Complement to:

"Adaptive Control: Stability, Convergence, and Robustness."

S. Sastry & M. Bodson, Prentice–Hall, 1989.

Homework Exercises

Dec. 1991

Problem 0.1

Consider the plant $\hat{P}(s) = 2/(s+1)$ and the reference model $\hat{M}(s) = 1/(s+1)$.

a) Plot the responses over 20 seconds of $\theta(t)$, $e_0(t)$ and $v(t) = e_0^2(t) + k_p\,\phi^2(t)$ for the M.I.T. rule, the Lyapunov redesign and the indirect scheme of chapter 0. Let $r(t) = \sin(t)$, $\theta(0) = 1$ and all other initial conditions be zero. Comment on the responses in view of the analytical results.

b) Repeat part a) with $r(t) = e^{-t}$.

Problem 0.2

Give an example of a control system where parametric adaptation is required or desirable. Identify the source of variation and indicate whether it is due to a nonlinearity, a slow state variable, an external factor, an initial uncertainty or an inherent change in the system under control.

Problem 0.3

Consider the MRAC algorithm of chapter 0, using the Lyapunov redesign. Let $r = r_0$ be constant. Show that the system with state variables (e_0, ϕ) is linear time–invariant. Calculate the poles of the system as functions of a, k_p and r_0. Plot the locus of the poles as r_0 varies from 0 to ∞. Discuss what happens when $r_0 = 0$, and when $k_p < 0$.

Problem 0.4

The stability properties of the MIT rule were investigated by D.J.G. James in "Stability of a Model Reference Control System," *AIAA Journal*, vol. 9, no. 5, pp. 950–952, 1971. The algorithm is the same as the one given by eqn. (0.3.6). The author calculated the stable and unstable regions of the algorithm, which are shown on Figure P0.4.

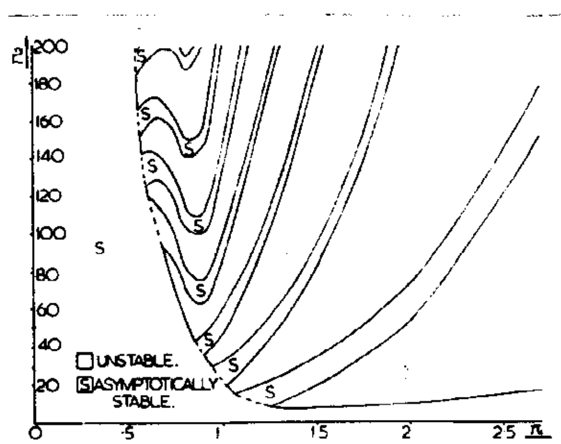

Figure P0.4: Stability Regions for the MIT Rule

Assuming that the plant is $\hat{P}(s) = 1/(s+1)$, the reference model is $\hat{M}(s) = 1/(s+1)$, and the reference input $r = \sin(\omega t)$, the x–axis of the figure becomes $\pi_1 = \omega$ and the y–axis becomes $\pi_2 = g$. Note that, for small enough ω, the MIT rule is stable for all g, while for small enough g, it is stable for all ω. Otherwise, the dynamic behavior of the algorithm is quite complex, with interleaving of the stable and unstable regions.

a) Write the differential equations of the overall adaptive system and show that it is a nonlinear time–varying system. Given that the equation for y_m can be solved independently of the others, show that the remaining equations constitute a *linear* time–varying system $\dot{x} = A(t)\,x$.

b) Simulate the responses of the adaptive system for a few values of the parameters to check the validity of the figure. In particular, check the responses for $\omega = 1$ and $g = 10, 35, 50$ and 80, then for $g = 10$ and $\omega = 1, 1.5$ and 2.5. Let all initial conditions be zero and plot the error e_0 for 50 seconds. Could such behavior be observed with a second-order linear time–invariant system ?

Problem 1.1

a) Determine which functions belong to L_1, L_2, L_∞, L_{1e}, L_{2e}, $L_{\infty e}$

$$f_1(t) = e^{-at} \quad \text{for} \quad a>0 \quad \text{and} \quad a \le 0$$

$$f_2(t) = \frac{1}{t+1}\,; \qquad f_3(t) = \frac{1}{t^2-1}\,; \qquad f_4(t) = \frac{1}{\sqrt{t+1}}$$

b) Show that $f \in L_1 \cap L_\infty$ implies $f \in L_2$.

Problem 1.2

a) Show that $A \in R^{2x2}$ is positive definite if and only if $a_{11} > 0$, $a_{22} > 0$, and $a_{11}\, a_{22} > (a_{12} + a_{21})^2/4$.

b) Find examples of 2x2 symmetric (but not diagonal) matrices such that $A \ge 0$, $A < 0$ and A is neither positive nor negative semidefinite. In each case, give $\lambda_i(A)$.

c) Find an example of two symmetric positive definite matrices A and B such that the product is neither symmetric nor positive definite. Give $\lambda_i(A\,B)$ and $\lambda_i((A\,B) + (A\,B)^T)$.

Problem 1.3

Prove the following facts. All matrices are real and square (except in a)). Use only the definitions of eigenvalues and eigenvectors.

a) For all $C \in R^{mxn}$, $C^T C \ge 0$ and $C^T C > 0$ if rank $(C) = n$

b) $A = A^T$ implies $\lambda_i(A) \in R$

c) $A = A^T$ and $\lambda_1(A) \ne \lambda_2(A)$ implies $x_1^T x_2 = 0$
where x_1, x_2 are the eigenvectors associated with $\lambda_1(A)$, $\lambda_2(A)$

d) $A \ge 0$ implies $\text{Re}\,\lambda_i(A) \ge 0$

Problem 1.4

Prove the following facts. All matrices are real and square. You may use the fact that $A = A^T$ implies that $A = U^T \Lambda U$ with $\Lambda = \text{diag}\,\lambda_i(A)$ and $U^T U = I$.

a) For $A = A^T$ $A \ge 0$ if and only if $\lambda_i(A) \ge 0$

b) $A \ge 0$ if and only if $\lambda_i(A_S) \ge 0$

c) For $A = A^T \ge 0$ $\lambda_{\min}(A)\,|x|^2 \le x^T A\, x \le \lambda_{\max}(A)\,|x|^2$

d) For $A = A^T > 0$ $\lambda_{\max}(A^{-1}) = (\lambda_{\min}(A))^{-1}$

e) For $A = A^T \ge 0$ $\|A\| = \lambda_{\max}(A)$

f) $\det(S) \ne 0$ implies $\lambda_i(A) = \lambda_i(S^{-1} A\, S)$

g) $A = A^T > 0$, $B = B^T \ge 0$ implies $\lambda_i(A\,B) \in R$ and $\lambda_i(A\,B) \ge 0$

Problem 1.5

Consider the plant $\hat{P}(s) = 1/(s + a_p)$ and the reference model $\hat{M}(s) = 1/(s + a_m)$. Let the controller be given by

$$u = d\, y_p + r$$

where r is the reference input (bounded), u and y_p are the input and output of the plant, and d is an adaptive gain with update law

$$\dot{d} = -g\, e_0\, y_p$$

a) Is the overall system described by a linear or nonlinear differential equation ? Is it time–invariant or time–varying ?

b) For constant $g > 0$, indicate whether the system is stable, uniformly stable, asymptotically stable, or uniformly asymptotically stable. Does $e_0 \to 0$ as $t \to \infty$?

c) Let $g(t)$ be a continuously differentiable function of time. What stability properties does the system have if $g(t) > 0$ for all t ?

Problem 1.6

Let $\dot{x} = f(t, x)$, $x(t_0) = x_0$, with $f(t, x)$ globally Lipschitz in x (with constant l) and $f(t, 0)$ bounded (with bound b). Show that

$$k_1 + k_2 e^{-l(t-t_0)} \le |x(t)| \le k_3 + k_4 e^{l(t-t_0)}$$

Find expressions for k_1, k_2, k_3 and k_4 as functions of $|x_0|$, b and l.

Problem 1.7

Consider the linear time–invariant system

$$\dot{x} = A x \qquad x(t_0) = x_0$$

Assume that there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$A^T P + P A + Q = 0$$

Find $m \ge 0$ and $\alpha > 0$ as functions of the matrices P and Q such that

$$|x(t)| \le m e^{-\alpha(t-t_0)} |x_0|$$

Problem 1.8

Consider the linear time–varying system (from Vidyasagar's *Nonlinear Systems Analysis*, Prentice–Hall, 1978)

$$\dot{x} = A(t) x$$

where

$$A(t) = \begin{pmatrix} -1 + a\cos^2 t & 1 - a \sin t \cos t \\ -1 - a \sin t \cos t & -1 + a \sin^2 t \end{pmatrix}$$

Show that the transition matrix $\Phi(t, \tau)$ satisfies

$$\Phi(t, 0) = \begin{pmatrix} e^{(a-1)t} \cos t & e^{-t} \sin t \\ -e^{(a-1)t} \sin t & e^{-t} \cos t \end{pmatrix}$$

and that the matrix $A(t)$ has eigenvalues that are independent of t and located in the left–half plane for $1 < a < 2$. Discuss the stability properties of the equilibrium point at $x = 0$. What conclusions can you draw from this example ?

Problem 1.9

Determine which of the following functions are globally Lipschitz, and give the Lipschitz constant. Let x and f be scalars.

a) $f(t, x) = \dfrac{x}{1 + x^2}$ b) $f(t, x) = \dfrac{e^t}{1 + e^t} x$

c) $f(t, x) = A x$ d) $f(t, x) = e^t x$

e) $f(t, x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ f) $f(t, x) = \sqrt{x}$

Hint: use the mean-value theorem of elementary analysis to relate the Lipschitz constant to $\partial f / \partial x$.

Problem 1.10

a) Given the linear time-invariant system $\dot{x} = A\,x$, determine whether the following A matrices correspond to an equilibrium point $x = 0$ that is stable in the sense of Lyapunov, asymptotically stable, or unstable.

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad A_5 = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \quad A_8 = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix} \quad A_9 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

b) For a general matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

give the conditions on a_{11}, a_{12}, a_{21} and a_{22} so that the equilibrium point is asymptotically stable, and so that it is stable in the sense of Lyapunov.

Hint: to find simple conditions on the elements of the matrix, recall the Routh–Hurwitz test.

Problem 1.11

Consider the circuit depicted in Fig. P1.11.

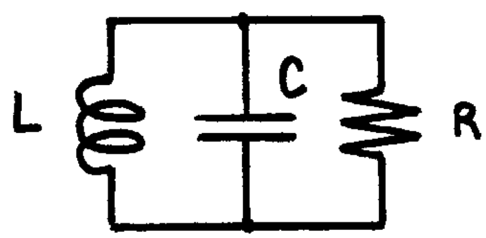

Figure P1.11: Circuit

a) Let x_1 be the voltage on the capacitor C and x_2 be the current in the inductor L. Show that the system is described by the differential equations

$$\frac{dx_1}{dt} = -\frac{1}{RC}\,x_1 - \frac{1}{C}\,x_2$$

$$\frac{dx_2}{dt} = \frac{1}{L}\,x_1$$

b) Consider the "natural" Lyapunov function consisting of the total energy stored in the system

$$v = \frac{C}{2}\,x_1^2 + \frac{L}{2}\,x_2^2$$

What stability properties of the equilibrium can you deduce from this Lyapunov function ?

c) Use the Lyapunov lemma to find a Lyapunov function such that

$$\alpha_1 |x|^2 \leq v \leq \alpha_2 |x|^2$$

for some strictly positive constants α_1, α_2, and such that

$$\dot{v} \leq -|x|^2$$

For $R=L=C=1$, give the specific values of the constants α_1, α_2. What stability properties of the equilibrium can you deduce from this Lyapunov function ?

Problem 1.12

a) Consider the system

$$\frac{dx}{dt} = -x^3 \qquad x(0)=x_0$$

Determine whether the function on the right–hand side is locally Lipschitz, or globally Lipschitz.

b) Show that the differential equation can be solved exactly and deduce what the stability properties of the equilibrium point are.

c) Find a Lyapunov function to confirm the conclusions of part b).

d) Repeat parts a) to c) for the system

$$\frac{dx}{dt} = -e^{-t}\, x \qquad x(0)=x_0$$

e) Repeat parts a) to c) for the system

$$\frac{dx}{dt} = -\frac{1}{t+1}\, x \qquad x(0)=x_0$$

Problem 1.13

a) Calculate the eigenvalues and eigenvectors of

$$A=\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$$

b) Find the decomposition $A=U^T \Lambda U$ and find a symmetric positive definite matrix $A^{1/2}$ such that $A=A^{1/2}\, A^{1/2}$.

c) Given an arbitrary *vector* $x \in R^n$, show that the matrix

$$A=x\, x^T$$

is always symmetric positive semi–definite, but never positive definite. What is the minimum nunber of vectors x_i needed so that the matrix

$$A=\sum_{i=1}^{N} x_i\, x_i^T$$

can be positive definite ?

Problem 1.14

Consider the linear time–invariant system with transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{b}{s+a}$$

with $a>0$. Show that, for zero initial conditions

$$\| y \|_\infty \le \| h \|_1 \; \| u \|_\infty$$

where $\| h \|_1$ is the L_1 norm of the impulse response $h(t)$ corresponding to $H(s)$, and $\| y \|_\infty$, $\| u \|_\infty$ are the L_∞ norms of $y(t)$ and $u(t)$ respectively. Give the value of $\| h \|_1$ as a function of a and b and show that there exists a nonzero $u(t)$ such that the inequality above becomes an equality.

Problem 2.1

a) Consider the identifier of section 2.2. Show that the plant transfer function may be realized by two state-space representations (R1) and (R2):

(R1) $\dot{x} = A_p x + b_p r$

$$y_p = c_p^T x$$

where

$$A_p = \Lambda + b_\lambda b^{*T}$$

$$b_p = b_\lambda \qquad c_p = a^*$$

and $\Lambda, b_\lambda, a^*, b^*$ are as defined in equations (2.2.9) and (2.2.10).

(R2) $\dot{w} = A_o w + b_o r$

$$y_p = c_o^T w$$

where

$$A_o = \begin{pmatrix} \Lambda & 0 \\ b_\lambda a^{*T} & \Lambda + b_\lambda b^{*T} \end{pmatrix}$$

$$b_o = \begin{pmatrix} b_\lambda \\ 0 \end{pmatrix} \qquad c_o = \begin{pmatrix} a^* \\ b^* \end{pmatrix}$$

b) Show that (R1) is controllable. Show that (R1) is observable if $\hat{n}_p$, $\hat{d}_p$ are coprime.

c) Show that (R2) is nonminimal and, from the eigenvalues of A_o, deduce that the extra modes are those of Λ.

d) Using the Popov-Belevitch-Hautus test, show that (R2) is controllable if $\hat{n}_p$, $\hat{d}_p$ are coprime. In the same manner, show that any unobservable mode must (indeed) be a mode of Λ, so that (R2) is detectable.

Hint: The Popov-Belevitch-Hautus test indicates that a state-space realization is controllable if and only if

$$\text{rank } (sI - A \;\; B) = \dim (A) \qquad \text{for all } s$$

and observable if and only if

$$\text{rank} \begin{pmatrix} C \\ sI - A \end{pmatrix} = \dim (A) \qquad \text{for all } s$$

Note that it is only necessary to check the rank for values of s for which $\det (sI - A) = 0$.

Problem 2.2

a) Let $w(t) = \sin (t)$. Find and plot

$$\lambda(t) = \int_0^t w^2(\tau)\, d\tau$$

b) Show that there exist α_1, α_2 and $\delta > 0$ such that

$$\alpha_2 \ge \int_{t_0}^{t_0+\delta} w^2(\tau)\, d\tau \ge \alpha_1$$

for all $t_0 \geq 0$.

c) Show that the following limit exists, independently of t_0

$$R_w = \lim_{\delta \to \infty} \frac{1}{\delta} \int_{t_0}^{t_0+\delta} w^2(\tau)\, d\tau$$

where $R_w > 0$.

d) Repeat part c) with

$$w(t) = \begin{pmatrix} \sin(t) \\ a \sin(t+\phi) \end{pmatrix}$$

and replacing $w^2(\tau)$ by $w(\tau)\, w^T(\tau)$. What conditions must a, ϕ satisfy so that the matrix R_w is positive definite ?

Problem 2.3

a) Consider the function $w(t)$ such that

$$w(t) = 1 \qquad t \in [\, n, n + (1/2)^{n+1}\,] \quad \text{for all positive integers } n$$

$$w(t) = 0 \qquad \text{otherwise.}$$

Plot $w(t)$ and show that there exists $\delta > 0$ such that

$$\int_{t_0}^{t_0+\delta} w^2(\tau)\, d\tau > 0$$

for all $t_0 \geq 0$. Show that, however, the solutions of

$$\dot{\phi} = -g\, w^2\, \phi \qquad \phi(0) = \phi_0$$

do *not* converge to zero exponentially. Is w persistently exciting ? Why ?

b) Consider the following proof of exponential convergence for $\dot{\phi} = -g\, w\, w^T \phi$, with w PE: Let $v = \phi^T \phi$, so that $\dot{v} = -2\, g\, (\phi^T w)^2 \leq 0$. Therefore, for all $t_0 \geq 0$, $\delta > 0$, and $\tau \in [\, t_0, t_0 + \delta\,]$

$$\phi^T(t_0)\, \phi(t_0) \geq \phi^T(\tau)\, \phi(\tau) \geq \phi^T(t_0+\delta)\, \phi(t_0+\delta) \tag{1}$$

and

$$v(t_0) - v(t_0+\delta) = 2\, g \int_{t_0}^{t_0+\delta} \phi^T(\tau)\, w(\tau)\, w^T(\tau)\, \phi(\tau)\, d\tau \tag{2}$$

$$\geq 2\, g\, \phi^T(t_0+\delta) \int_{t_0}^{t_0+\delta} w(\tau)\, w^T(\tau)\, d\tau \;\; \phi(t_0+\delta) \tag{3}$$

where the last inequality follows from (1). If w is PE, this implies that

$$v(t_0) - v(t_0+\delta) \geq 2\, g\, \alpha_1\, v(t_0+\delta) \tag{4}$$

and exponential convergence follows as in the proof of theorem 1.5.2. Why is this proof incorrect ?

Problem 2.4

This problem investigates the parameter convergence properties of the least-squares algorithm with forgetting factor

$$\dot{\theta}(t) = -P(t)\, w(t)\, (\theta^T(t)\, w(t) - y_p(t)) \qquad \theta(0) = \theta(0)$$

$$\dot{P}(t) = \lambda\, P(t) - P(t)\, w(t)\, w^T(t)\, P(t) \qquad P(0) = P^T(0) = P_0 > 0, \quad \lambda > 0$$

a) Let $\lambda = 0$. Use eqn (2.0.30) to show that $|\theta(t) - \theta^*| \to 0$ as $t \to \infty$ if

$$\int_0^t w(\tau)\, w^T(\tau)\, d\tau \ge \alpha(t)\, I$$

and $\alpha(t) \to \infty$ as $t \to \infty$. Show that if $\alpha(t) \ge k\, t$ for some $k > 0$, then $|\theta(t) - \theta^*|$ converges to zero as $1/t$. What if $\alpha \ge e^t$?

b) For $\lambda \ne 0$, find expressions similar to (2.0.29) and (2.0.30).

Hint: First derive $d(P^{-1})/dt$ and $d(P^{-1}\,\theta)/dt$.

c) What optimization criterion does the least-squares algorithm with forgetting factor solve ?

d) Show that $|\theta(t) - \theta^*|$ converges to zero exponentially with rate λ if $P(t)$ is bounded.

e) Show that $P(t)$ is bounded if w is PE.

Problem 2.5

This problem constructs discrete–time algorithms for the identification of sampled–data systems. The derivations follow lines parallel to the continuous–time case. However, see G.C. Goodwin & K.S. Sin (1984) for more details.

a) Consider the first order system

$$\dot{y}_p = -a_p\, y_p + k_p\, r$$

The continuous–time system is interfaced to a digital computer through discrete–time signals $r_D[k]$ and $y_D[k]$ such that

$$r(t) = r_D[k] \qquad \text{for } t \in [k\,T, (k+1)\,T]$$

$$y_D[k] = y_p(k\,T)$$

By solving the differential equation, show that y_D and r_D satisfy *exactly* a first–order *difference* equation

$$y_D[k] = a_D\, y_D[k-1] + k_D\, r_D[k-1]$$

and find expressions for a_D, k_D, as functions of k_p, a_p, and T (this is the so–called *step–response discrete–time equivalent* of the continuous–time system). What is the discrete–time transfer function $\hat{P}(z) = \hat{y}_D(z)/\hat{r}_D(z)$?

b) Extend the results of part a) to a general n–th order system

$$\dot{x} = A\,x + b\,r$$

$$y_p = C\,x$$

c) Adapt the results of section 2.2 to the discrete–time by describing an identifier structure for an n–th order plant with transfer function $\hat{P}(z)$. In particular, show that

$$y_D[k] = \theta^{*T}\, w[k]$$

where θ^* is a vector of unknown parameters and w is an observer state vector. Give conditions that the polynomial $\hat{\lambda}(z)$ must satisfy and indicate the simplifications that arise when $\hat{\lambda}(z) = z^n$, *i.e.*, when all the observer poles are at the origin. What is the effect of initial conditions in that case?

d) The so-called *projection algorithm* is the estimate $\theta[k]$ such that

$$|\theta[k] - \theta[k-1]|^2$$

is minimized, subject to

$$y_D[k] = \theta^T[k]\, w[k]$$

Using Lagrange multipliers, show that the solution of this optimization criterion is

$$\theta[k]=\theta[k-1]-w[k]\ \frac{\theta^T[k-1]\,w[k]-y_D[k]}{w^T[k]\,w[k]}$$

Show that $\theta[k]$ is also the projection of $\theta[k-1]$ on the set of $\theta[k]$'s such that $\theta^T[k]\,w[k]-y_D[k]=0$.

e) Find the (batch) *least-squares estimate*, which minimizes

$$J(\theta[k])=\sum_{j=1}^{k}\,(\theta^T[k]\,w[j]-y_D[j])^2$$

Then, show that the *recursive* formula for the least-squares estimate is

$$P[k]=P[k-1]-\ \frac{P[k-1]\,w[k]\,w^T[k]\,P[k-1]}{1+w^T[k]\,P[k-1]\,w[k]}$$

$$\theta[k]=\theta[k-1]-\ \frac{P[k-1]\,w[k]\,(\theta^T[k-1]\,w[k]-y_D[k])}{1+w^T[k]\,P[k-1]\,w[k]}$$

Hint: To find the difference equation for the covariance matrix, the following identity is useful

$$(A+B\,C)^{-1}=A^{-1}-A^{-1}\,B\,(I+C\,A^{-1}\,B)^{-1}\,C\,A^{-1}$$

f) Simulate the responses of the identifier with the projection algorithm. Replace the denominator $w^T[k]\,w[k]$ by $1+w^T[k]\,w[k]$ to avoid possible division by zero or by a very small number. Let $k_p=1$, $a_p=1$, $T=0.1\ s$, and all initial conditions be zero. Plot $r_D[k]$, $y_D[k]$, $\theta_1[k]$, and $\theta_2[k]$ for $r_D[k]=\sin\,(kT)$ and for $r_D[k]=1$. Compare the convergence properties.

g) Repeat part f) for the least-squares algorithm. Let $P[0]=I$ and plot $P_{11}[k]$, $P_{22}[k]$ and $P_{12}[k]$, in addition to the other variables.

Problem 2.6

a) Find conditions on α_1, α_2, β_1, β_2 such that the transfer function

$$\hat{M}(s)=\frac{\alpha_2\,s+\alpha_1}{s^2+\beta_2\,s+\beta_1}$$

is SPR.

b) Find conditions on R_1, R_2, C_1, C_2 (all >0) such that the impedance $\hat{Z}(s)=\hat{V}(s)/\hat{I}(s)$ (see figure P2.6) is SPR.

Problem 2.7

Let $\hat{M}(s)$ be a rational, strictly proper transfer function.

a) Let $s(t)$ be the step response associated with $\hat{M}(s)$. Show that if $\hat{M}(s)$ is SPR, then

$$\left(\frac{ds}{dt}\right)_{t=0}>0 \qquad \text{and} \qquad \left(\frac{d^2s}{dt^2}\right)_{t=0}<0$$

Hint: Use the Kalman-Yacubovitch-Popov lemma.

b) Show that $\hat{M}(s)$ is SPR if and only if there exists a minimal state-space representation

$$\dot{x}=A\,x+b\,u$$

$$y=c^T\,x$$

such that $c=b$ and $A+A^T=-Q$ for some $Q=Q^T>0$.

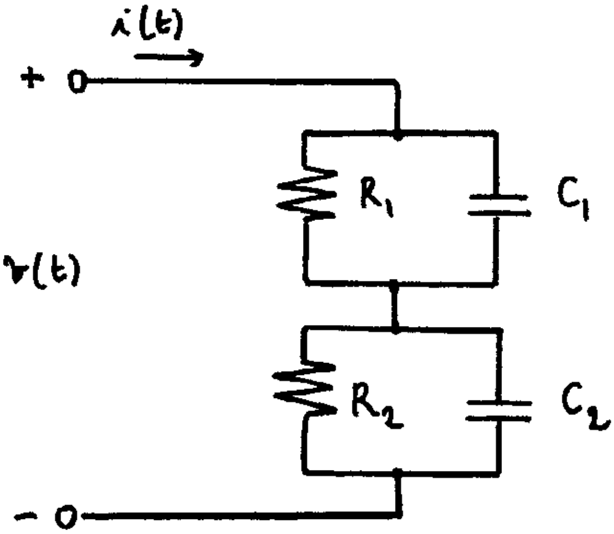

Figure P2.6: Circuit

Hint: Use again the Kalman-Yacubovitch-Popov lemma.

Problem 2.8

a) Prove one direction of the Kalman-Yacubovitch-Popov lemma, that is, show that

If there exist matrices $P = P^T > 0, Q = Q^T > 0$ such that

$$A^T P + P A = -Q \qquad \text{and} \qquad P b = c$$

with $[A, b]$ controllable

Then

$\hat{M}(s) = c^T (sI - A)^{-1} b$ is stable

Re $\hat{M}(j\omega) > 0$ for all ω and $\lim_{\omega \to \infty} \omega^2 \text{ Re } \hat{M}(j\omega) > 0$

Hint: Show first that

$$P(j\omega I - A)^{-1} + (-j\omega I - A^T)^{-1} P = (-j\omega I - A^T)^{-1} Q (j\omega I - A)^{-1}$$

b) Consider the system

$$\dot{x} = A x - b \theta^{*T} w(x) + b u$$

where $x, b \in R^n$, $\theta^*, w \in R^m$, $A \in R^{n \times n}$, and $u \in R$. A and b are known exactly, $w(x)$ is a known, globally Lipschitz function of x and the vector $x(t)$ itself is available. The vector θ^* is completely unknown. Assume that A is asymptotically stable and $[A, b]$ is controllable. Use the Kalman-Yacubovitch-Popov lemma to find an adaptive control law such that $x - x_m \to 0$ as $t \to \infty$ where

$$\dot{x}_m = A x_m + b r$$

where $r \in L_\infty$. Prove the result, including the boundedness of all the signals.

Problem 2.9

This problem examines the extension of the single–input single–output identification results to multivariable systems where all the states are available for measurement.

a) Consider the state-space representation

$$\dot{x} = A^* x + B^* u$$

where $x \in R^n$, $u \in R^m$, $A^* \in R^{n \times n}$ and $B^* \in R^{n \times m}$. Assume that x, $\dot{x}$ and u are available. A^* and B^* are completely unknown. Show that a multivariable, linear error equation can be obtained, and that a gradient identification algorithm can be derived. What stability properties can you establish?

b) Propose an algorithm for the case when $\dot{x}$ is not available and one does not wish to differentiate x explicitly for noise considerations.

Problem 2.10

This problem investigates the stability properties of the projection algorithm, which is the discrete–time equivalent of the gradient algorithm.

a) Consider the *modified projection algorithm*

$$\theta[k] = \theta[k-1] - \frac{\theta^T[k-1]\, w[k] - y_D[k]}{1 + w^T[k]\, w[k]}\; w[k]$$

Define the *a priori* error

$$e^-[k] = \theta^T[k-1]\, w[k] - y_D[k]$$

and the *a posteriori* error

$$e^+[k] = \theta^T[k]\, w[k] - y_D[k]$$

Show that

$$e^+[k] = e^-[k] / (1 + w^T[k]\, w[k])$$

so that

$$\theta[k] = \theta[k-1] - \frac{e^-[k]}{1 + w^T[k]\, w[k]}\; w[k]$$
$$= \theta[k-1] - e^+[k]\, w[k]$$

In other words, the modified projection algorithm is a *gradient* algorithm for the *a posteriori error* $e^+[k]$.

b) Let

$$y_D[k] = \theta^{*T} w[k] \qquad \text{and} \qquad \phi[k] = \theta[k] - \theta^*$$

Using the results of part a), show that

$$|\phi[k]|^2 = |\phi[k-1]|^2 \; - \; (e^+[k])^2\, (2 + w^T[k]\, w[k])$$

$$= |\phi[k-1]|^2 \; - \; (e^-[k])^2\, \frac{(2 + w^T[k]\, w[k])}{(1 + w^T[k]\, w[k])^2}$$

c) A sequence $x[k]$ is said to be in l_p if

$$\sum_{k=1}^{\infty} |x[k]|^p < \infty$$

and $x \in l_\infty$ if $x[k]$ is bounded for all k. Show that

(*i*) $|\phi[k]| \le |\phi[k-1]| \le |\phi[0]|$ for all k

(*ii*) $e^+[k]\,(1+w^T[k]\,w[k])^{1/2} \in l_2$

(*iii*) $e^+[k]\,(1+w^T[k]\,w[k])^{1/2} \to 0$ as $t \to \infty$

(*iv*) $\dfrac{e^-[k]}{(1+w^T[k]\,w[k])^{1/2}} \in l_2$

(*v*) $\dfrac{e^-[k]}{(1+w^T[k]\,w[k])^{1/2}} \to 0$ as $t \to \infty$

(*vi*) $|\phi[k]-\phi[k-1]| \to 0$ as $t \to \infty$

Does (*vi*) imply that $\theta[k]$ tends to some limit as $k \to \infty$ and, if so, is the limit necessarily θ^* ?

Problem 2.11

Show how the identification method described in section 2.2 can be extended to proper transfer functions of the form

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{\alpha_{n+1}\, s^n + \alpha_n\, s^{n-1} + \cdots + \alpha_1}{s^n + \beta_n\, s^{n-1} + \cdots + \beta_1}$$

Specialize the results to a first-order system ($n=1$) to check the method.

Problem 2.12

Consider the linear error equation $e_1 = \phi\, w$ where e_1, ϕ, w are all scalars. Further, let $w(t) = w_0$ be constant.

a) Consider the *gradient algorithm*

$$\dot{\phi} = -g\, w_0^2\, \phi$$

with $g > 0$ and $\phi(0) = \phi_0$ arbitrary. Find the solution of the differential equation. Indicate to what value ϕ converges and describe the type of convergence (exponential, asymptotic,...). Discuss the case when $w_0 = 0$.

b) Consider the *least-squares algorithm*

$$\dot{\phi} = -g\, p\, w_0^2\, \phi$$

$$\dot{p} = -g\, p^2\, w_0^2$$

with $g > 0$, $\phi(0) = \phi_0$, and $p(0) = p_0 > 0$. Find the solutions of the differential equations. Indicate to what values ϕ and p converge and describe the type of convergence. Discuss the case when $w_0 = 0$.

Hint: note that $\dfrac{d}{dt}\,(\phi\, p^{-1}) = 0$.

c) Consider the *least-squares algorithm with forgetting factor*

$$\dot{\phi} = -g\, p\, w_0^2\, \phi$$

$$\dot{p} = -g\,(-\lambda\,p + p^2\,w_0^2)$$

with $g > 0$, $\lambda > 0$, $\phi(0) = \phi_0$, and $p(0) = p_0 > 0$. Give the solutions of the differential equations. Indicate to what values ϕ and p converge and describe the type of convergence. Discuss the case when $w_0 = 0$.

Hint: start again from $\frac{d}{dt}\,(\phi\,p^{-1})$.

d) Consider the *stabilized least–squares algorithm with forgetting factor*

$$\dot{\phi} = -g\,p\,w_0^2\,\phi$$

$$\dot{p} = -g\,(-\lambda\,p + p^2\,w_0^2 + \alpha\,p^2)$$

with $g > 0$, $\lambda > 0$, $\alpha > 0$, $\phi(0) = \phi_0$, and $p(0) = p_0 > 0$. The stabilized algorithm was proposed in the discrete–time framework by G. Kreisselmeier, in "Stabilized Least–Squares Type Adaptive Identifiers," *IEEE Trans. on Automatic Control*, vol. 35, no. 3, pp. 306–310, 1990. The stabilizing term $\alpha\,p^2$ is an interesting alternative to covariance resetting.

Give the solution of the differential equation for p. Indicate to what values ϕ and p converge and describe the type of convergence. Discuss the case when $w_0 = 0$.

Hint: show that p is bounded away from 0 to establish the convergence properties of ϕ.

Problem 2.13

A periodic signal $y(t)$ is measured. Its period T is known and one wishes to design a recursive scheme to estimate the coefficients of its Fourier series. It is assumed that a finite number of terms is sufficient to represent $y(t)$ exactly, the number being known.

a) Show that the problem can be put in the form of a linear error equation of the type described in chapter 2. Give a recursive algorithm to estimate the coefficients of the Fourier series.

b) Determine whether the estimates are guaranteed to converge to the values of the coefficients of the Fourier series, and justify your answer.

Problem 2.14

a) Consider the assertion made in the book on p. 61 that the filters $(sI - \Lambda)^{-1}\,b_\lambda$ can be replaced by (at least) two other vector transfer functions. For the first transfer function, indicate how the vector θ^* is related to the polynomials $k_p\,\hat{n}_p(s)$ and $\hat{d}_p(s)$. Give explicit formulas relating the α's and β's to the components of θ^* for the transfer function

$$\hat{P}(s) = \frac{\alpha_2\,s + \alpha_1}{s^2 + \beta_2\,s + \beta_1}$$

b) Repeat part a) for the other transfer function.

Hint: it may be useful to remember partial fraction expansions.

Problem 2.15

This problem investigates the properties of the projection modification, introduced to keep the adaptive parameters within a specified region in which the nominal parameters are known to lie. It is shown that a different projection modification is necessary for the least–squares algorithm in order to preserve the properties of the Lyapunov function.

Assume that $\theta^* \in S$, where S is a closed set with a smooth, differentiable boundary given by $f(\theta_1, \theta_2, \ldots, \theta_n) = 0$. Assume that $\theta \in S$ if and only if $f(\theta_1, \ldots, \theta_n) \le 0$ and that outside S, $f(\theta_1, \ldots, \theta_n) > 0$. See figure P2.15 for $n = 2$. Let $x(\theta)$ be a vector perpendicular to the plane tangent to the boundary of S at θ, pointing outwards.

a) Show how $x(\theta)$ can be obtained from $f(\theta_1, \ldots, \theta_n)$. What condition must S satisfy so that $\phi^T\,x > 0$ for θ on the boundary ?

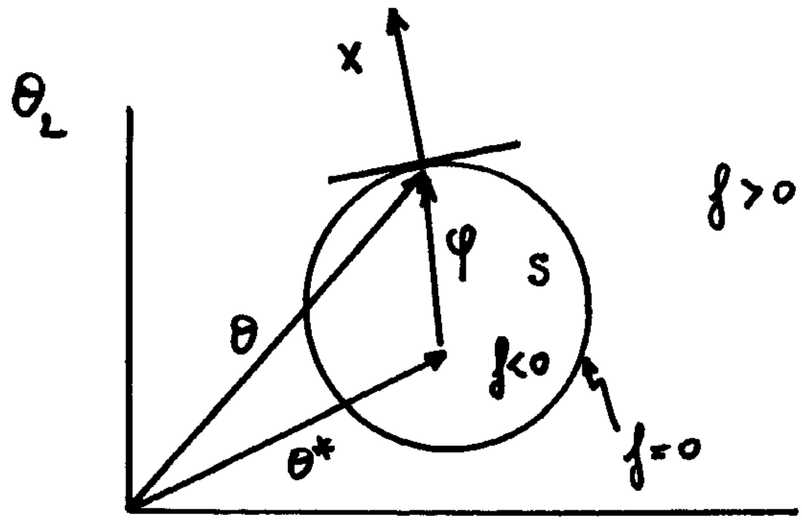

Figure P2.15: Projection modification.

b) Show that the modification

$$\dot{\theta} = z - \frac{(z^T x).\, x}{x^T x} \qquad \text{if } f(\theta)=0 \ \text{ and } \ (z^T x)>0$$

$$= z \qquad \text{otherwise}$$

consists in projecting the update vector z on the tangent plane when the boundary of the set S is reached and the vector is pointing outwards.

c) Show that, for the gradient algorithm ($z=-g\, e_1\, w$), the projection modification guarantees that $\theta(t)\in S$ for all t, assuming that $\theta(0)\in S$. Prove also that, for $v=\phi^T \phi$

$$-\dot{v} \ \geq\ 2\, g\, (\phi^T w)^2$$

d) Repeat part c) for the least-squares algorithm, letting $z=-g\, P\, e_1\, w$, $\dot{P}=-g\, P\, w\, w^T P$, and the Lyapunov function $v=\phi^T P^{-1}\phi$. However, note here that the properties of the Lyapunov function are *not* preserved by the projection modification.

e) Show that to preserve *both* properties for the least-squares algorithm, the following projection can be used

$$\dot{\theta} = z - \frac{(z^T x).\,(P\, x)}{x^T P\, x} \qquad \text{if } f(\theta)=0 \ \text{ and } (z^T x)>0$$

Optional: Show that this modification consists in projecting in the transformed space $P^{-1/2}\,\theta$ instead of the θ space.

f) Consider the case when $\theta\in R^2$ and the set S is the set of θ's such that $\theta_1 \leq \theta_1^+$, with θ_2 arbitrary. Write explicitly what the update laws for θ_1 and θ_2 are, as functions of z_1, z_2, P_{11}, P_{12}, P_{22}, and θ_1^+ (compare the gradient and the least-squares algorithms).

Problem 3.1

This problem considers the sampled–data control of a first–order system. The scheme is an indirect, model reference adaptive control algorithm.

a) Consider the first order system

$$\dot{y}_p = -a_p\, y_p + k_p\, u$$

The continuous time system is controlled by a digital computer through discrete time signals $u_D[k]$ and $y_D[k]$ such that

$$u(t)=u_D[k] \qquad \text{for } t \in [kT,(k+1)T]$$

$$y_D[k]=y_p(kT)$$

Let the control law be given by

$$u_D[k]=c[k]\,r_D[k]+d[k]\,y_D[k]$$

Show that there exist *nominal* values of the controller parameters c^* and d^* such that the discrete time transfer function from r_D to y_D matches the *reference model* transfer function

$$\hat{M}(z)=\frac{k_M}{z-a_M}$$

where k_M, a_M are arbitrary, but $|a_m|<1$.

b) Simulate the responses of an indirect adaptive control algorithm, combining a discrete time identifier using the modified projection algorithm (*cf.* problem 2.10) and a control law derived from part a). Let $k_p=1$, $a_p=1$, $T=0.1\,s$ and $\hat{M}(z)$ be the discrete time equivalent of $3/(s+3)$. Let $r_D[k]=\sin\,[kT]$. Include a standard modification to the identification algorithm so that the estimate of the gain is bounded away from 0 (assume that the sign and a lower bound of the gain is known).

Problem 3.2

a) Consider the vector $\overline{w}$, in section 3.2. Show that $\overline{w}$ is the state of $\hat{P}$ in a nonminimal state-space representation (give the A, b, and c matrices).

Hint: First show that there exist *unique* $f_0^*, g_0^* \in R$ and $f^*, g^* \in R^{n-1}$ such that

$$\dot{y}_p=f_0^*\,u+f^{*T}\,w^{(1)}+g_0^*\,y_p+g^{*T}\,w^{(2)}$$

by proving that there exist unique polynomials $\hat{f}^*$, $\hat{g}^*$ of degree at most $n-1$ such that

$$s\,\hat{P}=\frac{\hat{f}^*}{\hat{\lambda}}+\frac{\hat{g}^*}{\hat{\lambda}}\,\hat{P}$$

Indicate how $\hat{f}^*$ and $\hat{g}^*$ can be obtained from $\hat{P}$ and $\hat{\lambda}$.

b) Consider the polynomial $\hat{q}$ in section 3.2. Under the assumptions, is this polynomial guaranteed to be Hurwitz ? What can you conclude about the stability of the controller itself (as represented in Figure 3.2).

Problem 3.3

a) Consider the output error MRAC scheme of chapter 3. Let the reference model be given by

$$\hat{M}=\frac{s+3}{(s+1)(s+5)}$$

Simulate the response of the adaptive scheme for $r(t)=2\,\sin\,(t)$ and

$$\hat{P}=\frac{s+1}{s^2+1}$$

Let $g=1$ and all initial conditions be zero. Plot e_0 and the adaptive parameters for 1000 seconds. Calculate the nominal values of the adaptive parameters and comment on the responses.

b) Repeat part a) with the plant

$$\hat{P}=\frac{s+3}{(s-1)^2}$$

and the same adaptive controller. Again, comment on the responses.

Note: This problem illustrates the fact that two totally different unstable systems can be stabilized by a single adaptive controller. However, as the simulations show, some surprising responses can be observed within the confines of the Lyapunov stability results. Poor convergence results are often observed with gradient algorithms, and the interested student is invited to investigate other algorithms based on the least–squares estimates.

Problem 4.1

In this problem, the averaging results will be proved in the (simpler) linear periodic case, *i.e.*,

$$\dot{x}(t)=\varepsilon\, A(t)\, x(t) \qquad x(0)=x_0$$

where $A(t)\in R^{n\times n}$ is a piecewise continuous, bounded, and periodic function of t with period T.

a) Let

$$A_{av}=\frac{1}{T}\int_0^T A(\tau)\, d\tau$$

and

$$W(t)=\int_0^t (\, A(\tau)-A_{av})d\tau$$

Show that $W(t)$ is periodic and bounded.

b) Show that, for ε sufficiently small, the transformation from x to z defined by

$$x(t)=z(t)+\varepsilon\, W(t)\, z(t)$$

is one-to-one and is differentiable with bounded derivatives. Further

$$|\, x(t)-z(t)\,|<2\,\varepsilon\, T\, \|A(.)\|_\infty\, |\, z(t)\,|$$

c) Show that z satisfies a differential equation

$$\dot{z}=\varepsilon\, A_{av}\, z+\varepsilon^2\,(I+\varepsilon\, W)^{-1}\,(A\, W-W\, A_{av})\, z$$

d) Show that if

$$\dot{x}_{av}=A_{av}\, x_{av}$$

is exponentially stable, then

$$\dot{x}=\varepsilon\, A(t)\, x(t)$$

is exponentially stable for ε sufficiently small.

Hint: Use the Lyapunov lemma.

Problem 4.2

Calculate the averaged system and simulate the responses of the original and averaged system for

a) $\dot{x} = -g\sin^2(t)\,x \quad g = 1$ and $g = 0.25$

b) $\dot{x} = -g\,(\cos(t) + \sin(2t))^2 x \quad g = 0.5$

Plot $\ln(x^2)$ and $\ln(x_{av}^2)$.

Problem 4.3

a) Consider the least-squares algorithm

$$\dot{\phi} = -g\,P\,w\,w^T\phi \qquad \phi(0) = \phi_0$$

$$\dot{P} = -g\,P\,w\,w^T P \qquad P(0) = P(0)^T = P_0 > 0$$

where a gain $g > 0$ has been included. Assume that w is stationary. Find an expression for the averaged system. Solve for P_{av} and ϕ_{av} and discuss the convergence properties of ϕ_{av} when w is PE. In particular, show the asymptotic stability of the parameter vector and characterize the type of convergence.

b) Repeat part a) with the least-squares with forgetting factor

$$\dot{P} = -g\,(-\lambda P + P\,w\,w^T P) \qquad \lambda > 0$$

Problem 5.1

a) Consider the system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

Prove a special case of theorem 5.3.1, by showing that

If: $A(t)$, $B(t)$ are piecewise continuous and bounded, and

$$\dot{x}(t) = A(t)\,x(t)$$

is exponentially stable

Then: (i) $\|x\|_\infty \le \gamma_\infty \|u\|_\infty$ for $x_0 = 0$ and for $\|u\|_\infty < \infty$. Find an expression for γ_∞ as a function of the constants m and α of the exponential stability definition, and of the bounds on $A(t)$ and $B(t)$.
(ii) $|x(t)|$ converges to a ball of radius $\gamma_\infty \|u\|_\infty$ if $x_0 \neq 0$.

b) Find an example of a system

$$\dot{x} = f(t, x)$$

such that $x = 0$ is a stable equilibrium point, but $\dot{x} = f(t, x) + \varepsilon$ has unbounded trajectories for *any* $\varepsilon > 0$.

Problem 5.2

The following problem illustrates the tradeoff between speed of convergence and robustness.

a) Let the plant

$$\dot{y}_p = -a_p\,y_p + k_p\,r$$

where k_p is unknown and $a_p > 0$ is known. One wishes to identify k_p. Exploiting the fact that a_p is known, the following identifier is created

$$\dot{w} = -a_p\,w + r$$

$$\dot{\theta} = -g P w (\theta w - y_p) \qquad g > 0$$

$$\dot{P} = g (\lambda P - P^2 w^2) \qquad \lambda > 0$$

Explain how this identifier is related to the algorithms presented in the book and what the nominal value of the parameter θ is. Simulate the responses for $a_p = 1$, $k_p = 1$, $\lambda = 1$, $P(0) = 1$, and the other inital conditions be all zero. Let $r = \sin(t)$ and experiment with the values $g = 0.2, 1, 5, 25$.

b) Replace the equation for w by

$$\dot{w} = -a_{pi} w + r$$

Simulate again the responses, with $a_{pi} = 0.9$.