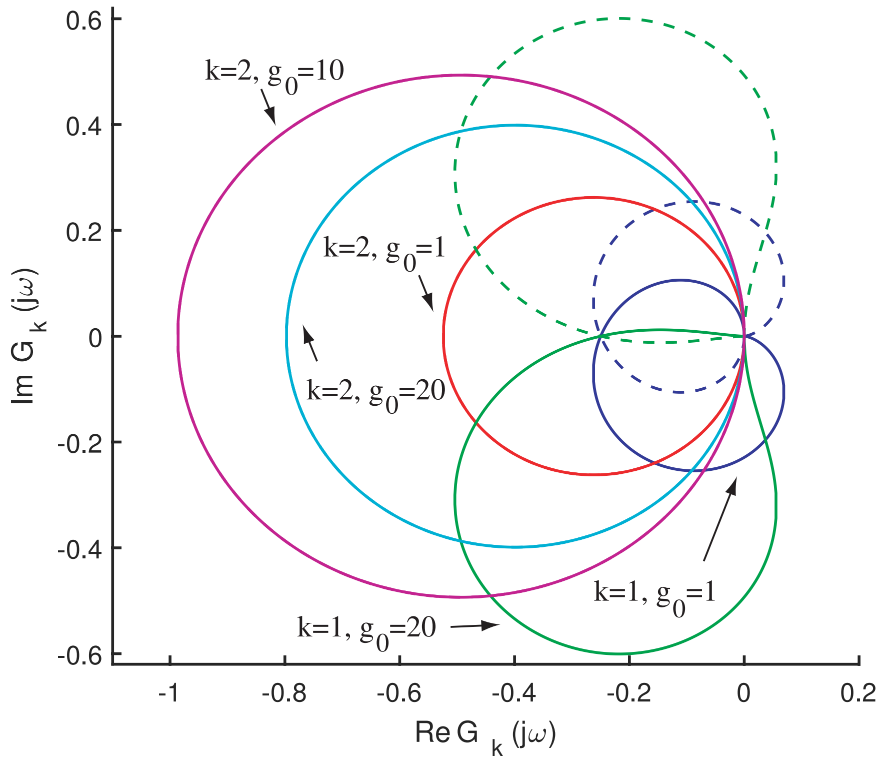

Figure 5. Nyquist plots associated with the second proof. All the Nyquist curves fit strictly inside a circle of magnitude one, implying that the number of right half-plane and left half-plane roots are the same in the two polynomials.

Sidebar: Summary

The Routh-Hurwitz criterion is a mathematical tool used to determine whether all the roots of a polynomial have negative real parts. The algorithm makes it possible to determine whether a closed-loop system is stable, including the conditions needed on plant and controller parameters so that stability is achieved. The procedure of the Routh-Hurwitz criterion is relatively simple, but the proof of the result has been elusive to students and their teachers. The paper shows that an explanation of the Routh-Hurwitz criterion can be presented shortly at the level of an introductory control course.

Sidebar: Applications of the Routh-Hurwitz criterion

Although the roots of polynomials are easily computed numerically nowadays, the Routh-Hurwitz criterion remains useful to determine how stability is affected by multiple plant and controller parameters. In [S1], a bound is derived for the gain of a DC-DC buck converter as a function of five system parameters. The minimum input voltage required for the stable operation of a type-3 PLL is obtained in [S2], while a condition relates the four circuit parameters of a constant-power load damper circuit in [S3]. Sometimes, the objective is to achieve *instability*, such as in the design of an oscillator in [S4]. For the control of a remotely piloted aircraft [S5], the Routh-Hurwitz criterion gives a condition to be satisfied by the load parameters so that stability is guaranteed. The condition is a function of the mass and inertia of the helicopter, the aerodynamic parameters, and the controller parameters. A set of inequalities is obtained in [S6] to ensure that a fixed structure/fixed order controller using Groebner bases is stabilizing.

Less conventional applications can be found such as the synchronization of fractional order chaotic systems, with application to cryptography [S7]. [S8] addresses the stability of the dynamics of HIV infection and drug therapy, and is representative of a class of papers where the Routh-Hurwitz criterion is used to evaluate the stability of a biological model. Similarly, the stability of genetic circuits is the focus of [S9]. The extension of the stability test to systems with complex parameters is considered in [S10], but uses the version using Hurwitz determinants instead of the Routh array. The 6^{th}-order model of a self-excited induction generator is transformed into an equivalent 3^{rd}-order system with complex coefficients, and analytic conditions are deduced for the instability of the zero equilibrium, a necessary condition for generation. In [S11], a simple condition is found to ensure the stability of a two-input two-output proportional integral control law applied to a doubly-fed induction generator.

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Author Information

Marc Bodson (bodson@eng.utah.edu) received the degree of Ingénieur Civil Mécanicien et Electricien from the Université Libre de Bruxelles, Belgium. Subsequently, he obtained two M.S. degrees, one in Electrical Engineering and Computer Science and the other in Aeronautics and Astronautics, both from the Massachusetts Institute of Technology, in Cambridge MA. He received a Ph.D. degree in Electrical Engineering and Computer Science from the University of California, Berkeley. Currently, he is a professor in the department of Electrical and Computer Engineering at the University of Utah, where he served as Chair between 2003 and 2009. He was the Editor-in-Chief of IEEE Trans. on Control Systems Technology from 2000 to 2003. He was elected Fellow of the IEEE in 2006, and Associate Fellow of the American Institute of Aeronautics and Astronautics in 2013. He received the Engineering Educator of the Year award from the Utah Engineers Council in 2007.