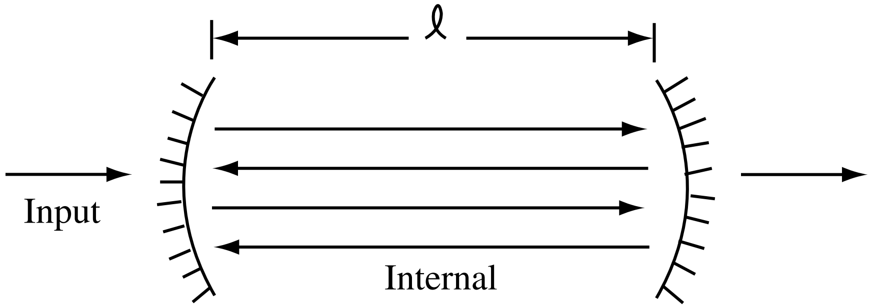
## 4. Optical Resonators

Optical resonators are used to build up large intensities with moderate input.




Resonators are typically characterized by their quality factor:

$$Q = w \times \frac{\text{stored energy}}{\text{dissipated energy}}$$

The field inside a resonator can be represented by $e(z,t) = E\sin(wt)\sin(kz)$ which is a standing wave. The average energy in the resonator is

$$\mathcal{E}_{elec} = \frac{A\epsilon}{2T}\int_0^T\int_0^\ell e^2(z,t)dzdt = \frac{1}{q}\epsilon E^2 V$$

where $A$ is the area cross-section of cavity, $\ell$ is the length, and $T = 2\pi/w$ is the period. $V = A\ell$ is the cavity volume. The total stored energy is

$$\mathcal{E} = 2\mathcal{E}_{elec} = \frac{1}{4}\epsilon E^2 V$$

If we assume the input power is $P$, then

$$Q = \frac{w\epsilon E^2 V}{4P}.$$

Note that at steady state, dissipated power equals input power. The peak field is therefore

$$E = \sqrt{\frac{4QP}{w\epsilon V}}.$$

A resonator cavity supports a large number of modes. For a laser resonator, many times we want only one mode to lase. This can be accomplished in a number of ways. One way is for the cavity to be $\sim \lambda^3$ in volume, but this is not always practical. Another way is to use mode selection mechanisms in the cavity. This will be discussed later.

A perfectly conducting box resonator has modes of the form

$$e(x,y,z) \propto \sin(k_x x)\sin(k_y y)\sin(k_z z)\sin(\omega t)$$

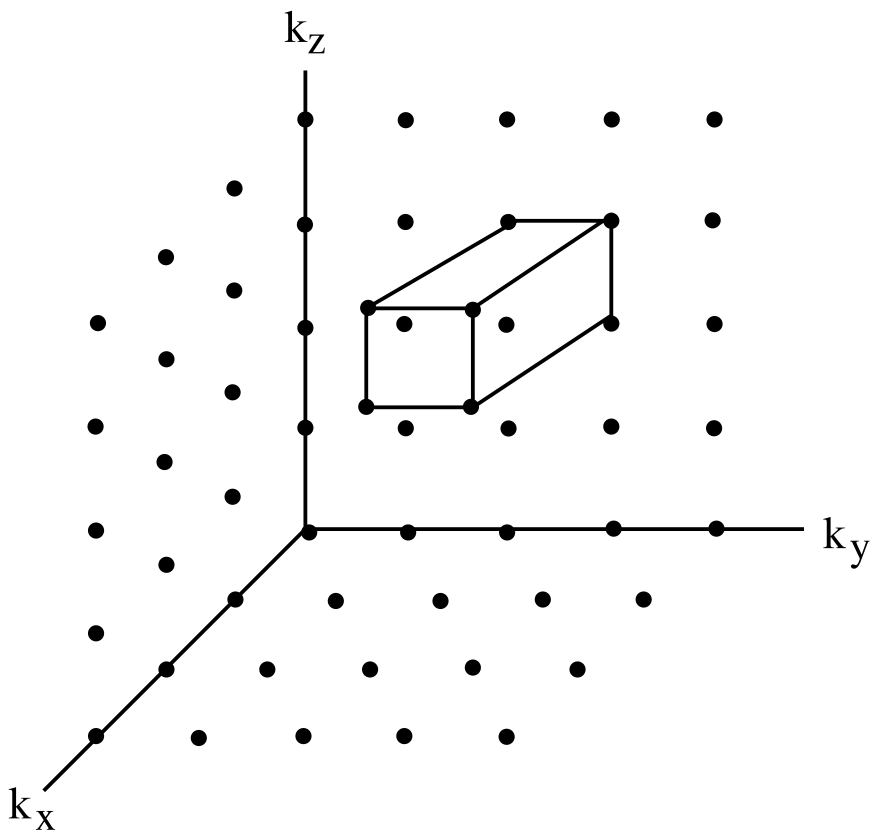
where

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{c}n\right)^2 = \left(\frac{2\pi\nu n}{c}\right)^2$$

The field must be zero at the boundaries (or, saying the same thing, an integer number of half wave lengths must fit between opposite walls). Therefore, we have the conditions

$$k_x = r\frac{\pi}{a} \qquad k_y = s\frac{\pi}{b} \qquad k_x = t\frac{\pi}{c}$$

where $a$, $b$, $c$ are the lengths of the sides of the box, the indices $(r, s, t)$ define a mode and are positive integers. The $k$ -space (momentum space) is discretely populated. Each point




(mode) is indexed by $(r, s, t)$.

The elemental volume is

$$V_m = \frac{\pi^3}{abc} = \frac{\pi^3}{V}$$

For any given mode,

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi\nu n}{c}$$

(Note that many modes have the same $k_i$. These modes are degenerate in frequency.) The number of modes that lie within the volume of a sphere of radius $k$ is

$$N(k) = \frac{\left(\frac{1}{8}\right)\frac{4\pi}{3}k^3}{\frac{\pi^3}{V}} = \frac{k^3 V}{6\pi^2}$$

The factor of (1/8) comes from $r, s, t > 0$, so we populate one octant of the sphere. Now, the number of modes with frequencies between 0 and $\nu$ is

$$N(\nu) = \frac{4\pi\nu^3 n^3 V}{3c^3}$$

The number of modes within a small frequency range about $\nu$ is

$$P(\nu) = 2\frac{dN(\nu)}{d\nu} = \frac{8\pi\nu^2 n^3 V}{3c^3}$$

or

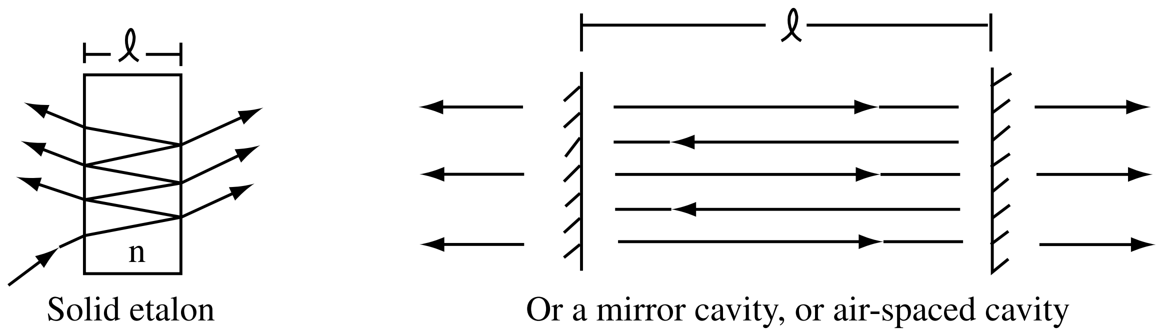
$$dN(\nu) = \left(\frac{8\pi\nu^2 n^3 V}{3c^3}\right) d\nu$$

The extra factor of 2 comes from 2 possible polarization states of the field.

As an example, if the volume of a resonator is $V = 1$ cm$^3$, the frequency taken as an optical frequency $\nu = 3 \times 10^{14}$, and we look over an interval $d\nu = 3 \times 10^{10}$ (which is typical of laser gain media), we have $N \sim 2 \times 10^9$ modes. For a laser, this may be too many modes (as most of them propagate in different directions). Therefore, lasers typically use open resonator cavities, such that modes can only exist in one direction.

### 4.1. Fabry-Perot Etalon

The Fabry-Perot etalon is a linear cavity consisting of two opposing, high reflectance mirrors. This can be considered the most basic laser cavity. These cavities can take a variety of physical forms We will analyze a solid etalon, but the results apply directly to a cavity by




setting the refractive index $n = 1$. The air-spaced cavity is essentially a laser cavity, or open resonator.

The reflection and transmission properties of a Fabry-Perot can be considered using the multiple reflections model. In this model, we must account for the amplitude and phase for each "ray," then sum up all of the reflected rays (labeled B) and all of the transmitted rays (labeled A). The incident ray has amplitude $A_i$.

The first reflected ray has amplitude

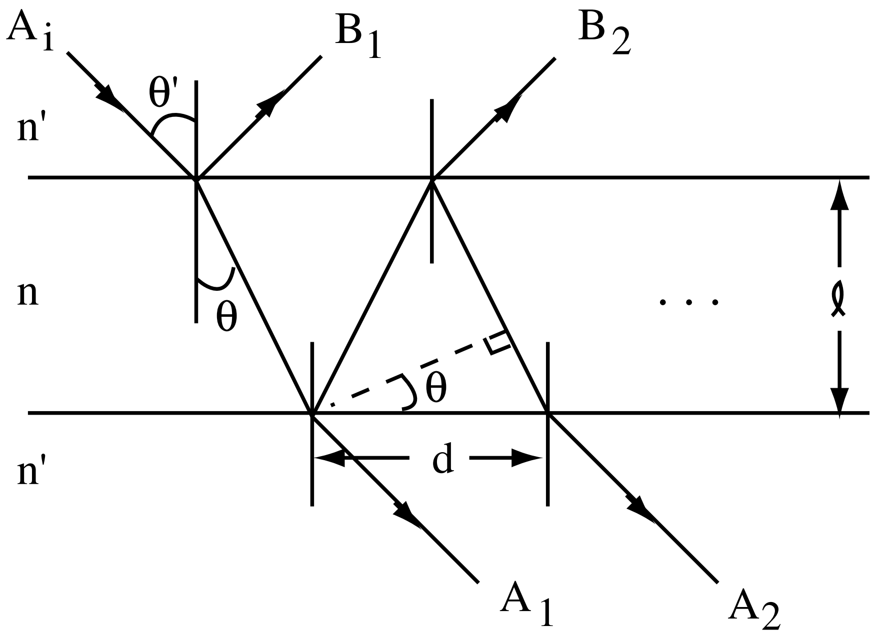
$$B_1 = rA_i,$$

where $r$ is the amplitude reflection coefficient from index n' to n. The first transmitted ray has amplitude

$$A_1 = tt'A_i,$$

where $t$ is the amplitude transmission coefficient from $n'$ to $n$ and $t'$ is the amplitude transmission coefficient from $n$ to $n'$. Following the same procedure

$$B_2 = tr't'A_ie^{i\phi}$$
$$A_2 = tr'r't'A_ie^{i\phi}$$




Now, the relative phase shifts $\phi$ between rays $B_1$ and $B_2$, and between $A_1$ and $A_2$ need to be included. The phase $\phi$ is due to the extra path length inside the etalon for each component of the wavefront. In the figure

$$\phi = \frac{4\pi n\ell\cos\theta}{\lambda}$$

and

$$\sin\theta = \frac{n'}{n}\sin\theta'$$

The total reflected wave is given by

$$\begin{aligned} A_r &= [B_1 + B_2 \cdots] = \left[r + tr't'e^{i\phi} + t(r')^3t'e^{2i\phi} + \cdots\right] A_i \\ &= \left[r + tr't'e^{i\phi}\left(1 + (r')^2e^{i\phi} + \cdots\right)\right] A_i \end{aligned}$$

and the transmitted wave

$$\begin{aligned} A_t &= [A_1 + A_2 \cdots] = \left[tt' + tr'r't'e^{i\phi} + \cdots\right] A_i \\ &= \left[tt'\left(1 + (r')^2e^{i\phi} + \cdots\right)\right] A_i \end{aligned}$$

The quantity () is common to both waves

$$\begin{aligned} \left(1 + (r')^2e^{i\phi} + (r')^4e^{2i\phi} + \cdots\right) &= 1 + x + x^2 + \cdots \\ &= \frac{1}{1-x} \quad \text{where} \quad x = (r')^2e^{i\phi} = R\,e^{i\phi}. \end{aligned}$$

Now, $r' = -r$, $t' = t$, $R = r^2$, $T = t^2$ and $R^2 + T^2 = 1$, so that

$$\begin{aligned} A_r &= r - \frac{Tre^{i\phi}}{1 - R\,e^{i\phi}}A_i = \frac{r\left(1 - R\,e^{i\phi}\right) - Tre^{i\phi}}{1 - R\,e^{i\phi}}A_i \\ &= \frac{r\left(1 - R\,e^{i\phi} - Te^{i\phi}\right)}{1 - R\,e^{i\phi}}A_i \end{aligned}$$

$$= \frac{r\left(1-e^{i\phi}\right)}{1-R\,e^{i\phi}}A_i = \frac{\sqrt{R}\left(1-e^{i\phi}\right)}{1-R\,e^{i\phi}}A_i$$

$$A_t = \frac{T}{1-R\,e^{i\phi}}A_i$$

The intensities of the reflected and transmitted waves are

$$\frac{I_r}{I_i} = \frac{A_r A_r^*}{A_i A_i^*} = \frac{4R\sin^2(\phi/2)}{(1-R)^2 + 4R\sin^2(\phi/2)}$$

$$\frac{I_t}{I_i} = \frac{A_t A_t^*}{A_i A_i^*} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\phi/2)}$$

Note that when you have mirrors of different reflectances, use the substitution $R = \sqrt{R_1 R_2}$. The transmission is 1 (i.e. $I_f = I_i$) when

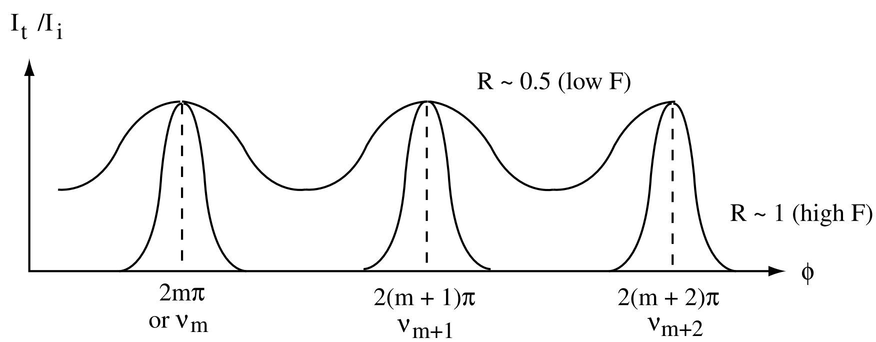
$$\phi = \frac{4\pi n\ell\cos\theta}{\lambda} = 2m\pi \qquad m = \text{integer},$$

which can be written in terms of frequency ($\nu = c/\lambda$) as

$$\nu_m = m\frac{c}{2n\ell\cos\theta}$$

For fixed $\ell$ and $\theta$, transmission peaks occur periodically in $\nu$ with spacing $c/2n\ell\cos\theta$ , which is called the free-spectral range. The frequencies $\nu_m$ are the only ones allowed to oscillate within the cavity.

$$\Delta\nu = \nu_{m+1} - \nu_m = \frac{c}{2n\ell\cos\theta}$$




Including losses in the cavity and on resonance ($\phi = 2\pi m$),

$$\frac{I_t}{I_i} = \frac{(1-R^2)A}{(1-RA)^2}$$

where $1-A$ is the fractional intensity loss per pass.