Introduction to Feedback Systems

Solutions to selected problems

Includes solutions to problems:

2.1 to 2.4, 3.1 to 3.8, 4.1 to 4.9, 5.1 to 5.5, 6.1 to 6.12, 7.1 to 7.3

2.1 (a) RESIDUE: $X(s) = \frac{c_{11}}{s+2} + \frac{c_{21}}{s-2}$

$$c_{11} = \left[(s+2)X(s)\right]_{s=-2} = \left[\frac{2s}{s-2}\right]_{s=-2} = 1 \Rightarrow X(s) = \frac{1}{s+2} + \frac{1}{s-2}$$

$$c_{21} = \left[(s-2)X(s)\right]_{s=+2} = \left[\frac{2s}{s+2}\right]_{s=2} = 1 \Rightarrow x(t) = e^{-2t} + e^{2t}$$

CLEARING FRACTIONS:

$$X(s) = \frac{c_{11}(s-2) + c_{21}(s+2)}{(s+2)(s-2)} \Rightarrow \begin{matrix} c_{11} + c_{21} = 2 \\ -2c_{11} + 2c_{21} = 0 \end{matrix}$$

$$\Rightarrow c_{11} = c_{21}, \quad 2c_{11} = 2 \Rightarrow c_{11} = 1 = c_{21} \Rightarrow \text{SAME ANSWER}$$

(b) RESIDUE $X(s) = \frac{c_{11}}{s} + \frac{c_{12}}{s^2} + \frac{c_{21}}{s+1} + \frac{c_{22}}{(s+1)^2}$

$$c_{12} = \left[s^2 X(s)\right]_{s=0} = \left[\frac{2s+1}{(s+1)^2}\right]_{s=0} = 1$$

$$c_{11} = \left[\frac{d}{ds}\left(s^2 X(s)\right)\right]_{s=0} = \left[\frac{2(s+1)^2 - 2(s+1)(2s+1)}{(s+1)^2}\right]_{s=0} = 0$$

$$c_{21} = \left[(s+1)^2 X(s)\right]_{s=-1} = \left[\frac{2s+1}{s^2}\right]_{s=-1} = -1$$

$$c_{22} = \left[\frac{d}{ds}(s+1)^2 X(s)\right]_{s=-1} = \left[\frac{2s^2 - 2s(2s+1)}{s^2}\right]_{s=-1} = 0$$

$$\Rightarrow X(s) = \frac{1}{s^2} + \frac{-1}{(s+1)^2} \Rightarrow x(t) = t - te^{-t}$$

CLEARING FRACTIONS:

$$X(s) = \frac{c_{11}s(s+1)^2 + c_{12}(s+1)^2 + c_{21}s^2(s+1) + c_{22}s^2}{s^2(s+1)^2}$$

$$\Rightarrow c_{11}s^3 + c_{11}2s^2 + c_{11}s + c_{12}s^2 + 2c_{12}s + c_{12} + c_{21}s^3 + c_{21}s^2 + c_{22}s^2$$

$$= 2s+1$$

$c_{11} + c_{21} = 0\ ,\quad 2c_{11} + c_{12} + c_{21} + c_{22} = 0,\ c_{11} + 2c_{12} = 2\ ,\ c_{12} = 1$

$\Rightarrow c_{12} = 1\ ,\ c_{11} = 0\ ,\ c_{21} = 0\ ,\ c_{12} = -1$

$\Rightarrow$ SAME ANSWER

(c) RESIDUE: $X(s) = \dfrac{c_{11}}{s} + \dfrac{c_{12}}{s^2} + \dfrac{c_{21}}{s-p} + \dfrac{c_{31}}{s-p^*}$

WITH $p = -1 + 2j,\ c_{31} = c_{21}^*$

$$c_{12} = \left[s^2 X(s)\right]_{s=0} = 1$$

$$c_{11} = \left[\frac{(3s^2+4s+2)(s^2+2s+5) - (2s^2+2)(s^3+2s^2+2s+5)}{(s^2+2s+5)^2}\right]_{s=0} = 0$$

$$c_{21} = \left[(s-p)X(s)\right]_{s=p} = \frac{p^3+2p^2+2p+5}{p^2\ (p-p^*)} = \frac{11-2j-6-8j-2+4j+5}{(-3-4j)(4j)} = \frac{8-6j}{16-12j} = \frac{1}{2}$$

$p^2 = +1-4-4j = -3-4j \qquad p^3 = 3+8-6j+4j = 11-2j$

$$\Rightarrow x(t) = c_{11} + c_{12}t + 2\,\mathrm{Re}(c_{21})\,e^{-t}\cos(2t) - 2\,\mathrm{Im}(c_{21})\,e^{-t}\sin(2t)$$
$$= t + e^{-t}\cos(2t)$$

CLEARING FRACTIONS: $X(s) = \dfrac{N(s)}{s^2(s^2+2s+5)}$

$$N(s) = c_{11}\, s\,(s^2+2s+5) + c_{12}\,(s^2+2s+5) + c_{21}\, s^2(s-p^*) + c_{31}\, s^2(s-p) = s^3+2s^2+2s+5$$

$$= s^3(c_{11}+c_{21}+c_{31}) + s^2\big(2c_{11} + c_{12} + c_{21}(1+2j) + c_{31}(+1-2j)\big) + s(5c_{11} + 2c_{12}) + 5c_{12} = s^3+2s^2+2s+5$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1+2j & 1-2j \\ 5 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 5 \end{pmatrix}$$

SOLVE SYSTEM:

$5c_{12} = 5 \Rightarrow c_{12} = 1$

$5c_{11} + 2c_{12} = 2 \Rightarrow c_{11} = 0$

$\left.\begin{array}{l} c_{11} + c_{21} + c_{31} = 1 \\ c_{31} = c_{21}^* \end{array}\right\} 2\,\mathrm{Re}(c_{21}) = 1 \Rightarrow \mathrm{Re}(c_{21}) = \frac{1}{2}$

$\underbrace{2c_{11}}_{0} + \underbrace{c_{12}}_{1} + \underbrace{(c_{21} + c_{31})}_{1} + 2j\underbrace{(c_{21} - c_{31})}_{2j(\mathrm{Im}(c_{21}))} = 2 \Rightarrow \mathrm{Im}(c_{21}) = 0$

$\Rightarrow c_{21} = \frac{1}{2}$

$\Rightarrow$ SAME RESULT

ALTERNATIVE APPROACH: let $A = 2\,\mathrm{Re}(c_{21})$, $B = 2\,\mathrm{Im}(c_{21})$

$\Rightarrow \quad (c_{21} + c_{31}) = A \qquad (c_{21} - c_{31}) = jB$

$\Rightarrow$ APPLY TO $N(s)$:

$N(s) = s^3(c_{11} + A) + s^2(2c_{11} + c_{12} + A - 2B) + s(5c_{11} + 2c_{12}) + 5c_{12} = s^3 + 2s^2 + 2s + 5$

$\Rightarrow$ SOLVE A SYSTEM OF EQUATIONS WITH $\underline{\underline{\text{REAL}}}$ COEFFICIENTS

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & -2 \\ 5 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 5 \end{pmatrix} \rightarrow \text{SAME ANSWER}$$

(d) $X(s) = \dfrac{c_{11}}{s-2j} + \dfrac{c_{12}}{(s-2j)^2} + \dfrac{c_{21}}{(s+2j)} + \dfrac{c_{22}}{(s+2j)^2}$

RESIDUE:

$c_{21} = c_{11}^*, \quad c_{22} = c_{12}^*$

$$c_{12} = \left[\frac{4s-8}{(s+2j)^2}\right]_{s=2j} = \frac{8j-8}{-16} = \frac{1}{2} - \frac{1}{2}j$$

$$c_{11} = \left[\frac{4(s+2j)^2 - 2(s+2j)(4s-8)}{(s+2j)^4}\right]_{s=2j} = \frac{4\times(-16) - 8j(8j-8)}{16\times 16} = \frac{1}{4}j$$

$$\Rightarrow x(t) = 2\,\mathrm{Re}(c_{11})\cos(2t) - 2\,\mathrm{Im}(c_{11})\sin(2t)$$
$$+ 2\,\mathrm{Re}(c_{12})\, t\cos(2t) - 2\,\mathrm{Im}(c_{12})\, t\sin(2t)$$
$$= -\frac{1}{2}\sin(2t) + t\cos(2t) + t\sin(2t)$$

CLEARING FRACTIONS:

$$X(s) = \frac{c_{11}(s-2j)(s+2j)^2 + c_{12}(s+2j)^2 + c_{21}(s-2j)^2(s+2j) + c_{22}(s-2j)^2}{(s-2j)^2(s+2j)^2}$$

$$\Rightarrow \quad c_{11}(s^3+2js^2+4s+8j) + c_{12}(s^2+4js-4)$$
$$+ c_{21}(s^3-2js^2+4s-8j) + c_{22}(s^2-4js-4) = 4s-8$$

NOTE: complex conjugate coefficients of previous line in this line

LET: $A = c_{11} + c_{21}$ (since $c_{21} = c_{11}^*$, $A = 2\,\mathrm{Re}(c_{11})$)

$jB = c_{11} - c_{21}$ (since $c_{21} = c_{11}^*$, $B = 2\,\mathrm{Im}(c_{11})$)

$C = c_{12} + c_{22}$ (since $c_{12} = c_{22}^*$, $C = 2\,\mathrm{Re}(c_{12})$)

$jD = c_{12} - c_{22}$ (since $c_{12} = c_{22}^*$, $D = 2\,\mathrm{Im}(c_{12})$)

$$\Rightarrow \quad s^3(A) + s^2(-2B + C) + s(4A - 4D) + (-8B + 4D) = 4s - 8$$

$A = 0$, $-2B + C = 0$, $4A - 4D = 4 \Rightarrow D = -1$, $-8B - 4C = -8 \Rightarrow C = 1$, $B = \frac{1}{2}$

$$c_{11} = \frac{1}{2}A + \frac{1}{2}Bj \qquad c_{12} = \frac{1}{2}C + \frac{1}{2}Dj$$
$$= +\frac{1}{4}j \qquad\qquad = \frac{1}{2} - \frac{1}{2}j$$

$\Rightarrow$ SAME ANSWER.

2.2 (a) $$X(s) = \frac{c_{11}}{s} + \frac{c_{12}}{s^2} + \frac{c_{13}}{s^3} + \frac{c_{21}}{s+1-2j} + \frac{c_{22}}{(s+1-2j)^2} + \frac{c_{21}^*}{s+1+2j} + \frac{c_{22}^*}{(s+1+2j)^2}$$

$$\Rightarrow x(t) = c_{11} + c_{12}\, t + c_{13}\, \frac{t^2}{2} + 2\,|c_{21}|\, e^{-t} \cos(2t + \angle c_{21}) + 2\,|c_{22}|\, t\, e^{-t} \cos(2t + \angle c_{22})$$

$c_{13} \neq 0$ $\quad c_{22} = 0$ (either Re c_{22} or Im $c_{22} \neq 0$)

other coefficients may be zero.

(b) $$X(s) = \frac{c_{11}}{s+2} + \frac{c_{12}}{(s+2)^2} + \frac{c_{13}}{(s+2)^3} + \frac{c_{14}}{(s+2)^4} + \frac{c_{21}}{s-3} + \frac{c_{22}}{(s-3)^2} + \frac{c_{23}}{(s-3)^3} + \frac{c_{31}}{(s+4)}$$

$$\Rightarrow x(t) = c_{11} e^{-2t} + c_{12}\, t\, e^{-2t} + c_{13}\, \frac{t^2}{2} e^{-2t} + c_{14}\, \frac{t^3}{6} e^{-2t} + c_{21} e^{3t} + c_{22}\, t\, e^{3t} + c_{23}\, \frac{t^2}{2} e^{3t} + c_{31} e^{-4t}$$

$c_{14} \neq 0$, $\quad c_{23} \neq 0$, $\quad c_{31} \neq 0$

other coefficients may be zero

2.3 (a) BOUNDED, LIM = 0

(b) BOUNDED, LIM = $-\frac{1}{2}$

(c) UNBOUNDED

(d) BOUNDED, LIM = 5

(e) BOUNDED, NO LIMIT

(f) UNBOUNDED

(g) BOUNDED, LIM = 0

(h) BOUNDED, LIM = 0

(i) BOUNDED, LIM = 0

2.4 (a) NO (NEED LOWER BOUND)

(b) TIME CONSTANT $\simeq$ 10 ms
IN 40 ms, THE SIGNAL HAS DECAYED TO 2% OF ITS VALUE AROUND t = 0

(c) IT IS BOUNDED, BUT IT DOES NOT CONVERGE.

3.1 (a) $sL\, I(s) = V(s) - R\, I(s) - K\omega(s)$

$$s J\, \omega(s) = K\, I(s)$$

(assume $i(t)$, $\omega(t)$ are zero at $t=0$ to compute the transfer function)

$$I(s) = \frac{V(s) - K\omega(s)}{sL + R}$$

$$\Rightarrow (sJ)(sL+R)\,\omega(s) = K\,(V(s) - K\omega(s))$$

$$\Rightarrow (JL\, s^2 + RJ\, s + K^2)\,\omega(s) = K\, V(s)$$

$$H(s) = \frac{\omega(s)}{V(s)} = \frac{K}{JL s^2 + RJ s + K^2} = \frac{K/JL}{s^2 + \frac{R}{L}s + \frac{K^2}{JL}}$$

POLES: $s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{K^2}{JL}}$

ZEROS: NONE

(b) $L = 0 \Rightarrow H(s) = \dfrac{K/RJ}{s + K^2/RJ}$

POLES: $s = -K^2/RJ$ ZEROS: NONE

(c) $L \neq 0$ $H(s) = \dfrac{49.1\ 10^3}{s^2 + 280s + 34.4\ 10^3}$ POLES: $-140 \pm 120j$

$L = 0$ $H(s) = \dfrac{1754}{s + 123}$ POLE: -123

(CLOSE TO OTHER 2 POLES)

3.2 (a) $X_1 = X - Y$ $\quad Y = H_2 X_2$

$X_2 = H_1 X_1 - H_3 Y$

ELIMINATE $X_1, X_2 \Rightarrow$ $X_2 = H_1 X - H_1 Y - H_3 Y =$

$Y = H_2 H_1 X - H_2 H_1 Y - H_2 H_3 Y$

$(1 + H_1 H_2 + H_2 H_3) Y = H_2 H_1 X$

$$\Rightarrow H(s) = \frac{H_1(s) H_2(s)}{1 + H_1(s) H_2(s) + H_2(s) H_3(s)}$$

(b) $X_1 = X - X_2$ $\quad Y = H_4 X_2 + H_3 H_1 X_1$

$X_2 = H_1 X_1 + H_2 X$

ELIMINATE X_1, X_2 $\quad X_2 = H_1 X - H_1 X_2 + H_2 X$

$Y = H_4 X_2 + H_3 H_1 X - H_3 H_1 X_2$

$(1 + H_1) X_2 = (H_1 + H_2) X$

$Y = (H_4 - H_3 H_1) X_2 + H_3 H_1 X$

$$= \frac{(H_4 - H_3 H_1)(H_1 + H_2)}{1 + H_1} X + H_3 H_1 X$$

$$\rightarrow H(s) = \frac{H_1(s) H_4(s) + H_2(s) H_4(s) - H_1(s) H_2(s) H_3(s) + H_1(s) H_3(s)}{1 + H_1(s)}$$

3.3 (a) STABLE (b) UNSTABLE (c) STABLE

(d) UNSTABLE (e) UNSTABLE (F) UNSTABLE

BOUNDED INPUT PRODUCING AN UNBOUNDED OUTPUT

(b) $x(t) = \cos(2t)$

(d) $x(t) = 1$

(e) $x(t) = 1$

(f) $x(t) = 1$

3.4 (a) DC GAIN $H(0) = 2$

STEP RESPONSE $Y(s) = \dfrac{2}{s\,(s^2+2s+1)}$

$$= \frac{c_{11}}{s} + \frac{c_{21}}{s+1} + \frac{c_{22}}{(s+1)^2}$$

$c_{11} = 2$, $c_{22} = -2$, $c_{21} = -2$ $\Rightarrow$ $y(t) = 2 - 2e^{-t} - 2t\,e^{-t}$

$\lim_{t\to\infty} y(t) = 2 = H(0) \times \underset{1}{\underset{\shortparallel}{x(\infty)}}$ $\Rightarrow$ SAME RESULT

(b) DC GAIN $H(0) = -1$ $\qquad p = -1 \pm j$

STEP RESPONSE $Y(s) = \dfrac{-s-2}{s(s^2+2s+2)}$

$$= \frac{c_{11}}{s} + \frac{c_{21}}{s+1-j} + \frac{c_{31}}{s+1+j}$$

$c_{11} = -1$, $c_{21} = \dfrac{1-j-2}{(-1+j)(2j)} = \dfrac{-1-j}{-2-2j} = \dfrac{1}{2} = c_{31}^{*}$

$\Rightarrow y(t) = -1 + e^{-t}\cos(t)$

$\lim_{t\to\infty} y(t) = -1 = H(0) \times x(\infty)$ $\Rightarrow$ SAME RESULT

3.5 (a) $H(s) = \dfrac{(\frac{1}{R} + sC)^{-1}}{(\frac{1}{R}+sC)^{-1} + R + sL} = \dfrac{R}{R + (1+sCR)(R+sL)}$

$R = L = C = 1 \Rightarrow H(s) = \dfrac{1}{s^2+2s+2}$

(b) $V_1(s) = \dfrac{5s}{s^2+1}$ $\qquad V_2(s) = \dfrac{5s}{(s^2+1)(s^2+2s+2)}$

$$V_2(s) = \frac{c_{11}}{s-j} + \frac{c_{21}}{s+j} + \frac{c_{31}}{s+1-j} + \frac{c_{41}}{s+1+j}$$

Pole at $+j$ $\qquad$ Pole at $-1+j$

$$c_{11} = \frac{5j}{2j(1+2j)} = \frac{5}{2}\,\frac{1-2j}{5} = \frac{1}{2} - j = c_{21}^*$$

$$c_{31} = \frac{5(-1+j)}{((-1+j)^2+1)\,2j} = \frac{5(-1+j)}{(1-2j)(2j)} = \frac{(-1+j)(1+2j)}{2j} = \frac{-3-j}{2j} = -\frac{1}{2} + \frac{3}{2}j$$

$$= c_{41}^*$$

$$\Rightarrow v_2(t) = \underbrace{\cos(t)}_{2\,\mathrm{Re}(c_{11})} + \underbrace{2\sin(t)}_{-2\,\mathrm{Im}(c_{11})} - \underbrace{e^{-t}\cos(t)}_{2\,\mathrm{Re}(c_{31})} - \underbrace{3e^{-t}\sin(t)}_{-2\,\mathrm{Im}(c_{31})}$$

STEADY-STATE: $\cos(t) + 2\sin(t)$

TRANSIENT : $-e^{-t}\cos(t) - 3e^{-t}\sin(t)$

(c) FREQUENCY RESPONSE at $s = j\omega = j$

$$\left[H(s)\right]_{s=j} = \frac{1}{1+2j} = \frac{1-2j}{5}$$

$$x(t) = 5\cos(t) \Rightarrow y_{ss}(t) = \mathrm{Re}\,H(j)\,5\cos(t) - \mathrm{Im}\,H(j)\,5\sin(t)$$

$$= \cos(t) + 2\sin(t)$$ SAME ANSWER

(d) $$H(j\omega) = \frac{1}{-\omega^2 + 2j\omega + 2}$$

$$|H(j\omega)| = \frac{1}{(2-\omega^2)^2 + 4\omega^2} \neq 0 \text{ for all } \omega$$

$\Rightarrow y_{ss}(t) \neq 0$ for any $\cos(\omega t)$ (except $\omega \to \infty$)

For $x(t) = \cos(\omega t)$, $y_{ss}(t) \doteq \sin(\omega t)$ if $\angle H(j\omega) = \pm 90°$ which occurs if $\omega^2 = 2$ i.e. $\omega = \sqrt{2}$.

3.6 (a) $v_1 = Ri_1 + L\dfrac{di_1}{dt} + v_c$

$$C\frac{dv_c}{dt} = i_c \qquad v_2 = R(i_1 - i_c) = v_c$$

$$\rightarrow i_c = i_1 - \frac{1}{R}v_c$$

LET $x = \begin{pmatrix} i_1 \\ v_c \end{pmatrix} \rightarrow \dot{x} = \underbrace{\begin{pmatrix} -R/L & -1/L \\ 1/C & -1/RC \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1/L \\ 0 \end{pmatrix}}_{B} v_1$

$$y = v_2 = \underbrace{(0 \quad 1)}_{C} x + \underbrace{0}_{D} \cdot v_1$$

(b)

$$H(s) = (0 \quad 1)\begin{pmatrix} s+R/L & 1/L \\ -1/C & s+1/RC \end{pmatrix}^{-1}\begin{pmatrix} 1/L \\ 0 \end{pmatrix}$$

$$= (0 \quad 1)\begin{pmatrix} s+1/RC & -1/L \\ 1/C & s+1/RC \end{pmatrix}\begin{pmatrix} 1/L \\ 0 \end{pmatrix} \cdot \frac{1}{(s+\frac{R}{L})(s+\frac{1}{RC})+\frac{1}{LC}}$$

$$= \frac{1/LC}{s^2+(\frac{R}{L}+\frac{1}{RC})s+\frac{2}{LC}} = \frac{1}{s^2+2s+2} \text{ for } L=C=R=1$$

(c)

$$Y_{ZI}(s) = C(sI-A)^{-1}x(0) = \frac{(1/C \quad s+1/RC)}{s^2+(\frac{R}{L}+\frac{1}{RC})s+\frac{2}{LC}}\begin{pmatrix} i_1(0) \\ v_c(0) \end{pmatrix}$$

$$= \frac{1}{s^2+2s+2}i_1(0) + \frac{s+1}{s^2+2s+2}v_c(0)$$

$$= \left(\frac{c_{11}}{s+1-j} + \frac{c_{21}}{s+1+j}\right)i_1(0) + \left(\frac{c_{11}'}{s+1-j} + \frac{c_{21}'}{s+1+j}\right)v_c(0)$$

$$C_{11} = \frac{1}{2j} = -\frac{1}{2}j \qquad C_{11}' = \frac{j}{2j} = \frac{1}{2}$$

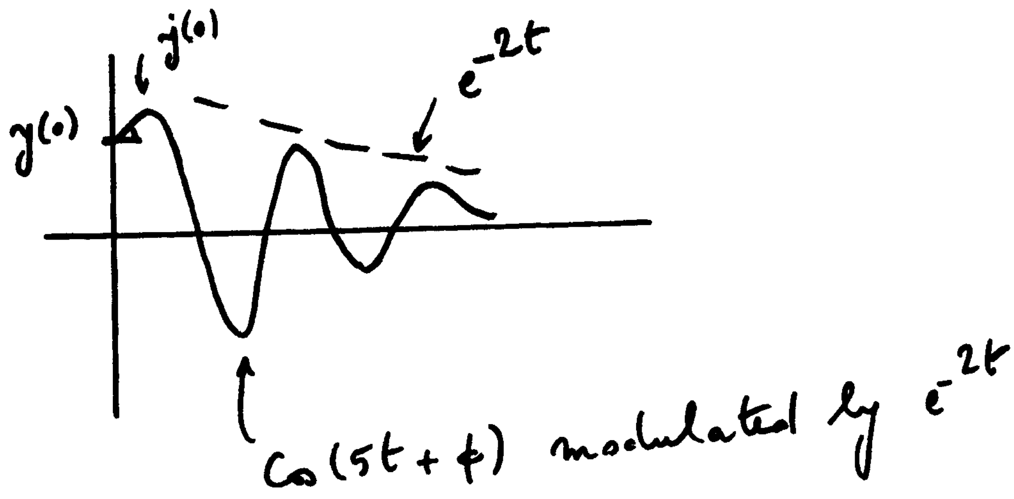
$$\rightarrow \quad y_{zi}(t) = e^{-t}\sin(t)\, i_1(0) + e^{-t}\cos(t)\, v_c(0)$$

USING 4 TIME CONSTANTS (2% DECAY) AN ESTIMATE OF THE DECAY TIME IS 4 SECONDS.

3.7 (a) $s^2 Y(s) - s\,y(0) - \dot{y}(0) + 4s\,Y(s) - 4y(0) + 29\,Y(s) = \dfrac{s}{s^2+1}$

$$Y(s) = \frac{s}{(s^2+1)(s^2+4s+29)} + \frac{s\,y(0) + \dot{y}(0) + 4y(0)}{s^2+4s+29}$$

(b) Poles: $\dfrac{-4 \pm \sqrt{16 - 4\times 29}}{2} = -2 \pm j5$



3.8 $v_1 = L \dfrac{di_1}{dt} + v_c \qquad C \dfrac{dv_c}{dt} = i_1 - i_2$

$v_c = L \dfrac{di_2}{dt} + R i_2$

Let the state vector $x = \begin{pmatrix} i_1 \\ i_2 \\ v_c \end{pmatrix}$, $y = v_2$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & -1/L \\ 0 & -R/L & 1/L \\ 1/C & -1/C & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 1/L \\ 0 \\ 0 \end{pmatrix}}_{B} v_1$$

$$y = \underbrace{\begin{pmatrix} 0 & R & 0 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

<u>Poles</u>: $\det \begin{pmatrix} s & 0 & 1/L \\ 0 & s+R/L & -1/L \\ -1/C & 1/C & s \end{pmatrix} = 0$

$\Rightarrow s\left(s^2 + R/L\, s + 1/LC\right) + 1/L \cdot 1/C \left(s + R/L\right) = 0$

or $s^3 + R/L\, s^2 + 2\, 1/LC\, s + R/L^2C = 0$

or $s^3 + s^2 + 2s + 1 = 0$ Roots = Poles of system

4.1 (a) $Y(s) = \underbrace{\dfrac{C(s)P(s)}{1+C(s)P(s)}} R(s) + \underbrace{\dfrac{P(s)}{1+C(s)P(s)}} D(s)$

$$H_1(s) = \frac{kk_p}{s+a+kk_p} \qquad H_2(s) = \frac{k}{s+a+kk_p}$$

STABILITY CONDITION: $a + kk_p > 0$

DC GAINS: $r \to y \quad H_1(0) = \dfrac{kk_p}{a+kk_p}$

$d \to y \quad H_2(0) = \dfrac{k}{a+kk_p}$

PERFECT TRACKING $H_1(0) = 1$? IN GENERAL, NO.

DIST. REJECTION $H_2(0) = 0$

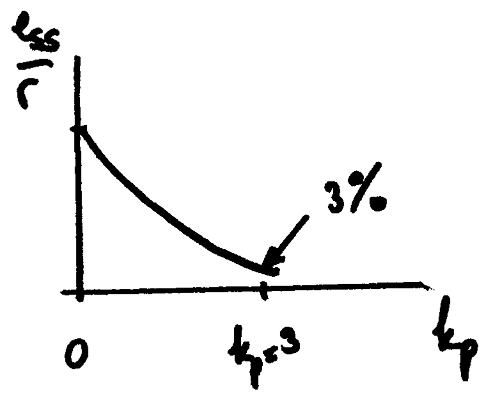
(b) $k = 1000 \quad a = 100 \quad k_p = 3 \quad r = 200 \quad d = 0$

$e_{ss} = r - y_{ss}$, $y_{ss} = H_1(0)\, r = \dfrac{3000}{3100} \cdot 200 = 193.5$ rad/s

$\to e_{ss} = 6.5$ rad/s

$$\frac{e_{ss}}{r} = \frac{a}{a+kk_p} = \frac{1}{1+10k_p}$$

$= 0.032$ or 3.2% for $k_p = 3$



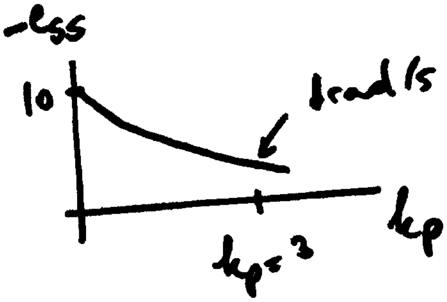
$y_{ss} = r$ if r is multiplied by $\dfrac{a+kk_p}{kk_p} = 1.03$ before being applied to the system.

LIMITATION: COMPENSATION IS NOT EFFECTIVE IF a, k ARE NOT PRECISE ENOUGH.

(c) $y_{ss} = H_2(0) \cdot d = \frac{1000}{3100} \times 3 = 0.98$ rad/s

$e_{ss} = -1$ rad/s

For $d = 1$, $e_{ss} = \frac{-k}{a + k k_p}$



FOR $y_{ss} = 0$, CHANGE r SUCH THAT

$$H_1(0) r + H_2(0) d = 0$$

i.e. $r = -\frac{H_2(0)}{H_1(0)} \cdot d = -\frac{1}{k_p} d$

LIMITATION: d MUST BE KNOWN
(NOT THE CASE IN GENERAL)

(d) $\frac{Y}{R} = \frac{k k_p}{s^2 + as + k k_p} = H_1$, $\frac{Y}{D} = \frac{ks}{s^2 + as + k k_p} = H_2$

STABILITY CONDITIONS: $a > 0$ $k k_p > 0$

DC GAINS: $H_1(0) = 1$ $H_2(0) = 0$

$\rightarrow$ PERFECT TRACKING AND DISTURBANCE REJECT.

(e) $H_1 = \frac{k k_p}{s^2 + as + k k_p}$ $H_2 = \frac{k}{s^2 + as + k k_p}$

STABILITY CONDITIONS: same as (d)

DC GAINS: $H_1(0) = 1$ $H_2(0) = \frac{1}{k_p}$

$\rightarrow$ PERFECT TRACKING BUT NOT DISTURBANCE REJ.

4.2 (a)

$$\begin{array}{c|cccc} s^4 & 1 & 3 & 1 & 0 \\ s^3 & 4 & 4 & 0 & 0 \\ s^2 & 2 & 1 & 0 & \\ s^1 & 4 & 0 & & \\ s^0 & 1 & & & \end{array}$$

ALL >0 → ALL ROOTS IN OPEN LEFT HALF PLANE

(b) SOME COEFFICIENTS <0 → NO NEED TO PROCEED
THERE ARE ROOTS IN CLOSED RIGHT HALF PLANE

(c)

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 1 \\ s^3 & 2 & 2 & 0 \\ s^2 & 1 & 1 & \\ s^1 & \boxed{0} & & \\ s^0 & & & \end{array}$$

4.3 SOME ROOTS IN CLOSED RIGHT HALF PLANE

$$\begin{array}{c|ccc} s^4 & 1 & 1 & b \\ s^3 & 1 & a & 0 \\ s^2 & 1-a & b & 0 \\ s^1 & a-b/1-a & 0 & 0 \\ s^0 & b & 0 & \end{array}$$

CONDITIONS FOR BIBO STABILITY:

$1-a>0$ $\quad a<1$

$a-\dfrac{b}{1-a}>0$ $\quad 0<b<a(1-a)$

$b>0$

4.4 $$\frac{P(s)C(s)}{1+P(s)C(s)} = \frac{k(k_p s + k_I + k_D s^2)}{s^2(s+a) + k(k_p s + k_I + k_D s^2)}$$

CL Poles: $s^3 + (a + k k_D) s^2 + k k_p s + k k_I = 0$

ROUTH - HURWITZ:

$$\begin{array}{c|cc} s^3 & 1 & k k_p \\ s^2 & a + k k_D & k k_I \\ s^1 & b & 0 \\ s^0 & c & 0 \end{array}$$

$$b = k k_p - \frac{k k_I}{a + k k_D}$$

$$c = k k_I$$

FOR STABILITY, ONE NEEDS (ASSUMING $k > 0$)

$$a + k k_D > 0 \qquad 0 < k_I < k_p (a + k k_D)$$

FOR $k_I = 0$, $$\frac{P(s)C(s)}{1+P(s)C(s)} = \frac{k k_D s + k k_p}{s^2 + s(a + k k_D) + k k_p}$$ ($s=0$ is not a pole of the transfer function)

ROUTH - HURWITZ:

$$\begin{array}{c|cc} s^2 & 1 & k k_p \\ s^1 & a + k k_D & 0 \\ s^0 & k k_p & \end{array}$$

NOW, ONE NEEDS

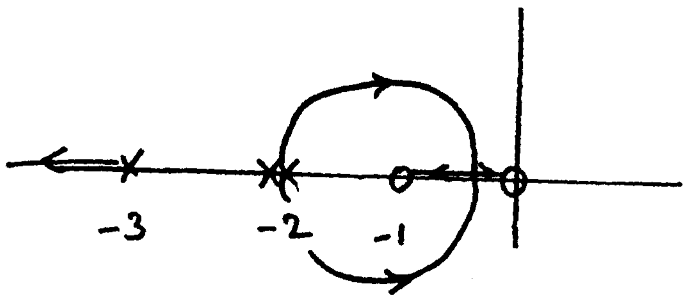
$$a + k k_D > 0 \qquad k_p > 0$$

ALTOGETHER, ONE NEEDS:

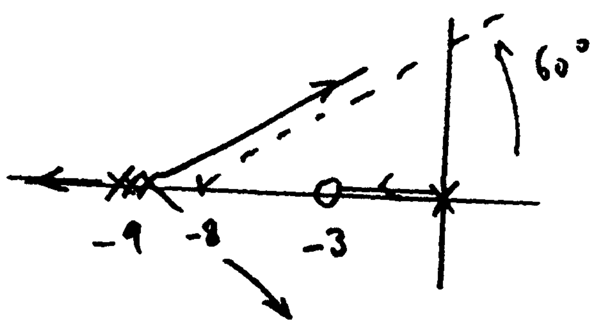
$$a + k k_D > 0$$
$$0 \le k_I < k_p (a + k k_D)$$

4.5

(a)

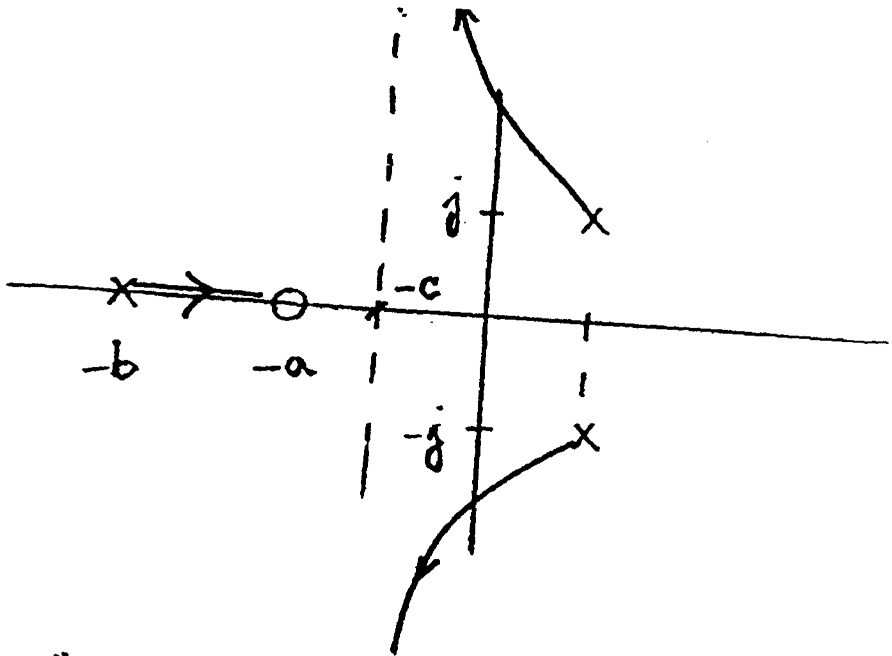


(b)



CENTROID : $\dfrac{3 \times (-9) + 1 \times (0) + (-1) \times (-3)}{4-1} = -8$

(c)



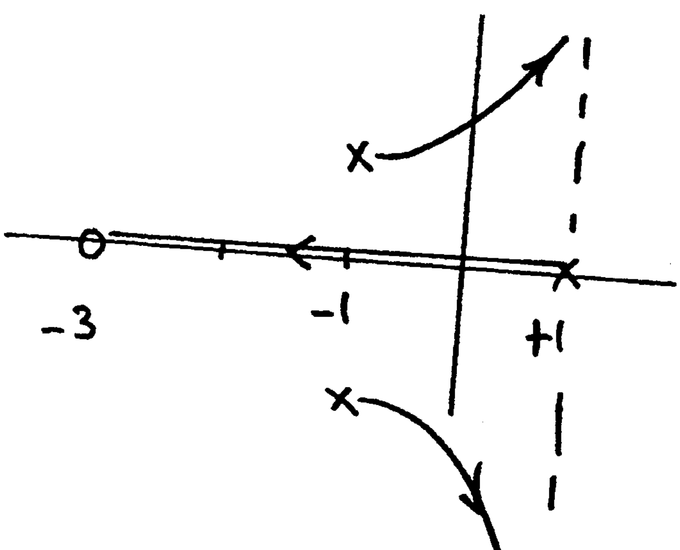
CENTROID:

$$(-c) = \frac{2(1) + (-b) - (-a)}{2+1-1} = \frac{2+a-b}{2}$$

STABILITY FOR HIGH-GAIN IF:

- $a > 0$
- $\frac{2+a-b}{2} < 0 \rightarrow b-a > 2$

4.6



ASYMPTOTES $\pm 90°$

CENTROID $\frac{(+1)+2(-1)-(-3)}{3-1} = 1$

RANGE OF k:

$$s^3 + 2s^2 + 2s - (s^2 + 2s + 2) = s^3 + s^2 - 2$$

$$\Rightarrow D_{CL}(s) = s^3 + s^2 + ks + (3k - 2)$$

1	k	0
1	3k-2	0
-2k+2	0	0
3k-2	0	0

↓

$-2k + 2 > 0$

$3k - 2 > 0$

$$\boxed{1 > k > \frac{2}{3}}$$

(THE REAL POLE CROSSES THE $j\omega$ AXIS BEFORE THE COMPLEX POLES)

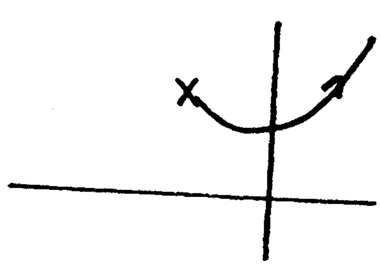
ANGLES OF DEPARTURE

$$\tan^{-1}\left(\frac{1}{2}\right) = 27°$$

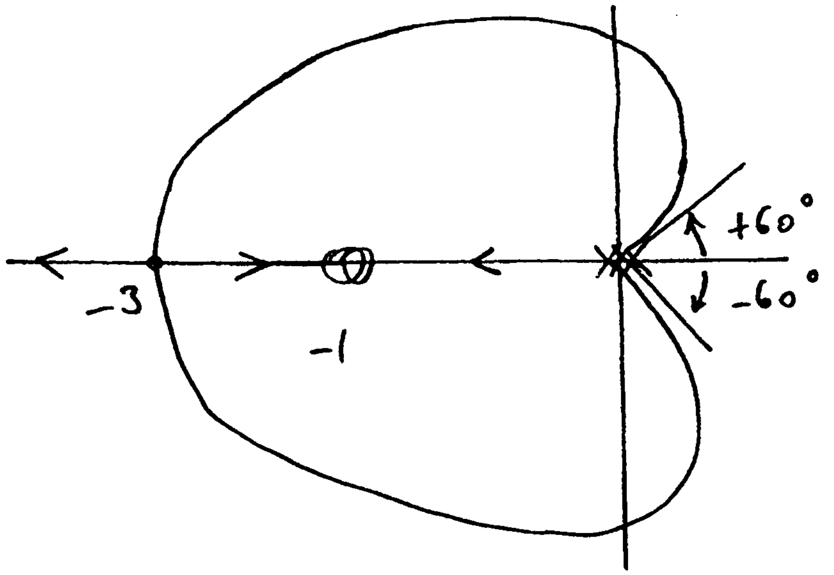
$$\theta + 90° + (180° - 27°) - 27° = 180°$$

$$\theta = -36°$$

LOCUS ACTUALLY LOOKS LIKE:



4.7



Breakaway points:

$$\frac{d}{ds}\left(\frac{(s+1)^2}{s^3}\right) = \frac{(2s+2)s^3 - (s+1)^2 \cdot 3s^2}{s^6}$$

$$= \frac{2s^2+2s-3s^2-6s-3}{s^4}$$

$$=0 \iff -s^2-4s-3=0$$
$$= -(s+1)(s+3))$$

$\rightarrow s=-3$ is the point on the real axis.

RANGE OF k:

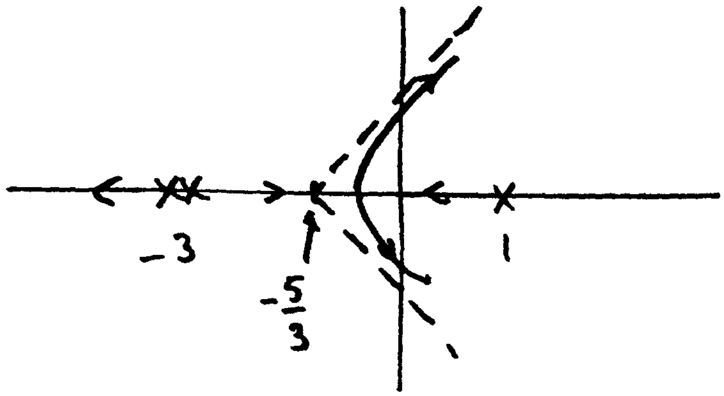
$$s^3+ks^2+2ks+k=0$$

s^3	1	$2k$	0
s^2	k	k	0
s	$2k^2-k/k$	0	
s^0	k		

$k>0$ and $2k-1>0$

$\Rightarrow k>\frac{1}{2}$

4.8



CENTROID: $\sigma = -5/3$

ASYMPTOTES: $\pm 60°, 180°$

RANGE OF k:

s^3	1	3	0
s^2	5	$k-9$	0
s	$\frac{24-k}{5}$	0	
s^0	$k-9$		

$\Rightarrow \quad 9 < k < 24$

(SYSTEM CAN BE STABILIZED)

BREAKAWAY: $\frac{d}{ds}(s^3+5s^2+3s-9) = 3s^2+10s+3 = 0$

$(\Rightarrow s = -\frac{5}{3} \pm \frac{\sqrt{16}}{3} = -1/3, -3)$

CROSSING: $s^3+5s^2+3s+24-9=0$

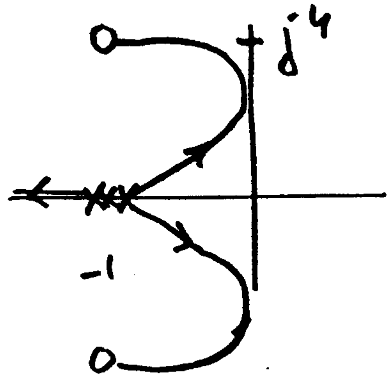
$(\overset{\text{ROOTS}}{\Rightarrow} s = \pm j\sqrt{3}, \ s = -5)$

OR

LET $s = j\omega \Rightarrow -j\omega^3 - 5\omega^2 + 3j\omega + 15 = -j\omega(\omega^2-3) - 5(\omega^2-3)$

$= (-j\omega-5)(\omega^2-3)$

4.9



ANGLES OF DEPARTURE: $180°, \pm 60°$

ANGLE OF ARRIVAL AT $-1+j4$:

$$-\theta - 90° + \underbrace{3 \times 90°}_{270°} = 180° \rightarrow \theta = 0°$$

RANGE OF k:

s^3	1	$3+2k$	0
s^2	$3+k$	$1+17k$	0
s	$\dfrac{9+9k+2k^2-1-17k}{3+k}$	0	
s^0	$1+17k$		

Need $2k^2 - 8k + 8 > 0$

$$\Updownarrow$$

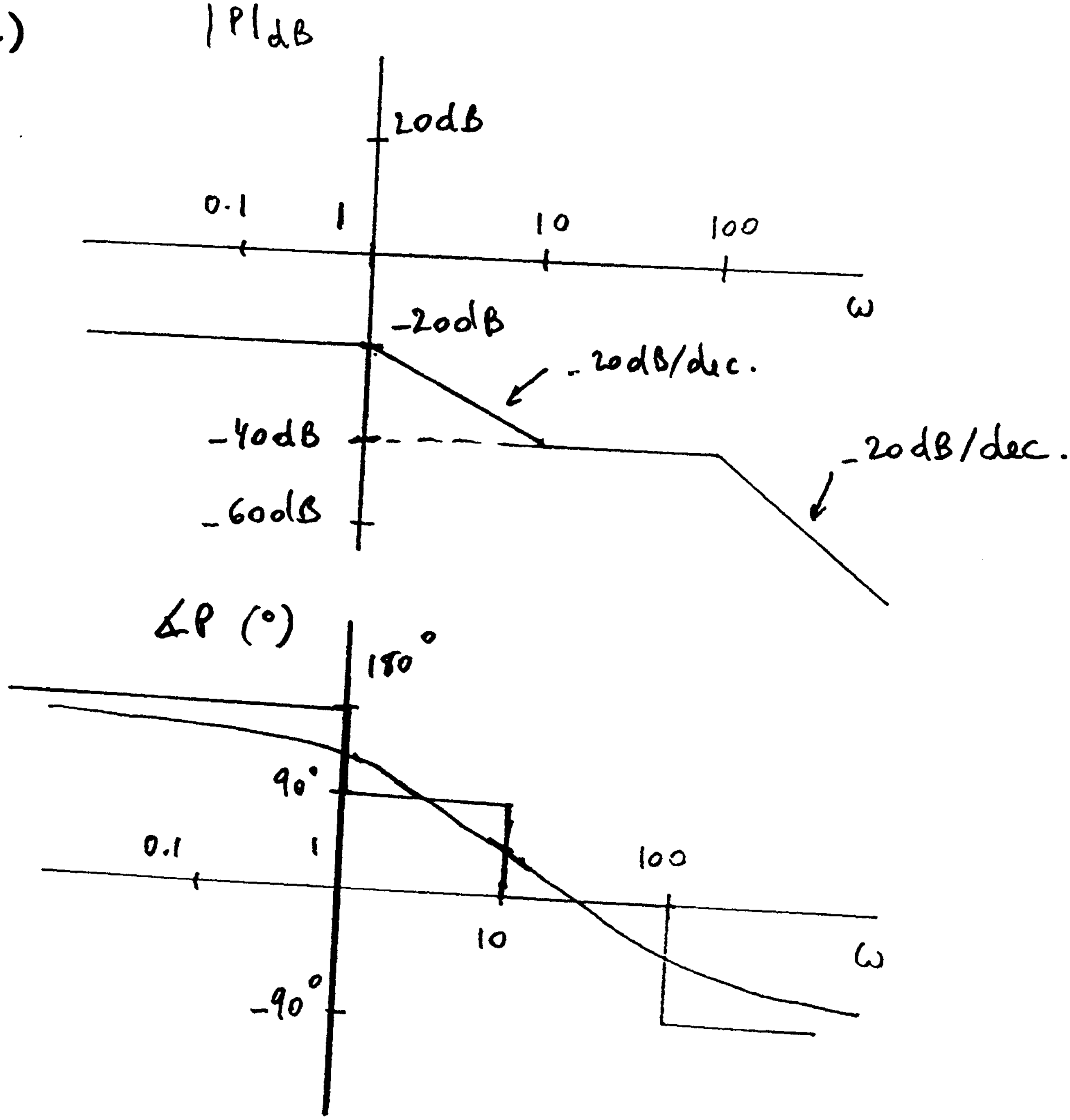
$$2(k-2)^2 > 0$$

$\rightarrow$ STABLE FOR ALL k EXCEPT $k=2$

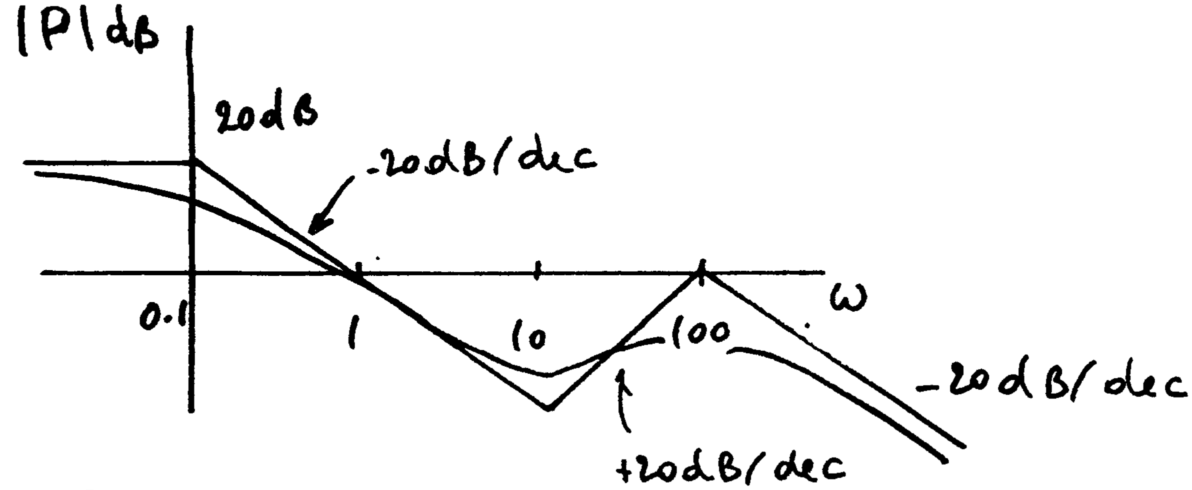
$\rightarrow$ ROOT-LOCUS "TOUCHES" $j\omega$-AXIS FOR $k=2$
(IS TANGENT TO)

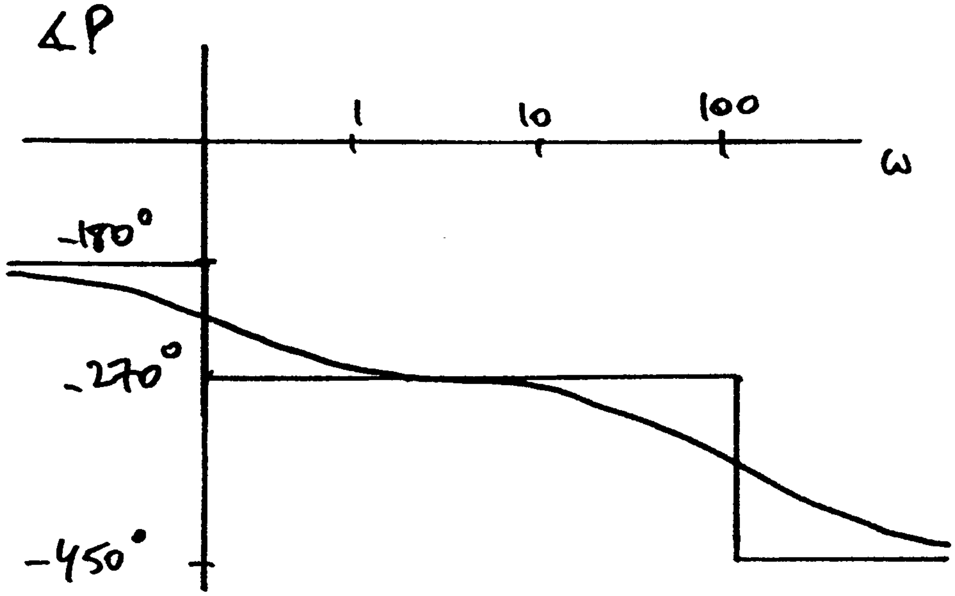
5.1

(a)

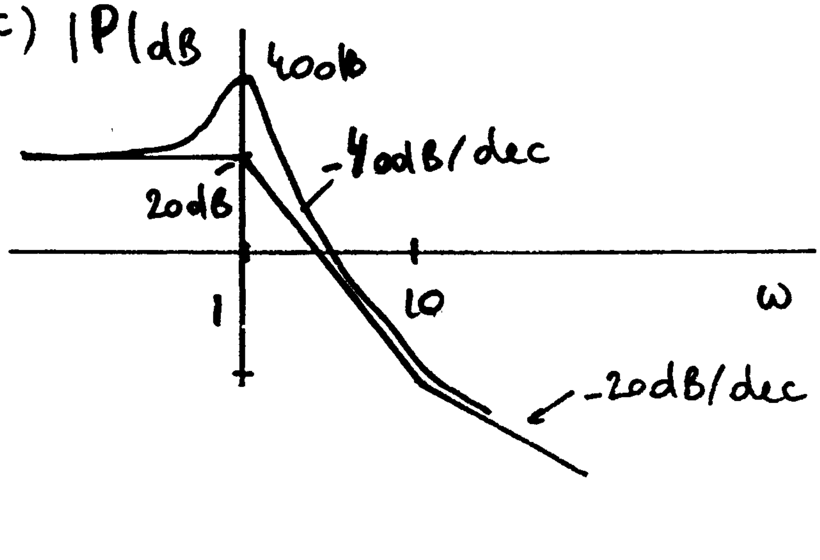

|P|dB
20dB
0.1
1
10
100
ω
-20dB
-20dB/dec.
-40dB
-20dB/dec.
-60dB
∠P (°)
180°
90°
0.1
1
100
10
ω
-90°

(b)





(c)



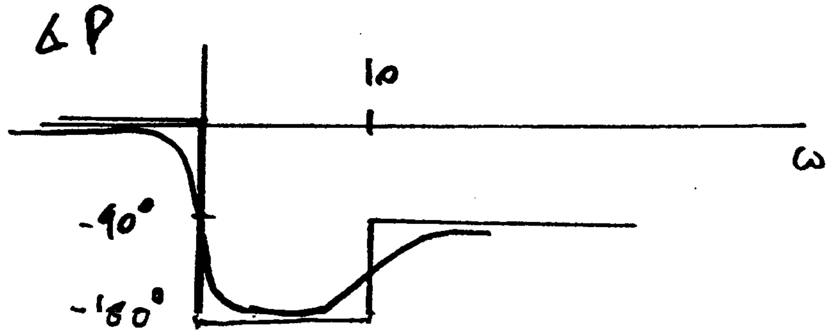
DAMPING FACTOR:

$\omega = 1$

$\xi = 0.05$

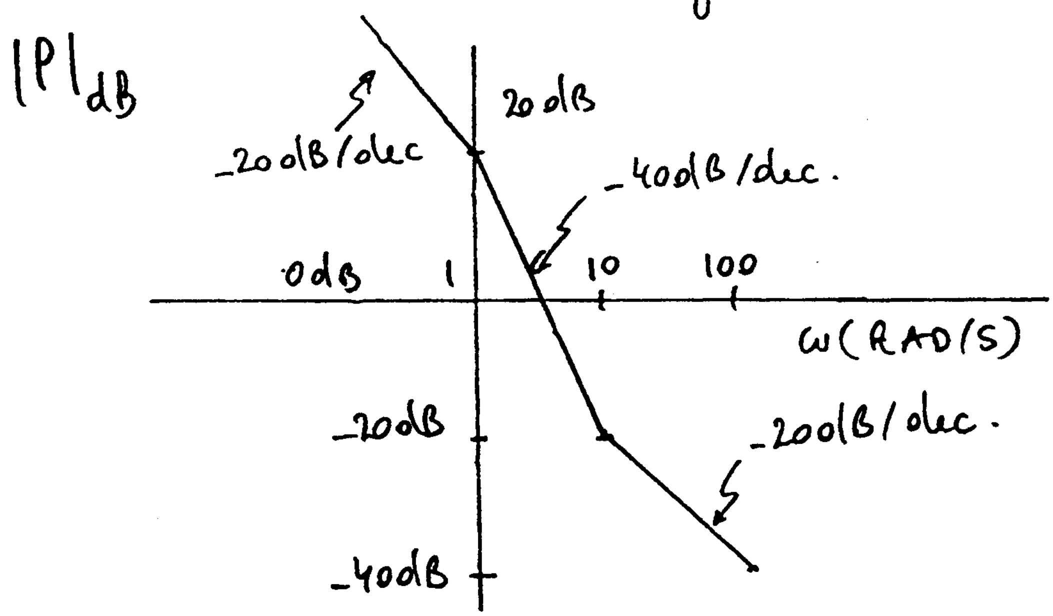
$\Downarrow$

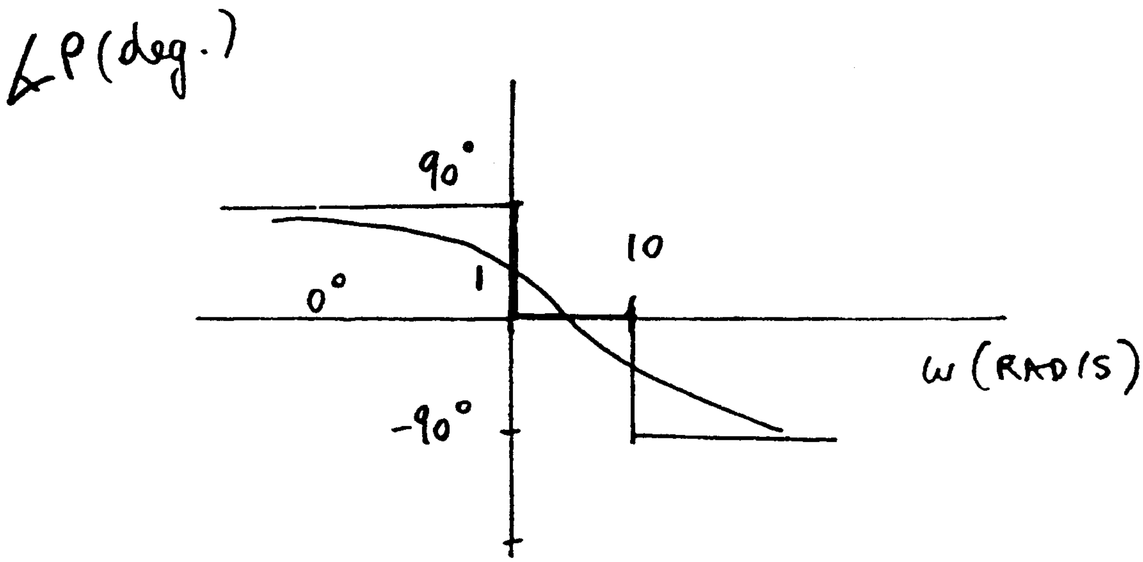
20dB peaking



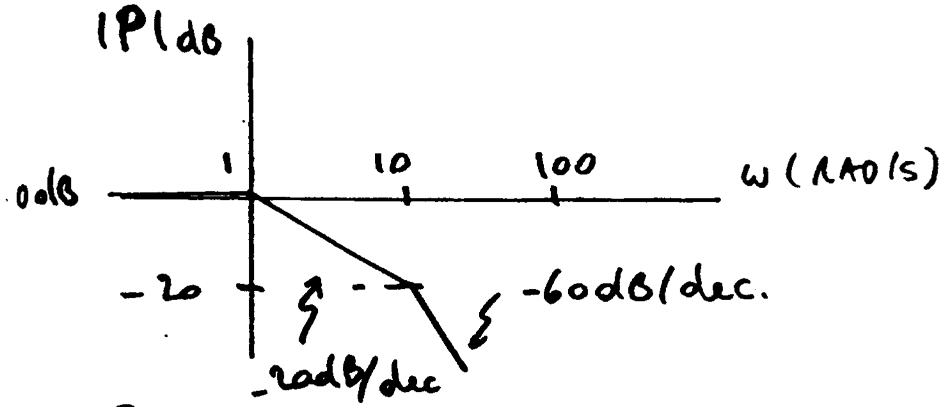
(d) LOW FREQUENCY:

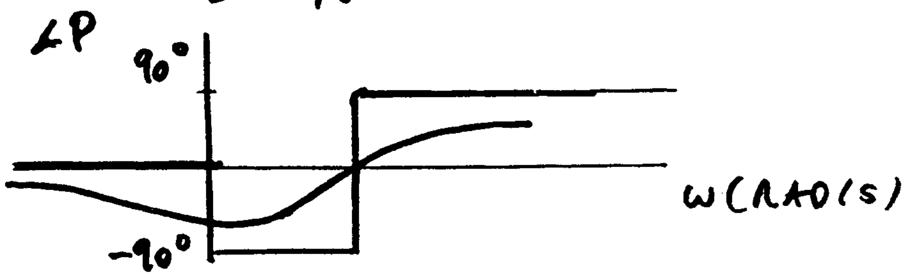
$$P(s) \simeq \frac{-10}{s} = \frac{-10}{j} \text{ at } s = j \quad \Rightarrow \begin{matrix} 20\,dB \\ +90^\circ \end{matrix}$$





(e) LOW FREQUENCY: $P(s) \simeq 1 \rightarrow 0\,dB,\ 0^\circ$



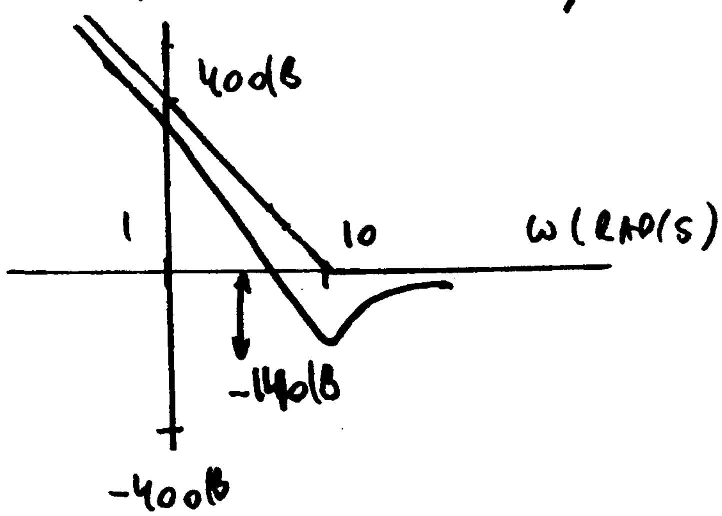


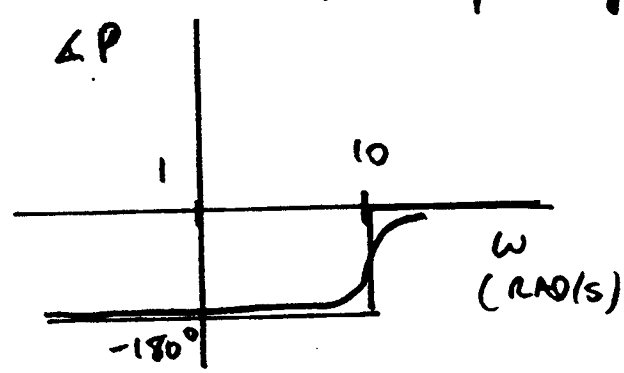
(F)

LOW FREQUENCY: $P(s) \simeq \frac{100}{s^2} \rightarrow$ 0dB at $\omega = 10$ rad/s
40dB at $\omega = 1$ rad/s
$\pm 180^\circ$

DAMPING FACTOR: $\omega_n^2 = 100 \rightarrow \omega_n = 10$

$2\xi\omega_n = 2 \rightarrow \xi = \frac{1}{10} \rightarrow \frac{1}{2\xi} = 5 \simeq 14\,dB$ peaking





5.2

(a) DC GAIN = 1 (0 dB)

$\Rightarrow P(s) = 10 \dfrac{s+1}{s+10}$

also $-10 \dfrac{s+1}{s+10}$, $10 \dfrac{s-1}{s+10}$, $-10 \dfrac{s-1}{s+10}$.

(b) LOW-FREQUENCY:

-20 dB/dec and 90° + 0 dB gain at 0.01 rad/s

$\Rightarrow P(s) = \dfrac{1}{100 s}$ at low-frequency

1st POLE/ZERO: ZERO AT 0.1 RAD/S (MIN. PHASE)

2nd: POLE PAIR AT 1 RAD/S

PEAKING ≃ 12 dB (UP) ≃ $4 = \dfrac{1}{2\xi} \Rightarrow \xi = \dfrac{1}{8}$ (ACTUALLY $\xi = 0.1$)

3rd: ZERO PAIR AT 2 RAD/S

PEAKING ≃ 12 dB (DOWN) $\Rightarrow \xi = \dfrac{1}{8}$ (ACTUALLY $\xi = 0.1$)

4th: POLE PAIR AT 20 RAD/S

PEAKING ≃ 18 dB ≃ $8 = \dfrac{1}{2\xi} \Rightarrow \xi = \dfrac{1}{16}$ (ACTUALLY $\xi = 0.05$)

$$P(s) = \frac{k}{s} \frac{(s+0.1)(s^2+0.4s+4)}{(s^2+0.2s+1)(s^2+2s+400)}$$

$$k \times \frac{0.1 \times 4}{1 \times 400} = \frac{1}{100} \Rightarrow k = 10$$

<u>NOTE:</u> THIS PROCEDURE IS VERY APPROXIMATE, IN GENERAL.

5.3

(a) GM $\simeq$ 20dB
PM $\simeq 45°$

(Note: $G(s) = 10(s+10)/(s+0.1)^2(s^2+s+100)$)

(b) THE SYSTEM WILL EXHIBIT OSCILLATIONS IN THE TRANSIENT RESPONSE (FREQ.: $\simeq$ 10 RAD/S) - TWO POLES OF THE CLOSED-LOOP SYSTEM WILL BE CLOSE TO $\pm j10$.

(c) YES, IT MUST BE STABLE BECAUSE IT IMPLIES $N=0$ (NO ENCIRC.), $P=0$ (OPEN-LOOP STABLE) $\rightarrow Z=0$ (CLOSED-LOOP STABLE) -

5.4

(a) FOR $k=1$, $P=0$, $N=2 \rightarrow Z=2 \rightarrow$ CLOSED-LOOP SYSTEM IS UNSTABLE

NEED $N=0$, SO THAT $Z=0$ -

$\rightarrow$ NEED: $(1.5)^{-1} > k$ $\boxed{\text{i.e.}}$ $k < \frac{2}{3}$

NOTE: $G(s) = \dfrac{s^2+2s+22}{(s+1)^3}$

OR $k > (0.2)^{-1}$ OR $k > 5$

(b) $P=1$ (1 UNSTABLE OPEN-LOOP POLE) $\rightarrow$ NEED ONE COUNTERCLOCKW. ENCIRCLEMENTS $N=-1$
$Z=0$ (WANT STABLE CLOSED LOOP)

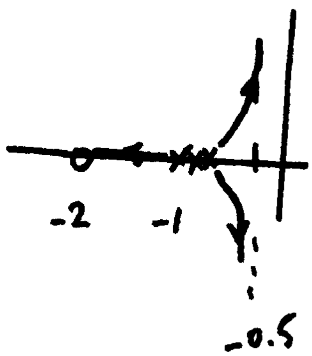
$\rightarrow$ NEED $(0.5)^{-1} > k > (0.75)^{-1}$

or $\frac{4}{3} < k < 2$

NOTE: $G(s) = \dfrac{1/2\,(s+3)}{(s-1)(s^2+2s+2)}$

5.5 (a) GAIN MARGIN = ∞, PHASE MARGIN $\simeq \sin^{-1}(0.5) = 30°$

ROOT-LOCUS: CENTROID = $\dfrac{3\times(-1)-(-2)}{3-1} = -\dfrac{1}{2}$



→ INFINITE GAIN MARGIN

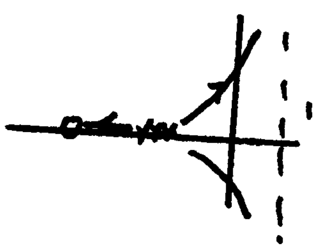
ROUTH-HURWITZ:

s^3	1	$3+k$
s^2	3	$2k+1$
s	$(8+k)/3$	0
1	$2k+1$	

$\Rightarrow$ STABLE FOR ALL $k>0$

(b) GAIN MARGIN = $\dfrac{1}{0.5} = 2$, PHASE MARGIN $\simeq \sin^{-1}(0.2) \simeq 10°$

ROOT-LOCUS: CENTROID = $\dfrac{3\times(-1)-(-5)}{3-1} = 1$



→ FINITE GAIN MARGIN

ROUTH-HURWITZ

s^3	1	$3+2k$
s^2	3	$1+10k$
s	$(8-4k)/3$	
1	$1+10k$	

$\Rightarrow$ STABLE FOR $k<2$ (assuming $k>0$)

6.1

(a) $x(0) = a$ (b) $x(0) = 0$

6.2

(a) $x(k) = x(k-1) - \dfrac{a x^2(k-1)}{2a\,x(k-1)} = \dfrac{1}{2}\, x(k-1)$

$X(z) = \dfrac{1}{2}\left(z^{-1} X(z) + x(-1)\right) \Rightarrow X(z) = \dfrac{z}{2z-1}\, x(-1) = \dfrac{z}{z-\frac{1}{2}}\left(\dfrac{x(-1)}{2}\right)$

$\Rightarrow x(k) = \left(\dfrac{1}{2}\right)^k \dfrac{x(-1)}{2} \Rightarrow x(k) \xrightarrow[k\to\infty]{} 0$ (NO CONDITION NEEDED)

(b) $x(k) = x(k-1) - 2a\, x(k-1)$

$X(z) = (1-2a)\left(z^{-1} X(z) + x(-1)\right) \Rightarrow X(z) = \dfrac{(1-2a)\, z}{z-(1-2a)}\, x(-1)$

$\rightarrow x(k) = (1-2a)^{k+1}\, x(-1) \Rightarrow x(k) \xrightarrow[k\to\infty]{} 0$ if $-1<1-2a<1$ or $0<a<1$

6.3

(a) $\dfrac{X(z)}{z} = \dfrac{1}{z(z-1)(z-2)} = \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{z-2}$ $A = \dfrac{1}{2}$ $B = -1$ $C = \dfrac{1}{2}$

$\rightarrow \dfrac{X(z)}{z} = \dfrac{1}{2} + \dfrac{-z}{z-1} + \dfrac{1}{2}\,\dfrac{z}{z-2}$

$\rightarrow x(k) = \dfrac{1}{2}\,\delta(k) - 1 + \dfrac{1}{2}\,(2)^k$

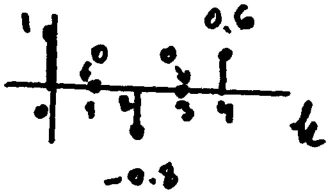
(b) $\dfrac{X(z)}{z} = \dfrac{c}{z-1-j} + \dfrac{c^*}{z-1+j} \rightarrow c = \dfrac{1}{2j}$ $p = 1+j$

$x(k) = 2.\,|c|\,|p|^k \cos(\angle p\, k + \angle c)$

$= 2.\,\dfrac{1}{2}\left(\sqrt{2}\right)^k \cos\left(\dfrac{\pi}{4}k - \dfrac{\pi}{2}\right) = \left(\sqrt{2}\right)^k \sin\left(\dfrac{\pi}{4}k\right)$

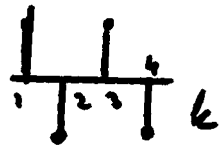
6.4

(a) $x(k) = (0.9)^k \cos(\frac{\pi}{2} k)$ (ACTUAL SIGNAL MAY HAVE DIFFERENT MAG/PHASE)



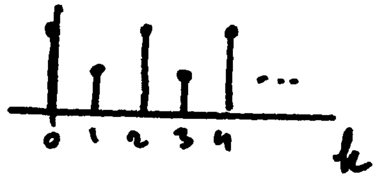
(b) $x(k) =$ SUM OF

AND

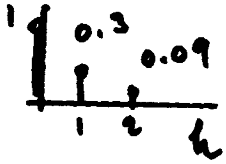


(SAME COMMENT: ACTUAL MAGNITUDE MAY VARY)

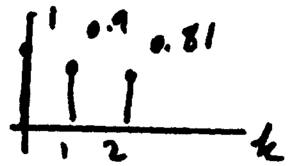
$\Rightarrow$



(c) $x(k) =$ SUM OF



AND

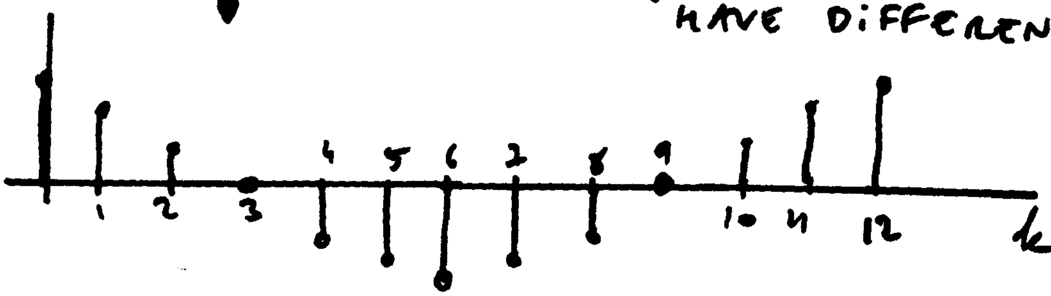


(SAME)

FAST DECAY SLOW DECAY

$\Rightarrow$

(d) $x(k) = \cos(\frac{\pi}{6} k)$ (ACTUAL SIGNAL MAY HAVE DIFFERENT MAG/PH.)

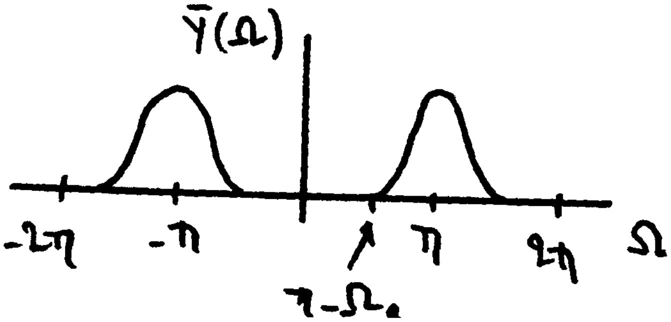


6.5

(a) $X(z) = \sum x(n) z^{-n}$

$Y(z) = \sum y(n) z^{-n} = \sum x(n) (-1)^n z^{-n} = X(-z)$

$\Rightarrow \bar{Y}(\Omega) = \bar{X}(\Omega + \pi)$ SINCE $-e^{j\Omega} = e^{j\Omega + \pi}$

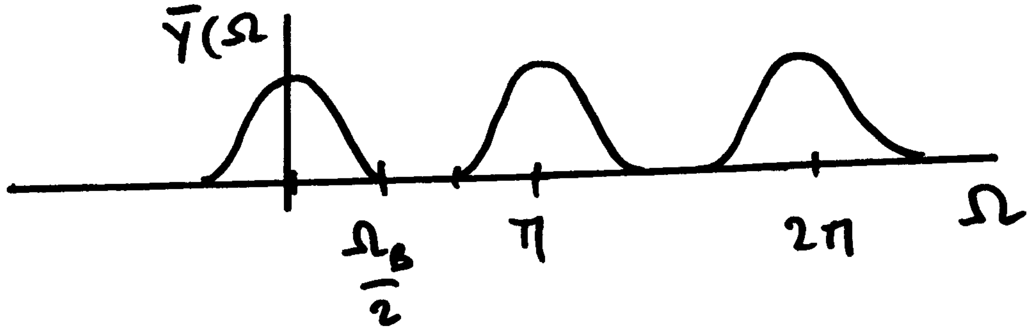


(b) $Y(z) = y(0) + y(1)z^{-1} + y(2)z^{-2}$

$= x(0) + x(1)z^{-2} + x(2)z^{-4} + \cdots$

$= X(z^2)$

$\rightarrow \bar{Y}(\Omega) = \bar{X}(2\Omega)$



6.6

(a) $\frac{X(z)}{z} = \frac{c_1}{z} + \frac{c_2}{z-1} + \frac{c_3}{z-j} + \frac{c_4}{z+j}$

$c_1 = -4 \quad c_2 = 2 \quad c_3 = 1 + j = c_4^*$

$|c_3| = \sqrt{2} \quad \angle c_3 = 45^\circ \ (\pi/4) \quad |p| = 1 \quad \angle p = 90^\circ \ (\pi/2)$

$x(k) = 4\,\delta(k) + 2 + 2\sqrt{2}\cos\left(\frac{\pi}{2}k + \frac{\pi}{4}\right)$

$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$x=0$	$x=0$	$x=0$	$x=4$	$x=5$	$x=0$	$x=0$
$k=7$	$k=8$					
$x=4$	$x=4$					

(b)

$$
\begin{array}{r l}
 & 4z^{-3} + 4z^{-4} + 4z^{-7} + 4z^{-8} \cdots \\
z^3 - z^2 + z - 1 & \overline{)\,4} \\
 & 4 - 4z^{-1} + 4z^{-2} - 4z^{-3} \\
\hline
 & 4z^{-1} - 4z^{-2} + 4z^{-3} \\
 & 4z^{-1} - 4z^{-2} + 4z^{-3} - 4z^{-4} \\
\hline
 & 4z^{-4} \\
 & 4z^{-4} - 4z^{-5} + 4z^{-6} - 4z^{-7} \\
\hline
 & 4z^{-5} - 4z^{-6} + 4z^{-7}
\end{array}
$$

6.7

(a) CONV. $\to 0$

(b) FINITE TIME

(c) BOUNDED, DOES NOT CONVERGE

(d) BOUNDED, CONV. $\to$ $2/(3/2)^2 = 8/3$

(e) BOUNDED, CONV. $\to 2$

(f) UNBOUNDED

(g) BOUNDED, DOES NOT CONVERGE

(h) BOUNDED, CONV. $\to 1$

6.8

(a) YES (b) YES (c) NO (d) YES (e) NO (f) YES

6.9

(a) $Y(z) = X(z) + a z^{-1} Y(z) - a^2 z^{-2} Y(z)$

$$H(z) = \frac{z^2}{z^2 - az + a^2} \qquad \text{POLES: } z = \frac{a}{2} \pm j \frac{\sqrt{3}}{2} a$$

$$|z| = a$$

STABLE $\Leftrightarrow |a| < 1$

(b) $X_1 = 3X + \frac{1}{2} z^{-1} X_1 \Rightarrow X_1 = \frac{6z}{2z-1} X$

$$X_2 = 2X_1 + z^{-1} X + 4z^{-1} X_1$$

$$Y = 2z^{-1} X + 3z^{-1} X_1 + X_2 = 2z^{-1} X + 3z^{-1} X_1 + z^{-1} X + 2X_1 + 4z^{-1} X_1$$

$$= 3z^{-1} X + (2 + 7z^{-1}) \frac{6z}{2z-1} X$$

$$z(2z-1) Y = \big(3(2z-1) + (2z+7) 6z\big) X$$

$$Y = \frac{12z^2 + 48z - 3}{z(2z-1)} X \qquad \text{STABLE}$$

6.10

(a) $Y = z^{-1} Y + z^{-2} Y + X \rightarrow H(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$

POLES: $\frac{1 \pm \sqrt{5}}{2}$ $\begin{cases} 1.62 & (z_1) \\ -0.61 & (z_2) \end{cases}$

UNSTABLE

(b) $x(k) = \delta(k) \rightarrow X(z) = 1$

$$\frac{Y(z)}{z} = \frac{c_1}{z-z_1} + \frac{c_2}{z-z_2} \qquad c_1 = (1+\sqrt{5})/2\sqrt{5}, \quad c_2 = -(1-\sqrt{5})/2\sqrt{5}$$

$$\rightarrow y(k) = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1-\sqrt{5}}{2\sqrt{5}}\cancel{\left(\frac{1-\sqrt{5}}{2}\right)^k}$$

GOES TO 0 AS $k \rightarrow \infty$

$$\Rightarrow \lim_{k\rightarrow\infty} \frac{y(k)}{y(k-1)} = \left(\frac{1+\sqrt{5}}{2}\right)$$

6.11

(a) SYSTEM IS STABLE $\Rightarrow$ STEADY-STATE EXISTS

$$H(1) = \frac{-1}{3/2} = -2/3 \qquad \text{(DC GAIN)}$$

$$\rightarrow y_{ss}(k) = -2$$

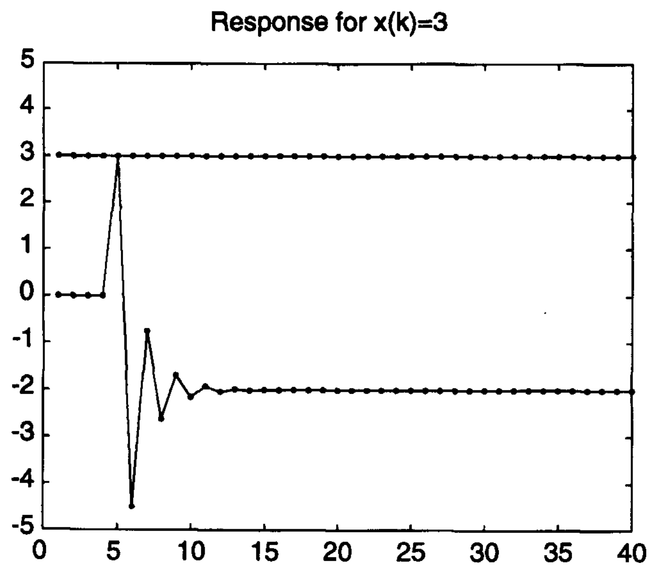
(b) AGAIN, STEADY-STATE EXISTS

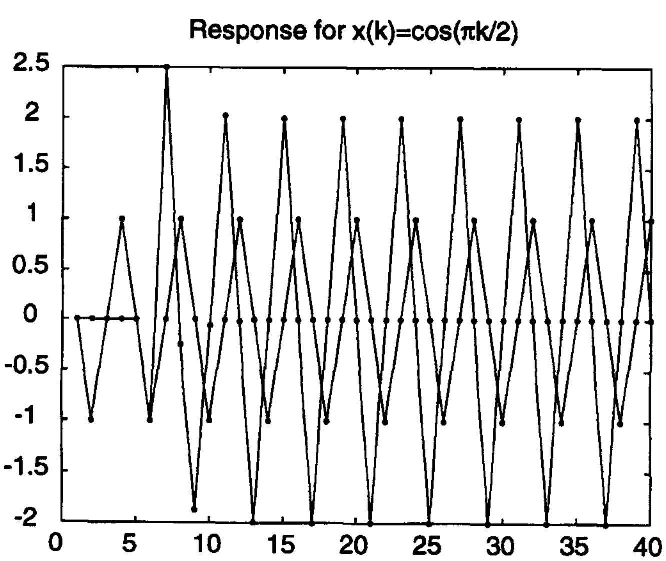
$$H(e^{j\pi/2}) = H(j) = \frac{j-2}{j^4(j+0.5)} = 2j = 2e^{j\pi/2} \qquad \text{(FREQ. RESP.)}$$

$$\Rightarrow y_{ss}(k) = 2\cos\left(\frac{\pi}{2}k + \frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}k\right)$$

M-FILE

```
npoints=40;
x=zeros(1,npoints);y=zeros(1,npoints);t=zeros(1,npoints);
xkm1=0;xkm2=0;xkm3=0;xkm4=0;xkm5=0;ykm1=0;
for k=1:npoints
t(k)=k;
x(k)=3;
%x(k)=cos(pi*k/2);
y(k)=-0.5*ykm1+xkm4-2*xkm5;
xkm5=xkm4;xkm4=xkm3;xkm3=xkm2;xkm2=xkm1;xkm1=x(k);
ykm1=y(k);
end
```





6.12

$$E(z) = \frac{(z-a)\,z}{z^2 - az + k} \cdot \frac{r^*}{z-1}$$

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1} (z-1)E(z) = \frac{1-a}{1-a+k} \cdot r^* = 0 \iff a = 1$$

THIS FACT ASSUMES THAT THE LIMIT EXISTS, i.E. THAT THE SYSTEM IS STABLE.

POLES: $z_{1,2} = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4k} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4k}$

$k < \frac{1}{4}$: 2 real poles with $|z| < 1$

$k > \frac{1}{4}$: 2 complex poles with $|z|^2 = \frac{1}{4} + \frac{1}{4}(4k-1)$
$= k$

$\Rightarrow$ need $k < 1$

7.1

(a) $Y(s) = H(s)\cdot\frac{1}{s} = \dots = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$ (PFE)

$y(t) = t - 1 + e^{-t}$ $\quad y_d(k) = kT - 1 + e^{-kT}$

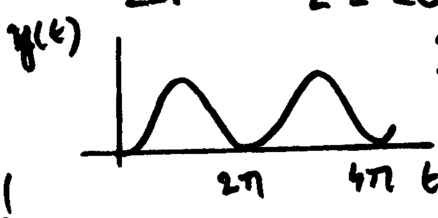
$Y_d(z) = \frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}$ $\quad H_d(z) = \frac{z-1}{z} Y_d(z)$

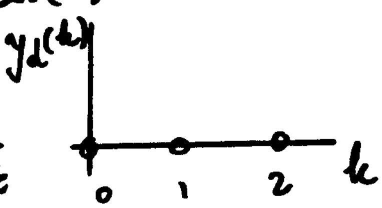
$$\Rightarrow \dots \quad H_d(z) = \frac{z(T-1+e^{-T}) + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}$$

(b) $y(t) = 1 - \cos(t)$ $\quad Y_d(z) = \frac{z}{z-1} - \frac{z^2 - \cos(T)z}{z^2 - 2\cos(T)z + 1}$

$$H_d(z) = \frac{(1-\cos T)(z-1)}{z^2 - 2\cos T\, z + 1}$$

$= 0$ FOR $T = 2\pi$!

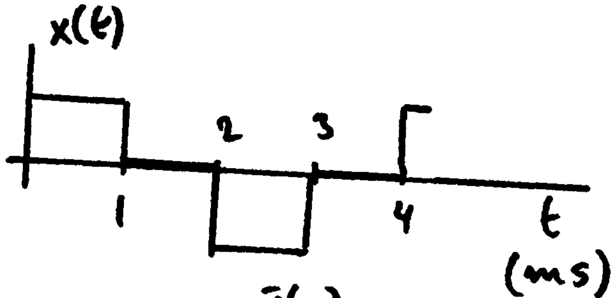




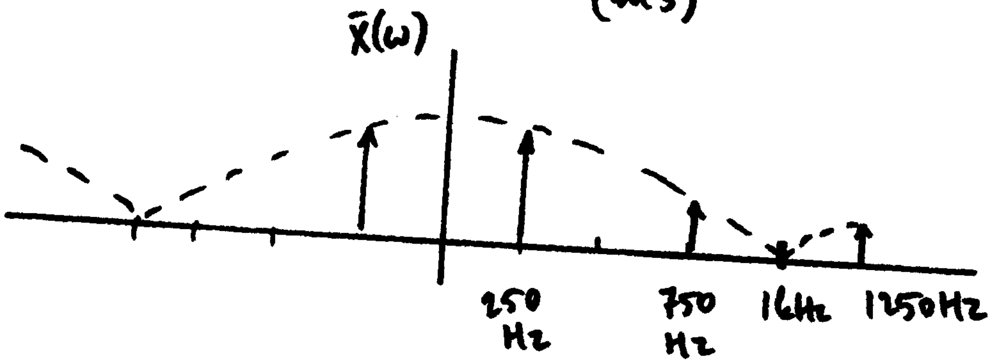
7.2 $f_s > 30\,Hz$

7.3

(a)



(b)



$f_1 = 250\,Hz \qquad m_1 = |\sin(\pi/4)/(\pi/4)| = 0.9003$

$f_2 = 750\,Hz \qquad m_2 = |\sin(3\pi/4)/(3\pi/4)| = 0.3001$

$f_3 = 1250\,Hz \qquad m_3 = |\sin(5\pi/4)/(5\pi/4)| = +0.1801$

$\phi_1 = -\frac{\pi}{4} = -45°$

COMPARE WITH FOURIER SERIES: $\quad \omega = \frac{2\pi}{T} \cdot \frac{1}{4}$

$$x(t) = \alpha_1 \cos(\omega t) + \beta_1 \sin(\omega t) + \alpha_2 \cos(2\omega t) + \ldots$$

$$\alpha_1 = \frac{2}{4T} \int_0^{4T} x(t) \cos(\omega t)\, dt = \frac{2}{\pi} \int_0^{\pi/2} \cos(\alpha)\, d\alpha = \frac{2}{\pi}$$

$$\alpha_2 = \ldots = \frac{2}{\pi}$$

$$\alpha_1 \cos(\omega t) + \beta_1 \sin(\omega t) = \underbrace{\frac{2\sqrt{2}}{\pi}}_{0.9003} \cos(\omega t - 45°)$$

SAME RESULT!