## 2.7. Temporal coherence

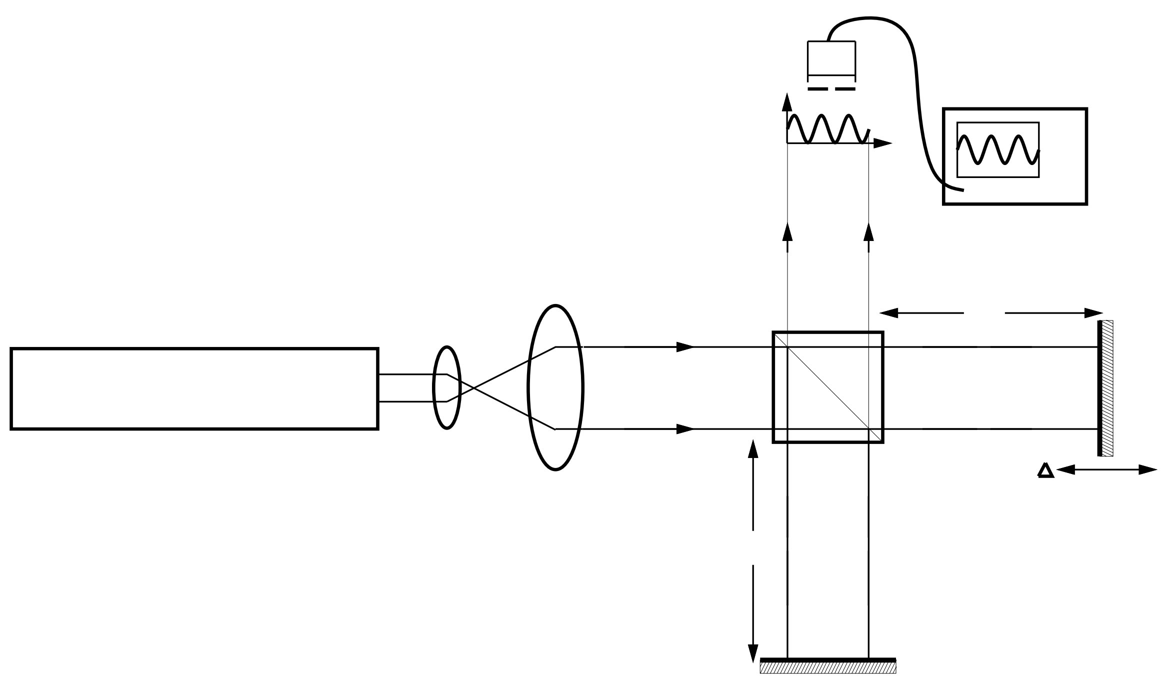
An ideal light source emits perfectly monochromatic (i.e. single frequency, or temporally coherent), and spatially coherent radiation, such as described by the simple plane wave solution

$$E(z,t) = E_0 e^{j(\omega_0 t - k_0 z)}.$$

This solution has a well defined angular frequency $\omega_0 = 2\pi\nu_0$, resulting from the fact that the solution extends sinusoidally over all values of time, and therefore exhibits perfect temporal coherence, which means that we can predict the phase of the wave for any time given the value of the phase at some known time. Therefore, the wavefront has perfect temporal correlation. In addition, being a plane wave with well defined wavefronts that cover all space transverse to the direction of propagation, this solution has perfect spatial coherence, which means that we can preduct the phase of the wave for any point in a transverse plane given the phase at some known point in that same plane.

No real light source has perfect coherence. We will first talk about limited temporal coherence. For a source with limited temporal coherence, there exists some frequency bandwidth $\delta\nu$ about the center frequency $\nu_0$. For an incoherent light source, $\delta\nu$ is determined by the natural linewidth of the source. A laser source, on the other hand, typically has multiple longitudinal modes $\nu_m$, each with linewidth $\delta\nu$ controlled by the cavity finesse $\mathcal{F}$ (and the effects of gain narrowing). The longitudinal modes are spaced in frequency by $\Delta\nu = c/2L$, which is set by the cavity length $L$. The number of longitudinal modes is determined by $\Delta\nu$ and the gain bandwidth, which arises from either a single homogeneously-broadened atomic or molecular resonance, or a collection of many such resonances, which is termed inhomogeneous broadening.

The best way to think about temporal coherence is to consider an amplitude-splitting Michelson interferometer, which is used to generate two identical wavefronts from the output of a single light source, as shown in the figure. One of these wavefronts can be delayed with




respect to the other by a movable mirror. Light from the source passes through the beamsplitter which breaks the beam into two paths - one beam for each arm of the interferometer. Each beam reflects off a mirror and passes through the beamsplitter again. Under square-law detection, the wavefronts combine to form intensity interference fringes at the output.

Assume that the electric field of the source output is written

$$\mathcal{R}e\left[A(\mathbf{r},t)\right]$$

The beamsplitter divides the intensity in half (and therefore divides the field by a factor $\sqrt{2}$). The electric field in the observation plane can be written

$$E(\mathbf{r},t)=\frac{1}{2}A(t)-\frac{1}{2}A(t-2\Delta/c)e^{ik_f\sin\theta x},$$

where $\Delta = l_1 - l_2$ and $2\Delta$ is the total path length difference between the two arms of the interferometer and $\theta$ is a small tilt angle between the two interfering beams that gives rise to a spatial fringe pattern. Any detector will only sense the time-averaged optical intensity, which is given by

$$\begin{aligned}\langle I(x,t)\rangle &= \frac{1}{2\eta}\langle|E(x,t)|^2\rangle \\ &= \frac{1}{2}\langle I\rangle\left[1+\gamma(\tau)\cos\left(k_x x\right)\right],\end{aligned}$$

where $\tau = 2\Delta/c$, $k_x = k_f\sin(\theta)$, and the degree of coherence

$$\gamma(\tau)=\frac{(1/2\eta)\Gamma(\tau)}{\langle I\rangle},$$

which is essentially the "coherence envelope." The mutual coherence function is defined as

$$\Gamma(\tau)=\langle A^* A(\tau)\rangle,$$

and is the autocorrelation of the electric field. The mutual coherence function can also be calculated from the inverse-Fourier transform of the spectral intensity of the field:

$$\Gamma(\tau)=\mathcal{F}^{-1}\{I(\nu)\}.$$

The "visibility" of the spatial interference fringes at the ouput of the interferometer is an easily measured quantity. The visibility is defined as the ratio

$$V=\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}},$$

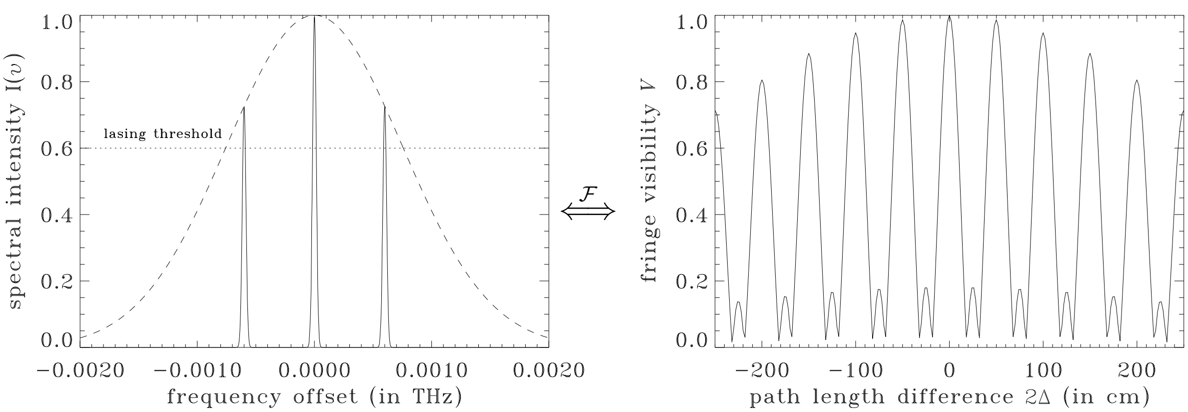
where $I_{\max} = (1/2)\langle I\rangle[1+\gamma(\tau)]$ is the maximum intensity of the fringe pattern and $I_{\min} = (1/2)\langle I\rangle[1-\gamma(\tau)]$ is the minimum intensity of the fringe pattern. Therefore, the visibility can be re-written

$$V(\tau)=\gamma(\tau)\propto\mathcal{F}^{-1}\{I(\nu)\},$$

which is the degree of coherence. Therefore, the measure of the fringe visibility as a function of path length difference maps out the degree of coherence of the source. It should also be mentioned that $0 \le V \le 1$ - for a perfectly monochromatic (temporally coherent) field, $V = 1$, and for a perfectly incoherent (infinitely broadband) field, $V = 0$.

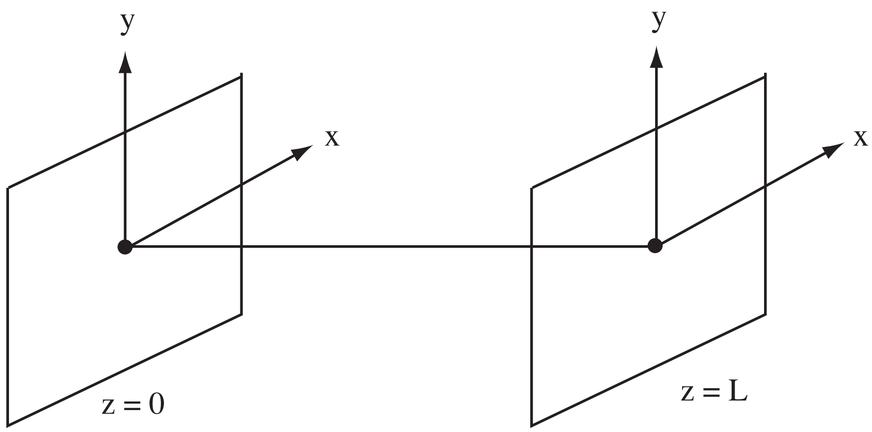
From the Fourier relationship between the source spectral density and the visibility, it is a relatively straightforward process to determine the coherence length from the spectral characteristics. The coherence time $t_c$ is defined as the difference in values of $\tau$ about $\tau = 0$ for which the fringe visibility drops to 0.5, i.e. $t_c$ is the temporal FWHM about zero path length. For an incoherent source of linewidth $\delta\nu$, the width of the coherence function $\gamma$ is $t_c = 1/\delta\nu$, and is the time over which the frequencies remain approximately in-phase. The coherence time gives rise to a coherence length $l_c = ct_c$, which is the physical distance over which coherence is maintained. Therefore, a spectrally pure source (such as a stabilized single longitudinal-mode laser) would have a long coherence time and length, while a spectrally broad-band source (such as white light or multi-mode laser) would have a short coherence length.

Most lasers oscillate in multiple longitudinal modes, where each longitudinal mode "samples" the gain bandwidth curve at intervals $\Delta\nu$. The lasing spectrum is shown on the left-hand side of the following figure. The coherence function for such a laser is periodic (due




to sampling in the Fourier domain), and modulated by a long envelope, as shown on the right-hand side of the figure. The coherence length is the width of the narrow periodic lobes, given by $l_c = c/(n-1)\Delta\nu$, where $n$ is the number of longitudinal modes and $(n-1)\Delta\nu$ is the total bandwidth of the source. The width of the envelope of the coherence (or visibility) function is given by $c/\delta\nu$, and results in reduced visibility over distances much longer than the actual coherence length.

### 2.8. Scalar Diffraction




We want to describe the propagation of a beam from a plane at $z = 0$ to a plane of $z = L$. Assume a monochromatic field $e(x, y, z, t) = A(x, y, z)e^{j\omega t}$. The scalar wave equation $\nabla^2 e = \mu\epsilon \frac{\partial^2 e}{\partial t^2}$ becomes $\nabla^2 A + k^2 A = 0$ where $k^2 = \omega^2 \mu\epsilon$ . The fundamental solutions of this equation are plane waves (i.e. plane waves are the eigenmodes, or the stationary solutions). Each plane wave progagates with a different propagation constant $k_z$. Therefore, it makes sense to decompose the electric field at $z = 0$ into a set of plane waves. This operation is known as the Fourier transform:

$$\tilde{A}\left(k_x, k_y\right) = \int\int A(x, y)e^{-j(k_x x + k_y y)} dxdy$$

Now, we have to figure out how to evolve the complex amplitude $\tilde{A}$ of each plane wave. We do this by transforming the wave equation:

$$\nabla^2 A + k^2 A = 0 \quad \langle \frac{F_{xy}}{F_{k_x k_y}} \rangle - \left(k_x^2 + k_y^2\right)\tilde{A} + k^2\tilde{A} + \frac{\partial^2 \tilde{A}}{\partial^2} = 0$$

The Fourier equation can be solved

$$\tilde{A}\left(k_x, k_y, z\right) = \tilde{A}\left(k_x, k_y, 0\right) e^{jk_z z}$$

where the wavenumber

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \quad H = \left(k_x, k_y, z\right) = e^{-jk_z z}$$

is the transfer function of propagation.

To recover the electric field at $z = L$ , we inverse transform $\tilde{A}\left(k_x, k_y, z = L\right)$ as follows:

$$A(x, y, z = L) = \frac{1}{4\pi^2} \int\int \tilde{A}\left(k_x, k_y, z = L\right) e^{-jk_z z} e^{j(k_x x + k_y y)} dk_y dk_y$$

For small angles of propagation $k_y, k_y << k$, then we can write

$$k_z = \sqrt{k^2 - k_k^2 - k_y^2} \approx k\left(1 - \frac{k_x^2 + k_y^2}{2k^2}\right)$$

and the Fourier integral can be evaluated

$$A(x,y,z=L) = \frac{1}{4\pi^2}\int\int A(x',y',0)\ p(x-x',y-y')dx'dy'$$

which is a convolution between $A$ and $p$, where $p$ is the impulse response function of propagation

$$p = \mathcal{F}^{-1}\left\{e^{-jk\left(1-\frac{k_x^2+k_y^2}{2k^2}\right)L}\right\} = \frac{2\pi ik}{L}e^{-jkL}e^{-jk\frac{x^2+y^2}{2L}}$$

Written out in full

$$A(x,y,z=L) = \frac{j}{\lambda L}e^{-jkL}\int\int A(x',y',0)e^{-\frac{jk}{2L}\left[(x-x')^2+(y-y')^2\right]}dx'dy'$$

which is the Fresnel-Kirchoff diffraction integral.

Now, let's use this integral to calculate the diffraction pattern from a rectangular slit (see Fig. 1.25 in the book). Since this is a 1-D problem, the integral reduces to

$$A(x,z=L) = \frac{j}{\lambda L}e^{-jkL}\int A(x',0)e^{-\frac{jk}{2L}(x-x')^2}dx'.$$

We assume that the field is constant across the aperture (i.e. the aperture is illuminated by a plane wave), so that

$$A(x,z=L) = \frac{jA_0}{\lambda L}e^{-jkL}\int_{-a/2}^{a/2} e^{-\frac{jk}{2L}(x-x')^2}dx'.$$

This integral can be evaluated in general using elliptic functions, but we will make one simplification that the size of the aperture divided by the observation distance is small, so that we can ignore $(x')^2/L$. This is called the Fraunhofer approximation, and the integral reduces to

$$A(x,z=L) = \frac{jA_0}{\lambda L}e^{-jkL}e^{jx^2/2L}\int_{-a/2}^{a/2} e^{-\frac{jk}{L}xx'}dx'.$$

After performing the integration, we get the following solution:

$$A(x,z=L) = aA_0\text{sinc}\left(\frac{kax}{2L}\right) = aA_0\text{sinc}\left(\frac{ka\sin\theta}{2}\right),$$

where $\sin\theta = x/L$ and we've neglected the phase terms.

### 2.9. Spatial coherence

For a source with limited spatial coherence, there exists some angular spread about the propagation direction $\vec{k}$. Therefore, a plane wave has perfect spatial coherence since it propagates in a single direction with a well-defined wavevector. Spatial coherence can be measured by interfering two different portions of a wavefront. This can be accomplished by using a wavefront splitting interferometer, as illustrated in the figure for the Young's double slit interferometer. The visibility of the fringe pattern is measured as function of the separation $\ell$ between the two slits. The value of $\ell$ for which the visibility drops to 1/2 is called the spatial coherence length.

From our diffraction integral, we can describe the field at the observation plane from the expression

$$A(x, z = L) = \frac{j}{\lambda L} e^{-jkL} \int \left[A(x' - \ell/2, 0) + A(x' + \ell/2, 0)\right] e^{-\frac{jk}{2L}(x-x')^2} dx'.$$

We could go ahead and evaluate this integral as before, but if we assume that the width of each slit is small, then each slit can be described by a delta function:

$$A(x, z = L) = \frac{j}{\lambda L} e^{-jkL} \int \left[A_0 \delta(x' - \ell/2) + A_0 \delta(x' + \ell/2)\right] e^{-\frac{jk}{2L}(x-x')^2} dx',$$

so that the amplitude on the screen is given by

$$A(x, z = L) \propto \cos\left(\frac{k\ell x}{2L}\right),$$

under the Fraunhofer approximation.

Taking into account spatial coherence, the time-averaged intensity would be

$$\begin{aligned} \langle I(x,t) \rangle &= \frac{1}{2\eta} \langle |E(x,t)|^2 \rangle \\ &= \frac{1}{2} \langle I \rangle \left[1 + \gamma(\ell) \cos\left(k\ell x/2L\right)\right], \end{aligned}$$

and the degree of coherence

$$\gamma(\ell) = \frac{(1/2\eta)\Gamma(\ell)}{\langle I \rangle},$$

which is spatial coherence envelope. The mutual coherence function is defined as

$$\Gamma(\ell) = \langle A^* A(\ell) \rangle,$$

and is the spatial autocorrelation of the electric field. The mutual coherence function can also be calculated from the inverse-Fourier transform of the transverse spatial frequency of the field:

$$\Gamma(\ell) = \mathcal{F}^{-1}\{I(k_x)\}.$$